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On monophonic pebbling number

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ABSTRACT

Given a connected graph G and a configuration D of pebbles on $V(G)$, a pebble move consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex. A monophonic path is a longest chordless path between two non-adjacent vertices u and v. The line segment that connects two vertices on a curve is known as a chord. The monophonic distance between u and v is the number of vertices in the longest u–v monophonic path, denoted by $d_\mu(u,v)$ in G. The monophonic pebbling number (MPN) of G is the least number of pebbles needed to guarantee that, from any distribution of pebbles on a graph G, one pebble can be moved to any specified vertex using monophonic paths through pebbling moves. The monophonic t-pebbling number (MtPN) of G is the least number of pebbles needed to guarantee that, from any distribution of pebbles, t pebbles can be moved to any specified vertex using monophonic paths. In this article, we determine the MPN and $MtPN$ of Dutch windmill graphs, square of cycles, tadpole graphs, lollipop graphs, double star path graphs, and fuse graphs, and we also discuss their t-pebbling versions.

Keywords: monophonic pebbling number, chord, monophonic distance, monophonic path, monophonic t-pebbling number

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1. Introduction

Pebbling, a recent development in graph theory introduced by Lagarias and Saks, has attracted considerable interest. Chung [2] was the first to introduce it into the literature, and many others have followed, including Hulbert, who published an overview of graph

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pebbling [3]. Graph pebbling is a concept that can be used to solve a variety of practical problems. "Graph pebbling" is a network optimization model used to move resources or materials that are absorbed during the journey. For example, when moving energy, heat, electricity, water, or information between locations, some loss may occur. Hence, the graph pebbling problem provides a way to measure the cost of pebble loss during conveyance. Because of its many applications, graph pebbling is currently one of the fastest-growing fields of research in graph theory.

Assume that G is a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Consider a connected graph with a fixed number of pebbles distributed on its vertices. Santhakumaran, A. P. et al. introduced the monophonic distance in graphs in [10]. Lourdusamy et al. introduced detour pebbling in [7]. By making use of these ideas, the monophonic pebbling number is defined. The monophonic distance between non-adjacent vertices u and v is the length of the longest chordless u–v monophonic path, denoted by $d_\mu(u,v)$ in G. For any two non-adjacent vertices u and v in a connected graph G, a longest u–v path is a monophonic path if it contains no chords [10]. The monophonic pebbling number and monophonic t-pebbling number of several typical graphs are determined in this study.

This paper is organized as follows. In Section 2, we give some preliminaries that are needed for the subsequent sections. In Section 3, we determine the MPN of Dutch windmill graphs. In Section 4, we determine the MPN of the square of cycles, tadpole graphs, lollipop graphs, double star path graphs, and fuse graphs. In Section 5, we determine the $MtPN$ of Dutch windmill graphs, square of cycles, tadpole graphs, lollipop graphs, double star path graphs, and fuse graphs.

2. Preliminaries

Definition 2.1. [5] A chord in a path is an edge joining two non-adjacent vertices. A monophonic path between u and v is a u–v path without a chord between u and v. The monophonic pebbling number of v in G, denoted by $\mu(G,v)$, is the least positive integer such that we can move a pebble to v through a monophonic path by a sequence of pebbling moves for all distributions of $\mu(G,v)$. The monophonic pebbling number of G is

$$\mu(G) = \max_{v\in V} \mu(G,v).$$

Instead of moving one pebble to a target, if we move t pebbles, it is called the $MtPN$ of G and is denoted by $\mu_t(G)$. Thus, the $MtPN$ of G is

$$\mu_t(G) = \max_{v\in V} \mu_t(G,v).$$

Note 2.2. A distribution f on a graph G is a function

$$f : V(G) \longrightarrow \mathbb{N} \cup \{0\}.$$

Suppose that m pebbles are distributed on the vertices of a connected graph G. A pebbling move, or shift, consists of removing two pebbles from one vertex and then placing one pebble on an adjacent vertex.

Definition 2.3. [1] A graph D_n^m, with $m \geq 1$ and $n \geq 3$, is known as a Dutch windmill graph. The graph D_n^m can be obtained by taking m copies of cycles C_n with one common vertex.

Definition 2.4. [12] The (m, n) tadpole graph is the graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge. It is denoted by $T_{(m,n)}$.

Definition 2.5. [11] The (m, n)-lollipop graph is the graph obtained by joining a complete graph K_m to a path graph P_n with a bridge. It is denoted by $L(m, n)$.

Definition 2.6. [8] A graph is called a double star path if the end vertices of a path P_n are joined to the center vertices of the star graphs $K_{1,l}$, where $l \geq 2$, and $K_{1,m}$, where $m \geq 2$, respectively. We denote it by $P_n(l, m)$.

Definition 2.7. [4] The class of fuses is defined as follows: the vertices of a fuse $F_l(k)$, where $l \geq 1$ and $k \geq 2$, are $v_0, v_1, v_2, \cdots, v_{n-1}$ with $n = l + k + 1$, such that the first $l + 1$ vertices form a path $v_0, v_1, v_2, \cdots, v_l$, and the remaining vertices $v_{l+1}, v_{l+2}, \cdots, v_{n-1}$ are independent and adjacent only to v_l.

Definition 2.8. Let G be a connected graph. For $u, v \in V(G)$, we denote by $d_G(u, v)$ the distance between u and v in G. The p^{th} power of G, denoted by G^p, is the graph obtained from G by adding an edge uv to G whenever $2 \leq d_G(u, v) \leq p$. That is,

$$E(G^p) = \{uv : 1 \leq d_G(u, v) \leq p\}.$$

In particular, the square of a path is denoted by P_n^2, and the square of a cycle is denoted by C_n^2.

Note 2.9. [9] The notation D_n^m, with $m \geq 1$ and $n \geq 3$, denotes a Dutch windmill graph and is taken from [1]. The notation $L(m, n)$ denotes a lollipop graph and is taken from [11]. The notation $T_{m,n}$ denotes the (m, n) tadpole graph and is taken from [12]. The notation $P_n(l, m)$ denotes a double star path graph and is taken from [8]. The notation $F_l(k)$, where $l \geq 1$ and $k \geq 2$, denotes the class of fuses and is taken from [4].

