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On domination cover edge pebbling number of generalized Petersen Graph, Jewel Graph and triangular snake graph

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ABSTRACT

Given a configuration of pebbles on the edges of a connected graph G, an edge pebbling move is defined as the removal of two pebbles off an edge and placing one on an adjacent edge. The domination cover edge pebbling number of a graph G is the minimum number of pebbles required such that the set of edges that contain pebbles form an edge dominating set S of G, for the initial configuration of pebbles can be altered by a sequence of pebbling moves and it is denoted by $\psi_e(G)$ for a graph G. In this paper, we determine $\psi_e(G)$ for Generalized Petersen graph, Jewel graph and Triangular snake graph.

Keywords: edge pebbling move, edge dominating set, generalized Petersen graph, Jewel graph, triangular snake graph, domination cover edge pebbling number

2020 Mathematics Subject Classification: 05C57, 05C70.

1. Introduction

Graph pebbling is a mathematical game and it is one of the developing concepts in Graph theory. In Graph pebbling, a small but significant number of fundamentals have been studied. F.R.K. Chung [2] recorded the pebbling concept in 1989. Elegant view of this concept is elaborated by G. Hurlbert [5]. Lourdusamy et al. [6, 7, 8] contribution to the field of Graph Pebbling is highly valued. Minimum number of pebbles that are required to reach any target vertex with one pebble regardless of the initial configuration of the pebbles is the pebbling number of graph G and it is denoted by $f(G)$ [2]. Distinct genres

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of pebbling numbers have been illustrated, as referenced in [1, 4, 14, 16, 18, 17]. In 2020, Priscilla Paul [10] strengthened the concept of pebbling by finding the edge pebbling number and cover edge pebbling number of graphs. The minimum number of pebbles such that any distribution of pebbles on the edges of G allows one pebble to be sent to any target edge is the edge pebbling number. In cover edge pebbling, the objective is to cover all the edges with at least one pebble irrespective of the initial allocation of pebbles. Priscilla Paul has also discovered the covering cover edge pebbling number in graphs. To have profound understanding in these concepts, please refer to [11, 13, 12]. The Concept of domination cover pebbling was established by J. Gardner et al. [3] and they have also analysed the domination cover pebbling number for complete graphs, wheel graphs, and for some standard graphs. For more about domination cover pebbling, see [19, 21]. We have introduced the domination cover edge pebbling number $\psi_e(G)$ and compute the domination cover edge pebbling number for the Path, Wheel, Comb, Complete, n-star, Fan and Twig graphs in [15]. In this paper, we determine $\psi_e(G)$ for the Generalized Petersen graph, Jewel graph and Triangular snake graph. Throughout this paper $P(e_1)$ = 1 refers as the pebble on the edge e_1 is 1.

2. Preliminaries

Definition 2.1. [10] An edge pebbling move on a graph G is defined to be the removal of two pebbles from one edge and the addition of one pebble to an adjacent edge.

Definition 2.2. [9] A set D of edges is called an edge dominating set for a graph G if every edge not in D has at least one neighbor in D.

Definition 2.3. [20] Generalized Petersen graph $GP_{n,k}$ for $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ consisting of an inner star polygon $\{n, k\}$ and outer regular polygon $\{n\}$ with corresponding vertices in the inner and outer polygons connected with edges i.e. $V(GP_{n,k}) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(GP_{n,k}) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k} : 1 \leq i \leq n$, subscripts modulo $n\}$.

Definition 2.4. [15] The domination cover edge pebbling number of a graph G is the minimum number of pebbles required such that the set of edges that contain pebbles form an edge dominating set S of G, for the initial configuration of pebbles can be altered by a sequence of pebbling moves. It is denoted by $\psi_e(G)$ for a graph G.

Theorem 2.5. [15] *The domination cover edge pebbling number for the path P_n graph is* $\psi_e(P_n) = \lceil \frac{2^n-2}{7} \rceil$; $n \geq 3$.

Theorem 2.6. [15] *The domination cover edge pebbling number for the n-star graph is* $\psi_e(S_n) = 1$.

Theorem 2.7. [15] *The domination cover edge pebbling number for the fan graph is* $\psi_e(F_n) = 2n\text{-}5$; $n \geq 5$, *when n is odd and* $\psi_e(F_n) = 2n\text{-}6$; $n \geq 6$, *when n is even.*

3. Main results

Theorem 3.1. *The domination cover edge pebbling number for the Jewel J_n graph is $\psi_e(J_n) = n+3$; $n \geq 1$.*

Proof. Let J_n be the Jewel graph with the edge set $\{e_1, e_2, ..., e_{2n+5}\}$. For Jewel graph J_n, consider a cycle C_4 with vertices a, b, c and d, then add an edge e_1 between the vertices b and d, and finally introduce n new vertices that connect to both vertices a and c. Let e_2, e_3, e_4, e_5, ..., e_{n+3} be the adjacent edges incident with a common vertex a and let e_{n+4}, e_{n+5}, ..., e_{2n+5} be the adjacent edges incident with a common vertex c.

Take $n+2$ pebbles for distribution. Place $n+2$ pebbles with the distribution such that one pebble on the edge e_1, and one pebble each on the edges e_2, e_3, e_4, e_5, ..., e_{n+2} of J_n. Now, the adjacent edge of e_{n+3} that is incident with a vertex c will be left undominated. Therefore, $\psi_e(J_n) \geq n+3$.