Result 2.10. Let G be a connected graph. The monophonic distance between u and v is 0 if and only if $u = v$, and it is 1 when uv is an edge of G.

Theorem 2.11. [5] *For the path graph of order n, the monophonic pebbling number is 2^{n-1}.*

Theorem 2.12. [6] *For the cycle graph C_n, the monophonic pebbling number is $2^{n-2} + 1$.*

Theorem 2.13. [6] *For the complete graph K_n, the monophonic pebbling number is n.*

Notation 2.14. Throughout this article, we use the following notation:

(1) M_i denotes a monophonic path, and $M_i^{\sim}$ contains the vertices that are not on M_i.

(2) We use MPN for the monophonic pebbling number and $MtPN$ for the monophonic t-pebbling number.

(3) d_μ denotes the monophonic distance.

(4) $v_k \xleftarrow{t}$ means retaining t pebbles on v_k, and $v_k \xrightarrow{t}$ means transferring t pebbles to $N(v_k)$.

(5) $N(v_0)$ denotes the neighborhood of v_0.

(6) Pebble transformation means removing two pebbles from a vertex and placing one pebble on an adjacent vertex.

3. The Monophonic Pebbling Number of Dutch Windmill Graphs

In this section, we determine the monophonic pebbling number of Dutch windmill graphs.

Theorem 3.1. *For the Dutch windmill graph D_3^1, the monophonic pebbling number is* 3.

Proof. Let

$$V(D_3^1) = \{v_0,\ v_1^1,\ v_2^1\}.$$

The graph $D_3^1 \cong K_3$, and by Theorem 2.13, the result follows. □

Theorem 3.2. *For the Dutch windmill graph D_n^1, the monophonic pebbling number is*

$$2^{n-2} + 1, \qquad n \geq 3.$$

Proof. Let

$$V(D_n^1) = \{v_0,\ v_1^1,\ v_2^1,\ v_3^1, \cdots, v_{n-1}^1\}.$$

The graph $D_n^1 \cong C_n$, and by Theorem 2.12, the result follows. □

Theorem 3.3. *For the Dutch windmill graph D_4^2, the monophonic pebbling number is* 16.

Proof. Let

$$V(D_4^2) = \{v_0,\ v_1^1,\ v_2^1,\ v_3^1,\ v_1^2,\ v_2^2,\ v_3^2\}$$

and

$$E(D_4^2) = \{v_0 v_k^j,\ v_s^j v_{s+1}^j\},$$

where $s, j = 1, 2$ and $k = 1, 3$. The distance $d_\mu(v_2^1, v) \leq 4$, where $v \in V(D_4^2)$. By placing 15 pebbles on v_2^2, we cannot reach v_2^1. Hence,

$$\mu(D_4^2) \geq 16.$$

Now we prove that $\mu(D_4^2) \leq 16$.

Case 1. Let the target be v_2^2.

Consider

$$M_1: \ v_1^2, \ v_1^1, \ v_0, \ v_3^2, \ v_2^2.$$

Note that $d_\mu(M_1)$ is 4. By Theorem 2.11, using at most 16 pebbles, we can pebble v_2^2. Suppose $p(V(M_1)) \leq 16$. If $p(v_1^2) \geq 2$ or $p(v_3^1) \geq 2$, then we can bring $\left\lfloor \frac{p(v_1^2)}{2} \right\rfloor$ pebbles to v_2^2 or $\left\lfloor \frac{p(v_3^1)}{2} \right\rfloor$ pebbles to v_0, which is sufficient to pebble v_2^2. If $p(v_0) \geq 4$, then we can pebble v_2^2. If $p(v_1^2) = 1$ or $p(v_3^2) = 1$ and $p(v_0) \geq 2$, then we can pebble v_2^2. If $p(v_0) \leq 4$ and $p(v_1^2) = p(v_3^2) = 0$, then at most 16 pebbles will be distributed on $v_1^1, \ v_2^1, \ v_3^1$. Now $d_\mu(v_0, v) \leq 2$, where $v \in \{v_1^1, \ v_2^1, \ v_3^1\}$, from which we can pebble v_2^2. Therefore, in every possible configuration, we can bring at least 4 pebbles to v_0, and v_2^2 is pebbled. By symmetry, the same argument applies to v_2^1.

Case 2. Let the target be v_0.

There are 4 vertices adjacent to v_0 and 2 vertices at monophonic distance 2. Thus, $d_\mu(v_0, v) \leq 2$ where $v \in V(D_4^2) - \{v_0\}$. Hence, if there are 4 pebbles on the vertices at distance 2 or 2 pebbles on the adjacent vertices, we can pebble v_0. Otherwise, if $N(v_0)$ has one pebble and an adjacent vertex of $N(v_0)$ that is at monophonic distance 2 has at least 2 pebbles, then we can pebble v_0. Thus, in this case, using at most 7 pebbles, we can pebble v_0.

Case 3. Let the target be $N(v_0)$.

Fix the target as v_3^1. The monophonic distance satisfies $d_\mu(v_3^1, v) \leq 3$, where $v \in V(D_4^2) - \{v_3^1\}$. There is one vertex at monophonic distance 3, three vertices at monophonic distance 2, and two vertices adjacent to v_3^1. If v_2^2 has at least 8 pebbles, or any vertex at distance 2 receives at least 4 pebbles, or any adjacent vertex receives at least 2 pebbles, then we can pebble v_3^1. Thus, in this case, using at most 10 pebbles, we can pebble v_3^1. In the same way, we can prove the result for $v_1^1, \ v_1^2$, and v_3^2. □

Theorem 3.4. *For* D_n^m*,*

$$\mu(D_n^m) = 2^{2n-4} + (m-2)(2^{n-2} - 1) + m,$$

where $m \geq 2$ *and* $n \geq 5$*.*

Proof. Let

$$V(D_n^m) = \{v_0, \ v_i^j : i = 1, 2, \cdots, n-1, \ j = 1, 2, \cdots, m\}$$

and

$$E(D_n^m) = \{v_0 v_m^j, \ v_0 v_1^j, \ v_k^j v_{k+1}^j : k = 1, 2, \cdots, n-2, \ j = 1, 2, \cdots, m\}.$$

There are many monophonic paths with the same length. Without loss of generality, consider

$$M_1: \ v_{n-2}^1, \ v_{n-3}^1, \cdots, \ v_1^1, \ v_0, \ v_1^2, \ v_2^2, \ v_3^2, \cdots, \ v_{n-3}^2, \ v_{n-2}^2,$$

whose length is $2n-4$. Suppose we have

$$2^{2n-4} - 1 + (m-2)(2^{n-2} - 1) + m$$

pebbles. By placing $2^{n-2}-1$ pebbles on each v_2^l, where $l = 3, 4, \cdots, m$, one pebble on each v_1^j, where $j = 1, 2, \cdots, m$, and $2^{2n-4}-1$ pebbles on v_{n-2}^1, we cannot pebble the vertex v_{n-2}^2. Therefore,

$$\mu(D_n^m) \geq 2^{2n-4} + (m-2)(2^{n-2}-1) + m.$$

Now let us prove the sufficient part.