For proving the sufficient part, place $n+2$ pebbles with the distribution such that one pebble each on the edges e_2, e_3, e_4, e_5, ..., e_{n+3}. So, we need $n+2$ pebbles in such a distribution to dominate the edges of J_n. Therefore, by this distribution of pebbles, the edges of J_n can be dominated with less than $n+3$ pebbles. Also, the claim is true if we place $n+3$ pebbles on any one of the edges of J_n. So, by all possible configurations, the domination cover edge pebbling number for the Jewel J_n graph is $\psi_e(J_n) = n+3$; $n \geq 1$. □

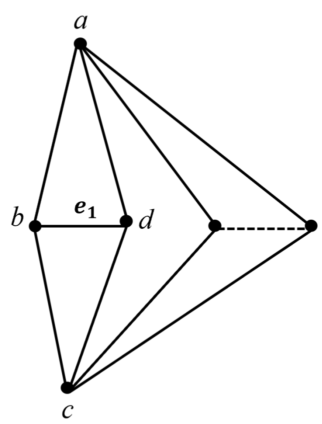

Fig. 1. Jewel graph J_n

Theorem 3.2. *The domination cover edge pebbling number for the triangular snake TS_n graph is $\psi_e(TS_n) = 2^n-(4k-1)$; $n \geq 5$ and $k = a_1, a_2, a_3, ...$; $a_1 = 3$, $a_2 = 4a_1-1$, $a_3 = 4a_2-1$, ... when n is odd and $\psi_e(TS_n) = 2^n-2(4k-1)$; $n \geq 6$ and $k = a_1, a_2, a_3, ...$; $a_1 = 3$, $a_2 = 4a_1-1$, $a_3 = 4a_2-1$, ... when n is even.*

Proof. Let TS_n be the triangular snake graph with the edge set $E(TS_n) = \{e_1, e_2, ..., e_{3(n-1)}\}$ and let $u_1, u_2, u_3, ..., u_n$ be the vertices of P_n and $\{e_1, e_4, e_7, ..., e_{3(n-1)-2}\}$ be the edges of P_n. Let the edges e_{i+1} and e_{i+2} be adjacent to e_i for every $i = 1, 4, 7, ...,$

3(n-1)-2 and these edges are incident with vertices in pairs, with edges e_2 and e_3 incident with v_1, the edges e_5 and e_6 incident with v_2, and so on, up to edges $e_{3(n-1)-1}$ and $e_{3(n-1)}$ incident with v_{n-1} i.e. the vertices u_1 and u_2 are adjacent to v_1.

Case 1. when n is odd.

Take 2^n-(4k-1)-1 pebbles for distribution. Place 2^n-(4k-1)-1 pebbles on the edge e_2. Now, the edge e_2 is left undominated after the pebble distribution. Therefore, $\psi_e(TS_n) \geq 2^n$-(4k-1), when n is odd. For proving the sufficient part, place the pebbles on any one of the edges of P_n except the edge $e_{3(n-1)-2}$ i.e. to dominate the edges of TS_n by the distribution of each edge in the edge set $\{e_2, e_4, e_{10}, e_{16}, e_{22}, ..., e_{3(n-1)-2}\}$ has one pebble, we need $9+\sum_{k=4}^{\lceil \frac{n}{2} \rceil} 2^k$ & $k \neq 5, 7, 9, 11, 13, ...$ pebbles on the edge e_4. Therefore, by this distribution of pebbles, the edges of TS_n can be dominated with less than 2^n-(4k-1)pebbles. Also, the claim is true if we place 2^n-(4k-1) pebbles on the edge e_2. So, by all possible configurations, $\psi_e(TS_n)= 2^n$-(4k-1); $n \geq 5$ and $k = a_1, a_2, a_3, ...; a_1 = 3, a_2 = 4a_1$-1, $a_3 = 4a_2$-1, ... when n is odd.

Case 2. when n is even.

Take 2^n-2(4k-1)-1 pebbles for distribution. Place 2^n-2(4k-1)-1 pebbles on the edge e_2. Now, the edge e_5 is left undominated after the pebble distribution. Therefore, $\psi_e(TS_n) \geq 2^n$-2(4k-1), when n is even. For proving the sufficient part, place the pebbles either on the edge e_1 or on the edge $e_{3(n-1)-2}$ i.e. place $5+\sum_{k=4}^{n-2} 2^k$ & $k \neq 5, 7, 9, 11, 13, ...$ pebbles with the distribution such that each edge in the edge set $\{e_1, e_7, e_{13}, e_{19}, e_{25}, ..., e_{3(n-1)-2}\}$ will have one pebble. So, we need $\sum_{k=4}^{n-2} 2^k$ & $k \neq 5, 7, 9, 11, 13, ...$ pebbles in such a distribution to dominate the edges of TS_n. Therefore, by this distribution of pebbles, the edges of TS_n can be dominated with less than 2^n-2(4k-1) pebbles. Also, the claim is true if we place 2^n-2(4k-1) pebbles on the edge e_2. So, by all possible configurations, $\psi_e(TS_n) = 2^n$-2(4k-1); $n \geq 6$ and $k = a_1, a_2, a_3, ...; a_1 = 3, a_2 = 4a_1$-1, $a_3 = 4a_2$-1, ... when n is even. □