Case 1. Let the target be v_2^1.

Without loss of generality, take

$$M_2 : \; v_2^2, \; v_3^2, \cdots, \; v_{n-1}^2, \; v_0, \; v_{n-1}^1, \; v_{n-2}^1, \cdots, \; v_3^1, \; v_2^1$$

with length $2n-4$. The vertex set consisting of vertices that are not on the monophonic path M_2 is $V(D_n^m) - V(M_2)$ and is denoted by $M_2^{\sim}$. By Theorem 2.11, using 2^{2n-4} pebbles, we can pebble v_2^1. Suppose $p(V(M_2)) < 2^{2n-4}$. If v_1^1 has at least 2 pebbles, then we can bring $\left\lfloor \frac{p(v_1^1)}{2} \right\rfloor$ pebbles to v_0, with which we can pebble v_2^1. Now consider the pebbling moves

$$\sum_{r=2,3,\cdots,m} v_1^r \xrightarrow{u_r} v_0, \tag{1}$$

and

$$\sum_{z=2,3,\cdots,m} v_2^z \longrightarrow v_3^z \longrightarrow \cdots \longrightarrow v_{n-1}^z \xrightarrow{y_z} v_0. \tag{2}$$

If

$$\sum_{r=2,3,\cdots,m} u_r + \sum_{z=2,3,\cdots,m} y_z + p(v_0) \geq 2^{n-2},$$

then we are done. Suppose

$$\sum_{r=2,3,\cdots,m} u_r + \sum_{z=2,3,\cdots,m} y_z + p(v_0) < 2^{n-2}.$$

Then the remaining pebbles will be on

$$V(M_2) - \{v_{n-1}^1, \; v_{n-2}^1, \cdots, \; v_3^1\},$$

and so we can reach v_2^1. By using the same argument, we can prove the result for v_2^j and v_{n-2}^j, where $j = 1, 2, \cdots, m$.

Case 2. Let the target be $N(v_0)$.

Fix the target as v_1^1. Without loss of generality, consider

$$M_3 : \; v_{n-2}^2, \; v_{n-3}^2, \cdots, \; v_2^2, \; v_1^2, \; v_0, \; v_1^1,$$

whose length is $n-1$. By Theorem 2.11, if $p(V(M_3)) \geq 2^{n-1}$, then we can pebble v_1^1. If $p(v_0) \geq 2$ or $p(v_2^1) \geq 2$, then we can pebble v_1^1. Otherwise, consider

$$M_4 : \; v_{n-1}^1, \; v_{n-2}^1, \cdots, \; v_2^1, \; v_1^1,$$

which has monophonic length $n-2$. By Theorem 2.11, if $p(V(M_4)) \geq 2^{n-2}$, then we can pebble v_1^1. Suppose

$$p(V(M_3)) < 2^{n-1}, \quad p(V(M_4)) < 2^{n-2}, \quad p(v_0) < 2, \quad p(v_2^1) < 2.$$

Consider

$$\sum_{r=2,3,\cdots,m} v_1^r \xrightarrow{u_r} v_0, \tag{3}$$

and

$$\sum_{z=2,3,\cdots,m} v_2^z \longrightarrow v_3^z \longrightarrow \cdots \longrightarrow v_{n-1}^z \xrightarrow{y_z} v_0. \tag{4}$$

If

$$\sum_{r=2,3,\cdots,m} u_r + \sum_{z=2,3,\cdots,m} y_z + p(v_0) \geq 2,$$

then we can pebble v_1^1. Suppose

$$\sum_{r=2,3,\cdots,m} u_r + \sum_{z=2,3,\cdots,m} y_z + p(v_0) < 2$$

and $p(V(M_4)) < 2^{n-2}$. This contradicts the total number of pebbles in the initial configuration. By the same argument, we can prove the result for v_1^j and v_{n-1}^j, where $j = 1, 2, \cdots, m$.

Case 3. Let the target be v_l^j, where $3 \leq l \leq n-3$ and $j = 1, 2, \cdots, m$.

Fix $j = 1$. The distance satisfies

$$d_\mu(v_l^1, v) \leq n + l - 2$$

or

$$d_\mu(v_l^1, v) \leq 2n - (l+2),$$

where $v \in V(D_n^m) - \{v_l^j\}$. If $\lfloor n/2 \rfloor \geq l$, then the monophonic path M_5 is

$$M_5 = \{v_l^1,\ v_{l+1}^1, \cdots,\ v_{n-2}^1,\ v_{n-1}^1,\ v_0,\ v_1^2, \cdots,\ v_{n-2}^2\}$$

with length $2n - (l+2)$. If $\lfloor n/2 \rfloor < l$, then the monophonic path M_6 is

$$M_6 = \{v_l^1,\ v_{l-1}^1, \cdots,\ v_2^1,\ v_1^1,\ v_0,\ v_1^2, \cdots,\ v_{n-2}^2\}$$

with length $n + l - 2$. Without loss of generality, consider M_5. Suppose $p(V(M_5)) \geq 2^{2n-(l+2)}$. Then, by Theorem 2.11, we can pebble v_l^j. Also, if $p(v_{l+1}^1) \geq 2$ or $p(v_{l-1}^1) \geq 2$, then we can reach v_l^j. Suppose $p(V(M_5)) < 2^{2n-(l+2)}$, $p(v_{l-1}^1) < 2$, and $p(v_{l+1}^1) \leq 1$. Then

$$\sum_{r=2,3,\cdots,m} v_1^r \xrightarrow{u_r} v_0, \tag{5}$$

$$\sum_{z=2,3,\cdots,m} v_2^z \longrightarrow v_3^z \longrightarrow \cdots \longrightarrow v_{n-1}^z \xrightarrow{y_z} v_0, \tag{6}$$

and

$$v_{l-2}^z \longrightarrow v_{l-3}^z \longrightarrow \cdots \longrightarrow v_1^z \xrightarrow{x} v_0. \tag{7}$$