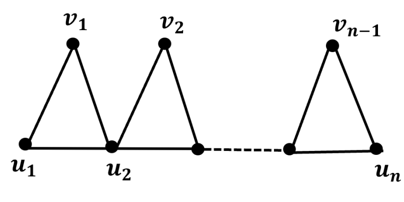

Fig. 2. Triangular snake graph TS_n

Theorem 3.3. *The domination cover edge pebbling number for the Generalized Petersen graph $GP_{3,1}$ is $\psi_e(GP_{3,1}) = 6$.*

Proof. For $GP_{3,1}$, let e_1, e_2, and e_3 be the outer edges of $GP_{3,1}$. Let e_4 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_2. Let e_5 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_2 and e_3. Let e_6 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_3. Let e_7 be the inner edge of $GP_{3,1}$ and adjacent to both the edges e_4 and e_5. Let e_8 be the inner edge of $GP_{3,1}$ and adjacent to both the edges e_5 and e_6. Let e_9 be the inner edge of $GP_{3,1}$ and adjacent to both the edges e_4 and e_6. Now, we need to find the minimum number of pebbles to cover the edges that form an edge dominating set.

If $P(e_1) = 1$, the edges e_2, e_3, e_4, and e_6 can be dominated and the edges e_5, e_7, e_8 and e_9 will be left undominated. So, 1 is not the required number of pebbles.

If $P(e_1) = 1$ and $P(e_9) = 1$, we can dominate all the edges of $GP_{3,1}$ with 2 pebbles but if $P(e_9) = 2$, we cannot dominate all the edges of $GP_{3,1}$ with 2 pebbles. So, 2 is not the required number of pebbles.

If $P(e_1) = 1$, $P(e_2) = 1$ and $P(e_3) = 1$, we cannot dominate all the edges of $GP_{3,1}$ with 3 pebbles. So, 3 is not the required number of pebbles.

If $P(e_1) = 2$ and $P(e_2) = 2$, any one of the edges of $GP_{3,1}$ will be left undominated. So, 4 is not the required number of pebbles.

If $P(e_1) = 1$ and place 4 pebbles on any one of the edges that connects both the inner and outer polygons, we can dominate all the edges of $GP_{3,1}$ with 5 pebbles but if $P(e_1) = 3$, $P(e_2) = 1$ and $P(e_3) = 1$, we cannot dominate all the edges of $GP_{3,1}$ with 5 pebbles. So, 5 is not the required number of pebbles. If we place 6 pebbles on any one of the edges of $GP_{3,1}$, we can dominate all the edges of $GP_{3,1}$ and if we also alter the configuration of pebbles, 6 pebbles are the minimum number of pebbles required. Thus, the domination cover edge pebbling number for the Generalized Petersen $GP_{3,1}$ graph is $\psi_e(GP_{3,1}) = 6$. □

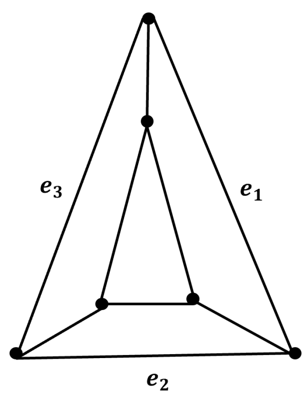

Fig. 3. Generalized petersen graph $GP_{3,1}$

Theorem 3.4. *The domination cover edge pebbling number for the Generalized Petersen graph $GP_{4,1}$ is $\psi_e(GP_{4,1}) = 9$.*

Proof. For $GP_{4,1}$, let e_1, e_2, e_3 and e_4 be the outer edges of $GP_{4,1}$. Let e_5 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_2. Let e_6 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_2 and e_3. Let e_7 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_3 and e_4. Let e_8 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_4. Let e_9 be the inner edge of $GP_{4,1}$ and adjacent to both the edges e_5 and e_6. Let e_{10} be the inner edge of $GP_{4,1}$ and adjacent to both the edges e_6 and e_7. Let e_{11} be the inner edge of $GP_{4,1}$ and adjacent to both the edges e_7 and e_8. Let e_{12} be the inner edge of $GP_{4,1}$ and adjacent to both the edges e_5 and e_8. Now, we need to find the minimum number of pebbles to cover the edges that form an edge dominating set.

If $P(e_1) = 1$, the edges e_1, e_2, e_4, e_5, and e_8 can be dominated and the edges e_3, e_6, e_7, e_9, e_{10}, e_{11} and e_{12} will be left undominated. So, 1 is not the required number of pebbles. If $P(e_1) = 1$ and $P(e_{12}) = 1$, the edges e_3, e_6, e_7, and e_{10} will be left undominated. So, 2 is not the required number of pebbles.

If $P(e_1) = 1$, $P(e_2) = 1$ and $P(e_3) = 1$, the edges e_9, e_{10}, e_{11} and e_{12} will be left undominated. So, 3 is not the required number of pebbles.

If $P(e_1) = 2$ and $P(e_3) = 2$ and consider the distribution of 1 pebble moves to each of the edges e_5 and e_7 then the edges e_6 and e_8 will be left undominated. So, 4 is not the required number of pebbles.