If

$$\sum_{r=2,3,\cdots,m} u_r + \sum_{z=2,3,\cdots,m} y_z + x + p(v_0) \geq 2^{n-l},$$

then we can pebble v_l^j. Suppose

$$\sum_{r=2,3,\cdots,m} u_r + \sum_{z=2,3,\cdots,m} y_z + x + p(v_0) < 2^{n-1}.$$

Then

$$\sum_{r=2,3,\cdots,m} u_r + \sum_{z=2,3,\cdots,m} y_z + x + p(v_0) + p(v_{n-1}^1) + p(v_{n-2}^1) + \cdots + p(v_{l+1}^1) \geq 2^{n-1},$$

and so we can pebble v_l^j. The same method applies when $\lfloor n/2 \rfloor < l$.

Case 4. Let the target be v_0.

Note that v_0 is a cut vertex. Let M_j be a monophonic path with length $n-2$. That is, M_j is either

$$v_2^j,\ v_3^j, \cdots,\ v_{n-1}^j,\ v_0$$

or

$$v_{n-2}^j,\ v_{n-3}^j, \cdots,\ v_1^j,\ v_0,$$

where $j = 1, 2, \cdots, m$. By Theorem 2.11, if $p(V(M_j)) \geq 2^{n-2}$, then we can pebble v_0. □

4. The Monophonic Pebbling Number (MPN) of Some Path- and Cycle-Related Graphs

In this section, we determine the monophonic pebbling number (MPN) of the square of a cycle, tadpole graph, lollipop graph, double star path graph, and fuse graph.

Theorem 4.1. *The monophonic pebbling number of the square of a cycle C_n^2 is*

$$\begin{cases} 2^{n-\left(\frac{n}{3}+2\right)} + 2, & \text{if } n \equiv 0 \pmod 3, \\ 2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)} + \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 1 \pmod 3, \\ 2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)} + 3, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let

$$V(C_n^2) = \{v_1,\ v_2, \ldots,\ v_n\}$$

and

$$E(C_n^2) = \{v_i v_{i+1},\ v_j v_{j+2},\ v_1 v_n,\ v_1 v_{n-1},\ v_2 v_n\},$$

where $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, n-2$.

Case 1. $n \equiv 0 \pmod 3$.

Consider

$$M_1 :\ v_{n-2},\ v_{n-4},\ v_{n-5},\ v_{n-7}, \cdots,\ v_5,\ v_4,\ v_2,\ v_1.$$

The vertices not on M_1 are

$$v_n,\ v_{n-1},\ v_{n-3},\ v_{n-6}, \cdots,\ v_6,\ v_3.$$

The length of M_1 is $n-\left(\frac{n}{3}+2\right)$, and $\frac{n}{3}+1$ vertices are not on M_1. Let

$$p(V(C_n^2)) = 2^{n-\left(\frac{n}{3}+2\right)}+1.$$

By placing $2^{n-\left(\frac{n}{3}+2\right)}-1$ pebbles on v_{n-2} and one pebble each on v_n and v_{n-1}, we cannot pebble v_1. Hence,

$$\mu(C_n^2) \geq 2^{n-\left(\frac{n}{3}+2\right)}+2.$$

Now we prove that

$$\mu(C_n^2) \leq 2^{n-\left(\frac{n}{3}+2\right)}+2.$$

Consider any distribution of $2^{n-\left(\frac{n}{3}+2\right)}+2$ pebbles on $V(C_n^2)$. Without loss of generality, fix v_1 as the target vertex. Now $d_\mu(v_1,v) \leq n-\left(\frac{n}{3}+2\right)$, where $v \in V(C_n^2)$. Consider

$$M_2:\ v_1,\ v_3,\ v_4,\ v_6,\cdots,\ v_{n-6},\ v_{n-5},\ v_{n-3},\ v_{n-2}.$$

The vertices not on M_2 are

$$\{v_2,\ v_5,\ v_8,\cdots,\ v_{n-7},\ v_{n-4},\ v_{n-1},\ v_n\},$$

and this set is denoted by $M_2^{\sim}$. The set $M_2^{\sim}$ has $\frac{n}{3}+1$ vertices. If $p(V(M_2)) \geq 2^{n-\left(\frac{n}{3}+2\right)}$, then by Theorem 2.11, we can pebble v_1. Otherwise, if $p(v_n) \geq 2$, $p(v_{n-1}) \geq 2$, $p(v_2) \geq 2$, or $p(v_3) \geq 2$, then we can pebble v_1. Suppose $p(V(M_2)) < 2^{n-\left(\frac{n}{3}+2\right)}$ and $p(v_n) < 2$, $p(v_{n-1}) < 2$, $p(v_2) < 2$, and $p(v_3) < 2$. Then the remaining pebbles will be on $V(M_2^{\sim})$. Now we bring the pebbles to $V(M_2)$, and so we can pebble v_1.

Case 2. $n \equiv 1 \pmod 3$.