If $P(e_6) = 1$, $P(e_1) = 2$ and $P(e_3) = 2$ and consider the distribution of 1 pebble moves to each of the edges e_4 and e_5 then the edge e_{11} will be left undominated. So, 5 is not the required number of pebbles.

If $P(e_1) = 3$ and $P(e_3) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_5 and e_7, we can dominate all the edges of $GP_{4,1}$ with 6 pebbles but if $P(e_1) = 5$ and $P(e_3) = 1$ and consider the distribution of 1 pebble moves to the edge e_9, the edge e_{11} will be left undominated. So, 6 is not the required number of pebbles.

If $P(e_{12}) = 7$ and consider the distribution of 1 pebble moves to each of the edges e_4 and e_9 then the edge e_2 will be left undominated. So, 7 is not the required number of pebbles.

If $P(e_1) = 4$ and $P(e_3) = 4$ and consider the distribution of 1 pebble moves to each of the edges e_2, e_4, e_6 and e_8, we can dominate all the edges of $GP_{4,1}$ with 8 pebbles but if $P(e_1) = 8$ and consider the distribution of 1 pebble moves to each of the edges e_2, e_4 and e_9, the edge e_{11} will be left undominated. So, 8 is not the required number of pebbles.

If $P(e_1) = 9$ and consider the distribution of 1 pebble moves to each of the edges e_6 and e_{11}, we can dominate all the edges of $GP_{4,1}$ with 9 pebbles. If we also place 9 pebbles on any other edges of $GP_{4,1}$ and alter the configuration of pebbles, we can dominate all the edges of $GP_{4,1}$ and 9 pebbles are the minimum number of pebbles required. Thus, the domination cover edge pebbling number for the Generalized Petersen $GP_{4,1}$ graph is $\psi_e(GP_{4,1}) = 9$. □

Theorem 3.5. *The domination cover edge pebbling number for the Generalized Petersen graph $GP_{5,1}$ is $\psi_e(GP_{5,1}) = 13$.*

Proof. For $GP_{5,1}$, let e_1, e_2, e_3, e_4 and e_5 be the outer edges of $GP_{5,1}$. Let e_6 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_2. Let e_7 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_2 and e_3. Let e_8 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_3 and e_4. Let e_9 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_4 and e_5. Let e_{10} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_5. Let e_{11} be the inner edge of $GP_{5,1}$ and adjacent to both the edges e_6 and e_7. Let e_{12} be the inner edge of $GP_{5,1}$ and adjacent to both the edges e_7 and e_8. Let e_{13} be the inner edge of $GP_{5,1}$ and adjacent to both the edges e_8 and e_9. Let e_{14} be the inner edge of $GP_{5,1}$ and adjacent to both the edges e_9 and e_{10}. Let e_{15} be the inner edge of $GP_{5,1}$ and adjacent to both the edges e_6 and e_{10}. Now, we need to find the minimum number of pebbles to cover the edges that form an edge dominating set.

If $P(e_1) = 1$, the edges e_1, e_2, e_5, e_6, and e_{10} can be dominated and the remaining edges of $GP_{5,1}$ will be left undominated. So, 1 is not the required number of pebbles.

If $P(e_1) = 1$ and $P(e_8) = 1$, the edges e_7, e_9, e_{11}, e_{14}, and e_{15} will be left undominated. So, 2 is not the required number of pebbles.

If $P(e_1) = 1$, $P(e_8) = 1$ and $P(e_{14}) = 1$, the edges e_7 and e_{11} will be left undominated. So, 3 is not the required number of pebbles.

If $P(e_1) = 2$ and $P(e_3) = 2$ and consider the distribution of 1 pebble moves to each of the edges e_6 and e_8 then the edges e_5, e_7, e_9, e_{10}, and e_{14} will be left undominated. So, 4 is not the required number of pebbles.

If $P(e_3) = 3$, $P(e_6) = 1$ and $P(e_8) = 1$ and consider the distribution of 1 pebble moves to the edge e_9 then the edges e_7 and e_{10} will be left undominated. So, 5 is not the required number of pebbles.

If $P(e_1) = 1$, $P(e_3) = 1$, $P(e_{13}) = 1$ and $P(e_2) = 3$ and consider the distribution of 1 pebble moves to the edge e_6, we can dominate all the edges of $GP_{5,1}$ with 6 pebbles but if $P(e_6) = 3$ and $P(e_8) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_{11} and e_{13} then the edges e_5 and e_{10} will be left undominated. So, 6 is not the required number of pebbles.

If $P(e_{12}) = 1$, $P(e_{14}) = 3$ and $P(e_{15}) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_6 and e_9 then the edge e_3 will be left undominated. So, 7 is not the required number of pebbles.

If $P(e_1) = 4$ and $P(e_2) = 4$ and consider the distribution of 1 pebble moves to each of the edges e_6, e_5, e_7, and e_3 then the edges e_{13} and e_{14} will be left undominated. So, 8 is not the required number of pebbles.

If $P(e_1) = 4$, $P(e_2) = 4$ and $P(e_{12}) = 1$ and consider the distribution of 1 pebble moves to each of the edges e_6, e_5, e_7, and e_3 then the edge e_{14} will be left undominated. So, 9 is not the required number of pebbles.