Consider

$$M_3:\ v_1,\ v_2,\ v_4,\ v_5,\cdots,\ v_{n-6},\ v_{n-5},\ v_{n-3},\ v_{n-2}.$$

The vertices not on M_3 are

$$v_n,\ v_{n-1},\ v_{n-4},\cdots,\ v_9,\ v_6,\ v_3,$$

and they form $M_3^{\sim}$. The length of M_3 is $n-\left(\left\lfloor\frac{n}{3}\right\rfloor+1\right)$, and $\left\lceil\frac{n}{3}\right\rceil$ vertices are in $M_3^{\sim}$. Let

$$p(V(C_n^2)) = 2^{n-\left(\frac{n}{3}+2\right)}-1+\left\lceil\frac{n}{3}\right\rceil.$$

By placing $2^{n-\left(\left\lfloor\frac{n}{3}\right\rfloor+2\right)}-1$ pebbles on v_1 and one pebble each on the $\left\lceil\frac{n}{3}\right\rceil$ vertices on $M_3^{\sim}$, we cannot pebble v_{n-2}. Hence,

$$\mu(C_n^2) \geq 2^{n-\left(\left\lfloor\frac{n}{3}\right\rfloor+2\right)}+\left\lceil\frac{n}{3}\right\rceil.$$

Now we prove the reverse inequality. Consider any distribution of

$$2^{n-\left(\left\lfloor\frac{n}{3}\right\rfloor+2\right)}+\left\lceil\frac{n}{3}\right\rceil$$

pebbles on $V(C_n^2)$. Without loss of generality, fix v_{n-2} as the target. We have

$$d_\mu(v_{n-2},v) \leq n-\left(\left\lfloor\frac{n}{3}\right\rfloor+2\right),$$

where $v \in V(C_n^2)$. Consider M_3. If

$$p(V(M_3)) \geq 2^{n-\left(\lfloor \frac{n}{3} \rfloor +2\right)},$$

then by Theorem 2.11, we can pebble v_{n-2}. Otherwise, if $p(v_n) \geq 2$, $p(v_{n-1}) \geq 2$, $p(v_{n-3}) \geq 2$, or $p(v_{n-4}) \geq 2$, then we can pebble v_{n-2}. Suppose

$$p(V(M_3)) < 2^{n-\left(\lfloor \frac{n}{3} \rfloor +2\right)}$$

and $p(v_n) < 2$, $p(v_{n-1}) < 2$, $p(v_{n-3}) < 2$, and $p(v_{n-4}) < 2$. Then the remaining pebbles will be on $V(M_3^{\sim})$. Now we bring the pebbles to $V(M_3)$, and so we pebble v_{n-2}.

Case 3. $n \equiv 2 \pmod 3$.

Consider

$$M_4 : \; v_{n-2},\; v_{n-4},\; v_{n-6},\; v_{n-7}, \cdots,\; v_5,\; v_4,\; v_2,\; v_1.$$

The vertices not on M_4 are

$$v_n,\; v_{n-1},\; v_{n-3},\; v_{n-5},\; v_{n-8}, \cdots,\; v_6,\; v_3,$$

and they form $M_4^{\sim}$. The length of M_4 is

$$n - \left(\left\lfloor \frac{n}{3} \right\rfloor + 2 \right),$$

and $\lceil \frac{n}{3} \rceil$ vertices are on $M_4^{\sim}$. Let

$$p(V(C_n^2)) = 2^{n-\left(\frac{n}{3}+2\right)} + 2.$$

By placing $2^{n-\left(\frac{n}{3}+2\right)} - 1$ pebbles on v_{n-2} and one pebble each on v_{n-1}, v_n, v_{n-3}, we cannot pebble v_1 using a monophonic path. Thus,

$$\mu(C_n^2) \geq 2^{n-\left(\lfloor \frac{n}{3} \rfloor +2\right)} + 3.$$

Now we prove the reverse inequality. Consider any distribution of

$$2^{n-\left(\lfloor \frac{n}{3} \rfloor +2\right)} + 3$$

pebbles on $V(C_n^2)$. Without loss of generality, let v_1 be the target vertex. Then

$$d_\mu(v_1, v) \leq n - \left(\left\lfloor \frac{n}{3} \right\rfloor + 2 \right),$$

where $v \in V(C_n^2)$. Consider M_4. If

$$p(V(M_4)) \geq 2^{n-\left(\lfloor \frac{n}{3} \rfloor +2\right)},$$

then by Theorem 2.11, we can pebble v_1. Otherwise, if $p(v_n) \geq 2$, $p(v_{n-1}) \geq 2$, $p(v_2) \geq 2$, or $p(v_3) \geq 2$, then we can pebble v_1. Suppose

$$p(V(M_4)) < 2^{n-\left(\lfloor \frac{n}{3} \rfloor +2\right)}$$

and $p(v_n) < 2$, $p(v_{n-1}) < 2$, $p(v_2) < 2$, and $p(v_3) < 2$. Then the remaining pebbles will be on $V(M_4^{\sim})$. Now we bring the pebbles to $V(M_4)$, and so we pebble v_1. □

Theorem 4.2. *The monophonic pebbling number of the lollipop graph is*

$$2^{n+1} + m - 2.$$

Proof. Let

$$V(L(m,n)) = \{v_1,\ v_2, \cdots,\ v_m,\ u_1,\ u_2, \cdots,\ u_n\},$$

where the m vertices form a complete graph and the n vertices form a path. Consider a bridge between u_1 and v_1 based on the definition. Let

$$p(V(L(m,n))) = 2^{n+1} + m - 3.$$

Consider

$$M_1 :\ u_n,\ u_{n-1}, \cdots,\ u_1,\ v_1,\ v_2.$$

If we place one pebble each on $v_3, v_4, \cdots, v_m$ and $2^{n+1} - 1$ pebbles on u_n, then we cannot place a pebble on v_2. Thus,

$$\mu(L(m,n)) \geq 2^{n+1} + m - 2.$$

Now we prove that

$$\mu(L(m,n)) \leq 2^{n+1} + m - 2.$$

Case 1. Let the target vertex be v_m.
Consider

$$M_2 :\ v_m,\ v_1,\ u_1,\ u_2, \cdots,\ u_n,$$

a monophonic path of length $n+1$. The vertex set consisting of vertices that are not on M_2 is

$$\{v_2,\ v_3, \cdots,\ v_{m-1}\}$$

and is denoted by $M_2^{\sim}$. By Theorem 2.11, if $p(V(M_2)) \geq 2^{n+1}$, then we can pebble v_m. Suppose $p(V(M_2)) < 2^{n+1}$. Then the remaining pebbles will be distributed in $V(M_2^{\sim})$. That is,

$$p(V(L(m,n))) - p(V(M_2)) = 2^{n+1} + m - 2 - p(V(M_2)) > m - 1.$$

Since $M_2^{\sim}$ has $m-2$ vertices, by the Pigeonhole principle, there will be vertices with more than one pebble. Now we bring the pebbles to v_1. Since $p(V(M_2)) = 2^{n+1}$, we can pebble v_m. By symmetry, we can prove the result for u_n, and by using the same argument, we can prove the result for $v_{m-1}, v_{m-2}, \cdots, v_2$.

Case 2. Let the target vertex be v_1.
Consider

$$M_3 :\ v_1,\ u_1,\ u_2, \cdots,\ u_n$$

with length n. Now either $p(V(M_3)) \geq 2^n$ or $p(v_j) \geq 2$, where $j = 2, 3, \cdots, m$. Hence, by Theorem 2.11, we can pebble v_1.

Case 3. Let the target vertex be u_l, where $l = 1, 2, 3, \cdots, n-1$.

There exist two monophonic paths:

$$M_4:\ u_l,\ u_{l-1},\cdots,\ u_1,\ v_1,\ v_2$$

of length $l+1$, and

$$M_5:\ u_l,\ u_{l+1},\cdots,\ u_n$$

of length $n-l$. Then either $p(V(M_4)) \geq 2^{l+1}$ or $p(V(M_5)) \geq 2^{n-l}$. Hence, by Theorem 2.11, we can pebble u_l. □

Theorem 4.3. *The monophonic pebbling number of the tadpole graph is*

$$2^{m+n-2}+1.$$

Proof. By the definition of the tadpole graph, consider a bridge between u_1 and v_1. Let

$$V(T_{m,n}) = \{v_1,\ v_2,\cdots,\ v_m,\ u_1,\ u_2,\cdots,\ u_n\}$$

and

$$E(T_{m,n}) = \{u_iu_{i+1},\ u_nv_1,\ v_kv_{k+1},\ v_mv_1\},$$

where $i = 1,2,\cdots,n-1$ and $k = 1,2,\cdots,m-1$. Let

$$p(V(T_{m,n})) = 2^{m+n-2}.$$

Consider

$$M_1:\ u_n,\ u_{n-1},\cdots,\ u_1,\ v_1,\ v_2,\cdots,\ v_{m-1}$$

of length $m+n-2$. If we place one pebble on v_m and $2^{m+n-2}-1$ pebbles on u_1, then we cannot place a pebble on v_{m-1} using a monophonic path. Thus,

$$\mu(T_{m,n}) \geq 2^{m+n-2}+1.$$

Now let us prove the sufficient condition.

Case 1. Let the target vertex be u_n.

Consider

$$M_2:\ u_n,\ u_{n-1},\cdots,\ u_1,\ v_1,\ v_m,\cdots,\ v_3$$

of length $m+n-2$. By Theorem 2.11, if $p(V(M_2)) \geq 2^{m+n-2}$, then we can pebble u_n. Suppose $p(V(M_2)) < 2^{m+n-2}$. Then $2^{m+n-2}+1-p(V(M_2))$ pebbles will be on v_2, and we bring pebbles to v_1. Now

$$p(V(M_2)) + \left\lfloor \frac{p(v_m)}{2} \right\rfloor \geq 2^{m+n-2},$$

and we can pebble u_n. By symmetry, the same argument proves the result for v_3 and v_{m-1}.

Case 2. Let the target vertex be v_j, where $j = 3,4,\cdots,m-2$.

If $\lceil \frac{m}{2} \rceil + 1 \leq j$, then there is a monophonic path

$$M_3: \ u_1, \ u_2, \cdots, \ u_n, \ v_1, \ v_2, \cdots, \ v_j$$

of length $n + j - 1$. If $\lceil \frac{m}{2} \rceil + 1 > j$, then there is a monophonic path

$$M_4: \ u_1, \ u_2, \cdots, \ u_n, \ v_1, \ v_m, \ v_{m-1}, \cdots, \ v_j$$

of length $m + n - j + 1$. By Theorem 2.11, if $p(V(M_3)) \geq 2^{n+j-1}$, then we can pebble v_j. Otherwise, if $p(V(M_4)) \geq 2^{m+n-j+1}$, then we can pebble v_j.

Case 3. Let the target vertex be u_l, where $l = 1, 2, 3, \cdots, n-1$.

Consider the monophonic paths

$$M_5: \ u_l, \ u_{l+1}, \cdots, \ u_n$$

of length $n - l$, and

$$M_6: \ u_l, \ u_{l-1}, \ u_{l-2}, \cdots, \ u_1, \ v_1, \ v_2, \cdots, \ v_{m-1}$$

of length $m + l - 2$. By Theorem 2.11, if $p(V(M_5)) \geq 2^{n-l}$, then we can pebble u_l. Otherwise, if $p(V(M_6)) \geq 2^{m+l-2}$, then we can pebble u_l. □

Theorem 4.4. *The monophonic pebbling number of the double star path graph is*

$$2^{n+3} + l + m - 2.$$

Proof. Let

$$V(P_n(l, m)) = \{x_0, \ x_1, \cdots, \ x_l, \ y_0, \ y_1, \cdots, \ y_m, \ v_1, \ v_2, \cdots, \ v_n\}$$

and

$$E(P_n(l, m)) = \{x_0 x_i, \ x_0 v_1, \ v_s v_{s+1}, \ v_n y_0, \ y_0 y_j\},$$

where $i = 1, 2, \cdots, l$, $s = 1, 2, \cdots, n-1$, and $j = 1, 2, \cdots, m$. Consider

$$M_1: \ x_1, \ x_0, \ v_1, \ v_2, \cdots, \ v_n, \ y_0, \ y_1.$$

Suppose

$$p(V(P_n(l, m))) = 2^{n+3} + l + m - 3.$$

By placing one pebble each on $x_2, x_3, \cdots, x_l$, one pebble each on $y_2, y_3, \cdots, y_m$, and $2^{n+3} - 1$ pebbles on x_1, we cannot pebble y_1 using a monophonic path. Thus,

$$\mu(P_n(l, m)) \geq 2^{n+3} + l + m - 2.$$

To prove the sufficient condition, consider a distribution of

$$2^{n+3} + l + m - 2$$

pebbles on $V(P_n(l,m))$.

Case 1. Let the target vertex be x_1.