If $P(e_1) = 4$, $P(e_2) = 4$, $P(e_{12}) = 1$ and $P(e_{14}) = 1$, we can dominate all the edges of $GP_{5,1}$ with 10 pebbles but if $P(e_1) = 5$ and $P(e_3) = 5$ and consider the distribution of 1 pebble moves to each of the edges e_6, e_7, e_8, and e_{10} then the edge e_9 will be left undominated. So, 10 is not the required number of pebbles.

If $P(e_1) = 5$, $P(e_3) = 5$ and $P(e_2) = 1$ and consider the distribution of 1 pebble moves to each of the edges e_7, e_8, e_5 and e_{10}, we can dominate all the edges of $GP_{5,1}$ with 11 pebbles but if $P(e_1) = 11$ and consider the distribution of 1 pebble moves to each of the edges e_6, e_7 and e_9 then the edge e_8 will be left undominated. So, 11 is not the required number of pebbles.

If $P(e_5) = 12$ and consider the distribution of 1 pebble moves to each of the edges e_1, e_8, e_9 and e_{15} then the edge e_7 will be left undominated. So, 12 is not the required number of pebbles.

If $P(e_4) = 13$ and consider the distribution of 1 pebble moves to each of the edges e_1, e_{12}, and e_{14}, we can dominate all the edges of $GP_{5,1}$ with 13 pebbles and if $P(e_{13}) = 13$ and consider the distribution of 1 pebble moves to each of the edges e_3, e_5 and e_{11}, we can dominate all the edges of $GP_{5,1}$ with 13 pebbles. If we also place 13 pebbles on any other edges of $GP_{5,1}$ and alter the configuration of pebbles, we can dominate all the edges of $GP_{5,1}$ and 13 pebbles are the minimum number of pebbles required. Thus, the domination cover edge pebbling number for the Generalized Petersen $GP_{5,1}$ graph is $\psi_e(GP_{5,1}) = 13$. □

Theorem 3.6. *The domination cover edge pebbling number for the Generalized Petersen graph $GP_{5,2}$ is $\psi_e(GP_{5,2}) = 11$.*

Proof. For $GP_{5,2}$, let e_1, e_2, e_3, e_4 and e_5 be the outer edges of $GP_{5,2}$. Let e_6 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_2. Let e_7 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_2 and e_3. Let e_8 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_3 and e_4. Let e_9 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_4 and e_5. Let e_{10} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_5. Let e_{11} be the inner edge of $GP_{5,2}$ and adjacent to both the edges e_6 and e_8. Let e_{12} be the inner edge of $GP_{5,2}$ and adjacent to both the edges e_8 and e_{10}. Let e_{13} be the inner edge of $GP_{5,2}$ and adjacent to both the edges e_{10} and e_7. Let e_{14} be the inner edge of $GP_{5,2}$ and adjacent to both the edges e_7 and e_9. Let e_{15} be the inner edge of $GP_{5,2}$ and adjacent to both the edges e_9 and e_6. Now, we need to find the minimum number of pebbles to cover the edges that form an edge dominating set.

If $P(e_1) = 1$, the edges e_1, e_2, e_5, e_6, and e_{10} can be dominated and the remaining edges of $GP_{5,2}$ will be left undominated. So, 1 is not the required number of pebbles.

If $P(e_1) = 1$ and $P(e_8) = 1$, the edges e_7, e_9, e_{13}, e_{14}, and e_{15} will be left undominated. So, 2 is not the required number of pebbles.

If $P(e_1) = 1$, $P(e_8) = 1$ and $P(e_7) = 1$, the edges e_9 and e_{15} will be left undominated. So, 3 is not the required number of pebbles.

If $P(e_1) = 2$ and $P(e_3) = 2$ and consider the distribution of 1 pebble moves to each of the edges e_6 and e_8 then the edges e_5, e_7, e_9, e_{10}, e_{13} and e_{14} will be left undominated. So, 4 is not the required number of pebbles.

If $P(e_3) = 3$, $P(e_6) = 1$ and $P(e_8) = 1$ and consider the distribution of 1 pebble moves to the edge e_7 then the edges e_5, e_9, and e_{10} will be left undominated. So, 5 is not the required number of pebbles.

If $P(e_1) = 1$, $P(e_3) = 1$, $P(e_9) = 1$ and $P(e_{11}) = 3$ and consider the distribution of 1 pebble moves to the edge e_{12}, we can dominate all the edges of $GP_{5,2}$ with 6 pebbles but if $P(e_6) = 3$ and $P(e_8) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_{11} and e_{12} then the edges e_5, e_7, e_9 and e_{14} will be left undominated. So, 6 is not the required number of pebbles.

If $P(e_{12}) = 1$, $P(e_{14}) = 3$ and $P(e_{15}) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_7 and e_6 then the edges e_4, and e_5 will be left undominated. So, 7 is not the required number of pebbles.

If $P(e_1) = 4$ and $P(e_2) = 4$ and consider the distribution of 1 pebble moves to each of the edges e_6, e_5, e_7, and e_3 then the edge e_{12} will be left undominated. So, 8 is not the required number of pebbles.

If $P(e_1) = 4$, $P(e_2) = 4$ and $P(e_4) = 1$ and consider the distribution of 1 pebble moves to each of the edges e_6, e_5, e_7, and e_3 then the edge e_{12} will be left undominated. So, 9 is not the required number of pebbles.

If $P(e_1) = 4$, $P(e_2) = 4$, $P(e_4) = 1$ and $\mathrm{P}(e_{12}) = 1$, we can dominate all the edges of $GP_{5,2}$ with 10 pebbles but if $P(e_1) = 5$ and $P(e_3) = 5$ and consider the distribution of 1 pebble moves to each of the edges e_6, e_7, e_8, and e_{10} then the edge e_9 will be left undominated. So, 10 is not the required number of pebbles.

If $P(e_4) = 11$ and consider the distribution of 1 pebble moves to each of the edges e_1, e_8, and e_{14}, we can dominate all the edges of $GP_{5,2}$ with 11 pebbles and if $\mathrm{P}(e_{10}) = 11$ and consider the distribution of 1 pebble moves to each of the edges e_6, e_7 and e_4, we can dominate all the edges of $GP_{5,2}$ with 11 pebbles. If we also place 11 pebbles on any other edges of $GP_{5,2}$ and alter the configuration of pebbles, we can dominate all the edges of $GP_{5,2}$ and 11 pebbles are the minimum number of pebbles required. Thus, the domination cover edge pebbling number for the Generalized Petersen $GP_{5,2}$ graph is $\psi_e(GP_{5,2}) = 11$. □

Theorem 3.7. *The domination cover edge pebbling number for the Generalized Petersen graph $GP_{6,1}$ is $\psi_e(GP_{6,1}) = 18$.*

Proof. For $GP_{6,1}$, let e_1, e_2, e_3, e_4, e_5 and e_6 be the outer edges of $GP_{6,1}$. Let e_7 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_2. Let e_8 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_2 and e_3. Let e_9 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_3 and e_4. Let e_{10} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_4 and e_5. Let e_{11} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_5 and e_6. Let e_{12} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_6. Let e_{13} be the inner edge of $GP_{6,1}$ and adjacent to both the edges e_7 and e_{12}. Let e_{14} be the inner edge of $GP_{6,1}$ and adjacent to both the edges e_7 and e_8. Let e_{15} be the inner edge of $GP_{6,1}$ and adjacent to both the edges e_8 and e_9. Let e_{16} be the inner edge of $GP_{6,1}$ and adjacent to both the edges e_9 and e_{10}. Let e_{17} be the inner edge of $GP_{6,1}$ and adjacent to both the edges e_{10} and e_{11}. Let e_{18} be the inner edge of $GP_{6,1}$ and adjacent to both the edges e_{11} and e_{12}. Now, we need to find the minimum number of pebbles to cover the edges that form an edge dominating set.

If P(e_1) = 17 and consider the distribution of 1 pebble moves to each of the edges e_8, e_9 and e_{10} then the edge e_6 will be left undominated. So, $\psi_e(GP_{6,1}) \geq 18$.

For proving the sufficient part, If $P(e_i) = 1$, 2 or 3 for $1 \leq i \leq 18$, we cannot dominate all the edges of $GP_{6,1}$. So, $\psi_e(GP_{6,1}) \geq 4$. If $P(e_1) = 1$, $P(e_4) = 1$, $P(e_{15}) = 1$ and $P(e_{18}) = 1$, we can dominate all the edges of $GP_{6,1}$ with 4 pebbles and if $P(e_1) = 3$, $P(e_4) = 3$, $P(e_{14}) = 1$ and $P(e_{18}) = 1$ and consider the distribution of 1 pebble moves to each of the edges e_9 and e_{12}, we can dominate all the edges of $GP_{6,1}$ with 8 pebbles. So, in this distribution of pebbles, we need at most 8 pebbles to dominate all the edges of $GP_{6,1}$.

If $P(e_3) = 1$, $P(e_5) = 2$, $P(e_9) = 3$ and $P(e_{12}) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_6, e_{13} and e_{16}, we can dominate all the edges of $GP_{6,1}$ with 9 pebbles. If $P(e_4) = 4$, $P(e_5) = 4$ and $P(e_2) = 5$ and consider the distribution of 1 pebble moves to each of the edges e_3, e_6, e_{10}, e_{11} and e_{14}, we can dominate all the edges of $GP_{6,1}$ with 13 pebbles. So, in this distribution of pebbles, we need at most 13 pebbles to dominate all the edges of $GP_{6,1}$.

If $P(e_1) = 2$, $P(e_2) = 4$, $P(e_4) = 4$ and $P(e_5) = 4$ and consider the distribution of 1 pebble moves to each of the edges e_3, e_6, e_{10}, e_{11}, e_{12} and e_{14}, we can dominate all the edges of $GP_{6,1}$ with 14 pebbles. If $P(e_4) = 5$, $P(e_5) = 5$, $P(e_6) = 5$ and $P(e_2) = 2$ and consider the distribution of 1 pebble moves to each of the edges e_3, e_9, e_{10}, e_{13} and e_{17}, we can dominate all the edges of $GP_{6,1}$ with 17 pebbles. So, in this distribution of pebbles, we need at most 17 pebbles to dominate all the edges of $GP_{6,1}$.