Consider

$$M_2: \; y_1, \; y_0, \; v_n, \; v_{n-1}, \cdots, \; v_1, \; x_0, \; x_1$$

of length $n+3$. If $p(V(M_2)) \geq 2^{n+3}$, then by Theorem 2.11, we can pebble x_1. Suppose $p(V(M_2)) < 2^{n+3}$. Then

$$p(V(P_n(l,m))) - p(V(M_2))$$

pebbles will be distributed on the remaining pendant vertices of the graph. We bring the pebbles to $V(M_2)$ to reach x_1. In the same way, we can prove the result for $x_2, x_3, \cdots, x_l, y_1, y_2, \cdots, y_m$.

Case 2. Let the target vertex be y_0.

Consider

$$M_3: \; y_0, \; v_n, \; v_{n-1}, \cdots, \; v_1, \; x_0, \; x_1$$

of length $n+2$. If $p(V(M_3)) \geq 2^{n+2}$, then by Theorem 2.11, we can pebble y_0. Suppose $p(V(M_3)) < 2^{n+2}$. Then

$$p(V(P_n(l,m))) - p(V(M_3))$$

pebbles will be distributed on the remaining pendant vertices of the graph. Thus, by using pebbling moves, we can transfer pebbles from the pendant vertices to $V(M_3)$, which is sufficient to reach y_0. By the same argument, we can prove the result for the vertex x_0.

Case 3. Let the target vertex be v_k.

There exist two monophonic paths:

$$M_4: \; v_k, \; v_{k+1}, \cdots, \; v_n, \; y_0, \; y_1$$

of length $n-k+2$, and

$$M_5: \; v_k, \; v_{k-1}, \cdots, \; v_1, \; x_0, \; x_1$$

of length $k+1$. By Theorem 2.11, if $p(V(M_4)) \geq 2^{n-k+2}$ or $p(V(M_5)) \geq 2^{k+1}$, then we can pebble v_k.

Hence,

$$\mu(P_n(l,m)) \leq 2^{n+3} + l + m - 2.$$

□

Theorem 4.5. *The monophonic pebbling number of the class of fuses is*

$$2^{l+1} + k - 1.$$

Proof. Let

$$V(F_l(k)) = \{v_1, \; v_2, \cdots, \; v_l, \; v_{l+1}, \cdots, \; v_{n-1}\}$$

and

$$E(F_l(k)) = \{v_i v_{i+1}, \; v_l v_s\},$$

where $i = 0, 1, \cdots, l-1$, $s = l+1, l+2, \cdots, n-1$, and $n = l+k+1$. Suppose

$$p(V(F_l(k))) = 2^{l+1} + k - 2.$$

By placing $k-1$ pebbles on $v_{l+2}, v_{l+3}, \cdots, v_{n-1}$ and $2^{l+1}-1$ pebbles on v_{l+1}, we cannot pebble v_0 using a monophonic path. Thus,

$$\mu(F_l(k)) \geq 2^{l+1} + k - 1.$$

Now let us prove the sufficient condition.

Case 1. Let the target vertex be v_0.

Consider

$$M_1 : \; v_0, \; v_1, \cdots, \; v_l, \; v_{l+1}$$

of length $l+1$. By Theorem 2.11, if $p(V(M_1)) \geq 2^{l+1}$, then we can pebble v_0. Suppose $p(V(M_1)) < 2^{l+1}$. Then

$$2^{l+1} + k - 1 - p(V(M_1))$$

pebbles will be on the vertices that are adjacent to v_l and are not on M_1. Now we can bring the pebbles to v_l to reach v_0. By symmetry, we prove the result for $v_{l+1}, v_{l+2}, \cdots, v_{n-1}$.

Case 2. Let the target vertex be v_a, where $a = 1, 2, \cdots, l$.

Then the monophonic path

$$M_2 : \; v_a, \; v_{a+1}, \cdots, \; v_l, \; v_j,$$

where $j = l+1, l+2, \cdots, n-1$, has length $l+1-a$, and the monophonic path

$$M_3 : \; v_a, \; v_{a-1}, \cdots, \; v_0$$

has length a. By Theorem 2.11, if $p(V(M_2)) \geq 2^{l+1-a}$, then we can pebble v_a, and if $p(V(M_3)) \geq 2^a$, then we can pebble v_a.

Thus,

$$\mu(F_l(k)) = 2^{l+1} + k - 1.$$

□

5. The *MtPN* of Some Cycle and Path Graphs

In this section, we determine the *MtPN* of Dutch windmill graphs, square of cycles, tadpole graphs, lollipop graphs, double star path graphs, and fuse graphs.

Theorem 5.1. *The monophonic t-pebbling number of* D_n^m *is*

$$t2^{2n-4} + (m-2)(2^{n-2}-1) + m,$$

where $m \geq 2$ *and* $n \geq 5$.

Proof. By placing 2^{n-2} pebbles on v_2^l, where $l = 3, 4, \cdots, m$, one pebble each on v_1^j, where $j = 1, 2, \cdots, m$, and $t2^{2n-4} - 1$ pebbles on v_{n-2}^1, we cannot place t pebbles on v_{n-2}^2. Therefore,

$$\mu_t(D_n^m) \geq t2^{2n-4} + (m-2)(2^{n-2}-1) + m.$$

Now let us prove the sufficient part. If $t = 1$, then the result follows from Theorem 3.4. Assume that the result is true for $2 \leq t \leq t-1$. Let us fix any vertex u as the target.

Clearly, the graph D_n^m has at least

$$2^{2n-3} + (m-2)(2^{n-2}-1) + m$$

pebbles. Also, $d_\mu(u, v) \leq 2n-4$, where $v \in V(D_n^m)$. We can move a pebble to u at a cost of at most 2^{2n-4} pebbles. Hence, the number of pebbles remaining on $V(D_n^m) - \{u\}$ is at least

$$\begin{aligned} \mu_t(D_n^m) - 2^{2n-4} &= t2^{2n-4} + (m-2)(2^{n-2}-1) + m - 2^{2n-4} \\ &= (t-1)2^{2n-4} + (m-2)(2^{n-2}-1) + m \\ &= \mu_{t-1}(D_n^m). \end{aligned}$$