If $P(e_1) = 18$ and consider the distribution of 1 pebble moves to each of the edges e_3, e_6, e_{14} and e_{17}, we can dominate all the edges of $GP_{6,1}$ with 18 pebbles. If we also place 18 pebbles on any other edges of $GP_{6,1}$ and alter the configuration of pebbles, we can dominate all the edges of $GP_{6,1}$. Thus, the domination cover edge pebbling number for the Generalized Petersen $GP_{6,1}$ graph is $\psi_e(GP_{6,1}) = 18$. □

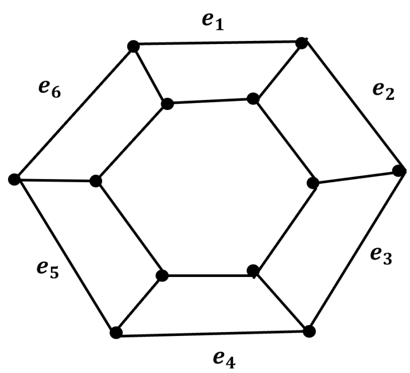

Fig. 4. Generalized petersen graph $GP_{6,1}$

Theorem 3.8. *The domination cover edge pebbling number for the Generalized Petersen graph $GP_{6,2}$ is $\psi_e(GP_{6,2}) = 25$.*

Proof. For $GP_{6,2}$, let e_1, e_2, e_3, e_4, e_5 and e_6 be the outer edges of $GP_{6,2}$. Let e_7 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1

and e_2. Let e_8 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_2 and e_3. Let e_9 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_3 and e_4. Let e_{10} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_4 and e_5. Let e_{11} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_5 and e_6. Let e_{12} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_6. Let e_{13} be the inner edge of $GP_{6,2}$ and adjacent to both the edges e_7 and e_9. Let e_{14} be the inner edge of $GP_{6,2}$ and adjacent to both the edges e_9 and e_{11}. Let e_{15} be the inner edge of $GP_{6,2}$ and adjacent to both the edges e_7 and e_{11}. Let e_{16} be the inner edge of $GP_{6,2}$ and adjacent to both the edges e_8 and e_{10}. Let e_{17} be the inner edge of $GP_{6,2}$ and adjacent to both the edges e_{10} and e_{12}. Let e_{18} be the inner edge of $GP_{6,2}$ and adjacent to both the edges e_8 and e_{12}. Now, we need to find the minimum number of pebbles to cover the edges that form an edge dominating set.

If $P(e_{14}) = 24$, the edge e_4 and the adjacent edges of e_4 will be left undominated. So, $\psi_e(GP_{6,2}) \geq 25$.

For proving the sufficient part, If $P(e_i) = 1$, 2 or 3 for $1 \leq i \leq 18$, we cannot dominate all the edges of $GP_{6,2}$. So, $\psi_e(GP_{6,2}) \geq 4$. If $P(e_1) = 1$, $P(e_4) = 1$, $P(e_{15}) = 1$ and $P(e_{16}) = 1$, we can dominate all the edges of $GP_{6,2}$ with 4 pebbles and if $P(e_2) = 1$, $P(e_5) = 1$, $P(e_9) = 3$ and $P(e_{12}) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_{13} and e_{17}, we can dominate all the edges of $GP_{6,2}$ with 8 pebbles. So, in this distribution of pebbles, we need at most 8 pebbles to dominate all the edges of $GP_{6,2}$.

If $P(e_1) = 1$, $P(e_5) = 2$, $P(e_8) = 3$ and $P(e_{11}) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_4, e_{14} and e_{16}, we can dominate all the edges of $GP_{6,2}$ with 9 pebbles. If $P(e_1) = 17$ and consider the distribution of 1 pebble moves to each of the edges e_4, e_{15} and e_{18}, we can dominate all the edges of $GP_{6,2}$ with 17 pebbles. So, in this distribution of pebbles, we need at most 17 pebbles to dominate all the edges of $GP_{6,2}$.

If $P(e_1) = 16$, $P(e_2) = 2$ and consider the distribution of 1 pebble moves to each of the edges e_1, e_4, e_{15} and e_{18}, we can dominate all the edges of $GP_{6,2}$ with 18 pebbles. If $P(e_{14}) = 23$ and $P(e_6) = 1$ and consider the distribution of 1 pebble moves to each of the edges e_1, e_8, and e_{10}, we can dominate all the edges of $GP_{6,2}$ with 24 pebbles. So, in this distribution of pebbles, we need at most 24 pebbles to dominate all the edges of $GP_{6,2}$.

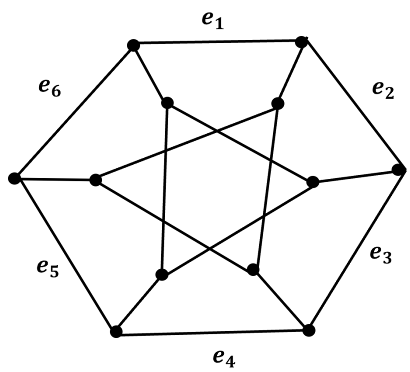

Fig. 5. Generalized petersen graph $GP_{6,2}$

If $P(e_{14}) = 25$ and consider the distribution of 1 pebble moves to each of the edges e_1, e_8, and e_{10}, we can dominate all the edges of $GP_{6,2}$ with 25 pebbles. If we also place 25 pebbles on any other edges of $GP_{6,2}$ and alter the configuration of pebbles, we can dominate all the edges of $GP_{6,2}$. Thus, the domination cover edge pebbling number for the Generalized Petersen $GP_{6,2}$ graph is $\psi_e(GP_{6,2}) = 25$. □

Theorem 3.9. *The domination cover edge pebbling number for the Generalized Petersen graph* $GP_{7,1}$ *is* $\psi_e(GP_{7,1}) = 28$.

Proof. For $GP_{7,1}$, let e_1, e_2, e_3, e_4, e_5, e_6 and e_7 be the outer edges of $GP_{7,1}$. Let e_8 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_2. Let e_9 be the edge that connects both the inner and outer polygons and adjacent to both the edges e_2 and e_3. Let e_{10} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_3 and e_4. Let e_{11} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_4 and e_5. Let e_{12} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_5 and e_6. Let e_{13} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_6 and e_7. Let e_{14} be the edge that connects both the inner and outer polygons and adjacent to both the edges e_1 and e_7. Let e_{15} be the inner edge of $GP_{7,1}$ and adjacent to both the edges e_8 and e_{14}. Let e_{16} be the inner edge of $GP_{7,1}$ and adjacent to both the edges e_8 and e_9. Let e_{17} be the inner edge of $GP_{7,1}$ and adjacent to both the edges e_9 and e_{10}. Let e_{18} be the inner edge of $GP_{7,1}$ and adjacent to both the edges e_{10} and e_{11}. Let e_{19} be the inner edge of $GP_{7,1}$ and adjacent to both the edges e_{11} and e_{12}. Let e_{20} be the inner edge of $GP_{7,1}$ and adjacent to both the edges e_{12} and e_{13}. Let e_{21} be the inner edge of $GP_{7,1}$ and adjacent to both the edges e_{13} and e_{14}. Now, we need to find the minimum number of pebbles to cover the edges that form an edge dominating set.

If $P(e_1) = 27$ and consider the distribution of 1 pebble moves to each of the edges e_4, e_7, e_{17} and e_{20}, the edge e_{15} will be left undominated. So, $\psi_e(GP_{7,1}) \geq 28$.

For proving the sufficient part, If $P(e_i) = 1, 2, 3$ or 4 for $1 \leq i \leq 21$, we cannot dominate all the edges of $GP_{7,1}$. So, $\psi_e(GP_{7,1}) \geq 5$. If $P(e_3) = 1$, $P(e_6) = 1$, $P(e_8) = 1$, $P(e_{18}) = 1$ and $P(e_{21}) = 1$, we can dominate all the edges of $GP_{7,1}$ with 5 pebbles and if $P(e_8) = 1$, $P(e_3) = 3$, $P(e_7) = 3$ and $P(e_{18}) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_4, e_{14} and e_{19}, we can dominate all the edges of $GP_{7,1}$ with 10 pebbles. So, in this distribution of pebbles, we need at most 10 pebbles to dominate all the edges of $GP_{7,1}$.

If $P(e_8) = 1$, $P(e_3) = 4$, $P(e_7) = 3$ and $P(e_{18}) = 3$ and consider the distribution of 1 pebble moves to each of the edges e_4, e_9, e_{14} and e_{19}, we can dominate all the edges of $GP_{7,1}$ with 11 pebbles. If $P(e_4) = 17$, $P(e_1) = 1$ and $P(e_{19}) = 1$ and consider the distribution of 1 pebble moves to each of the edges e_{13}, e_{16} and e_{19}, we can dominate all the edges of $GP_{7,1}$ with 19 pebbles. So, in this distribution of pebbles, we need at most 19 pebbles to dominate all the edges of $GP_{7,1}$.

If $P(e_4) = 17$, $P(e_{11}) = 2$ and $P(e_1) = 1$ and consider the distribution of 1 pebble moves to each of the edges e_{13}, e_{16} and e_{19}, we can dominate all the edges of $GP_{7,1}$ with 20 pebbles. If $P(e_1) = 26$ and $P(e_7) = 1$ and consider the distribution of 1 pebble moves to each of the edges e_4, e_8, e_{17} and e_{20}, we can dominate all the edges of $GP_{7,1}$ with 27 pebbles. So, in this distribution of pebbles, we need at most 27 pebbles to dominate all the edges of $GP_{7,1}$.

If $P(e_1) = 28$ and consider the distribution of 1 pebble moves to each of the edges e_4, e_7, e_8, e_{17} and e_{20}, we can dominate all the edges of $GP_{7,1}$ with 28 pebbles. If we also place 28 pebbles on any other edges of $GP_{7,1}$ and alter the configuration of pebbles, we can dominate all the edges of $GP_{7,1}$.

Thus, the domination cover edge pebbling number for the Generalized Petersen $GP_{7,1}$ graph is $\psi_e(GP_{7,1}) = 28$. □

4. Conclusion

In this paper, domination cover edge pebbling number for some of the Generalized Petersen graphs, Jewel graph and Triangular snake graph are determined. Investigating the domination cover edge pebbling number for the remaining Generalized Petersen graphs and other families of graphs is still an open research question.

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