By induction on t, we can shift $t-1$ additional pebbles to u. Suppose $p(u) = x$, where $1 \leq x \leq t-1$. Thus, the number of pebbles distributed on $V(D_n^m) - \{u\}$ is

$$\begin{aligned} \mu_t(D_n^m) - p(u) &= t2^{2n-4} + (m-2)(2^{n-2}-1) + m - x \\ &\geq (t-x)2^{2n-4} + (m-2)(2^{n-2}-1) + m \\ &= \mu_{t-x}(D_n^m). \end{aligned}$$

Then we can move $t-x$ additional pebbles to u. Thus,

$$\mu_t(D_n^m) \leq t2^{2n-4} + (m-2)(2^{n-2}-1) + m.$$

Hence, the proof is complete. □

Theorem 5.2. *The MtPN of C_n^2 is*

$$\begin{cases} t2^{n-\left(\frac{n}{3}+2\right)} + 2, & \text{if } n \equiv 0 \pmod 3, \\ t2^{n-\left(\left\lfloor\frac{n}{3}\right\rfloor+2\right)} + \left\lceil\frac{n}{3}\right\rceil, & \text{if } n \equiv 1 \pmod 3, \\ t2^{n-\left(\left\lfloor\frac{n}{3}\right\rfloor+2\right)} + 3, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. To prove this theorem, we consider the following three cases.

Case 1. $n \equiv 0 \pmod 3$.

Let

$$p(V(C_n^2)) = t2^{n-\left(\frac{n}{3}+2\right)} + 1.$$

By placing $t2^{n-\left(\frac{n}{3}+2\right)} - 1$ pebbles on v_{n-2} and one pebble each on v_n and v_{n-1}, we cannot place t pebbles on v_1. Hence,

$$\mu_t(C_n^2) \geq t2^{n-\left(\frac{n}{3}+2\right)} + 2.$$

Now we prove the sufficient condition. Consider any distribution of

$$t2^{n-\left(\frac{n}{3}+2\right)}+2$$

pebbles on the vertices of C_n^2. If $t=1$, then the result follows from Theorem 4.1. Assume that the result is true for $2 \leq t \leq t-1$. Let us fix any vertex v as the target.

We notice that the graph has at least

$$2^{n-\left(\frac{n}{3}+1\right)}+2$$

pebbles. The monophonic distance from v to any vertex in the graph is at most

$$n-\left(\frac{n}{3}+2\right).$$

Hence, using at most $2^{n-\left(\frac{n}{3}+2\right)}$ pebbles, we can place one pebble on the target. Now

$$\begin{aligned}\mu_t(C_n^2)-2^{n-\left(\frac{n}{3}+2\right)} &= t2^{n-\left(\frac{n}{3}+2\right)}+2-2^{n-\left(\frac{n}{3}+2\right)}\\ &=(t-1)2^{n-\left(\frac{n}{3}+2\right)}+2\\ &=\mu_{t-1}(C_n^2).\end{aligned}$$

Thus, by induction, we can move $t-1$ additional pebbles to v. Suppose we have y pebbles on v, where $y<t$. Then the total number of pebbles on $V(C_n^2)-\{v\}$ is

$$\begin{aligned}\mu_t(C_n^2)-p(v) &= t2^{n-\left(\frac{n}{3}+2\right)}+2-y\\ &\geq (t-y)2^{n-\left(\frac{n}{3}+2\right)}+2\\ &=\mu_{t-y}(C_n^2).\end{aligned}$$

Thus, we can move $t-y$ pebbles to v. Therefore,

$$\mu_t(C_n^2) \leq t2^{n-\left(\frac{n}{3}+2\right)}+2,$$

and the case is proved.

Case 2. $n \equiv 1 \pmod 3$.

Consider the monophonic path

$$M_3:\ v_1,\ v_2,\ v_4,\ v_5,\cdots,\ v_{n-6},\ v_{n-5},\ v_{n-3},\ v_{n-2}.$$

The vertices that are not on M_3 are

$$v_n,\ v_{n-1},\ v_{n-4},\cdots,\ v_9,\ v_6,\ v_3,$$

and these elements form the set $M_3^{\sim}$. Let

$$p(V(C_n^2))=t2^{n-\left(\frac{n}{3}+2\right)}-1+\left\lceil\frac{n}{3}\right\rceil.$$

By placing $t2^{n-\left(\lfloor\frac{n}{3}\rfloor+2\right)}-1$ pebbles on v_1 and one pebble each on the $\lceil\frac{n}{3}\rceil$ vertices on $M_3^{\sim}$, we cannot place t pebbles on v_{n-2}. Hence,

$$\mu_t(C_n^2) \geq t2^{n-\left(\lfloor\frac{n}{3}\rfloor+2\right)}+\left\lceil\frac{n}{3}\right\rceil.$$

Now we prove the sufficient condition. Consider any distribution of

$$t2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}+\left\lceil \frac{n}{3} \right\rceil$$

pebbles on the vertices of C_n^2. If $t=1$, then the result follows from Case 2 of Theorem 4.1. Assume that the result is true for $2 \le t \le t-1$. Let us fix any vertex u as the target. Clearly, the graph has at least

$$2^{n-\left(\lfloor \frac{n}{3} \rfloor+1\right)}+\left\lceil \frac{n}{3} \right\rceil$$

pebbles. Note that

$$d_\mu(u,v) \le n-\left(\left\lfloor \frac{n}{3} \right\rfloor+2\right),$$

where $v \in V(C_n^2)$, and we can move a pebble to u at a cost of

$$2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}$$

pebbles. Hence, the number of pebbles remaining on $V(C_n^2)-\{u\}$ is at least

$$\begin{aligned}\mu_t(C_n^2)-2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)} &= t2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}+\left\lceil \frac{n}{3} \right\rceil-2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}\\ &=(t-1)2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}+\left\lceil \frac{n}{3} \right\rceil\\ &=\mu_{t-1}(C_n^2).\end{aligned}$$

Thus, by induction, we can move $t-1$ additional pebbles to u. Suppose $p(u)=x$, where $1 \le x \le t-1$. Then the total number of pebbles distributed on $V(C_n^2)-\{u\}$ is

$$\begin{aligned}\mu_t(C_n^2)-p(u) &= t2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}+\left\lceil \frac{n}{3} \right\rceil-x\\ &\ge (t-x)2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}+\left\lceil \frac{n}{3} \right\rceil\\ &=\mu_{t-x}(C_n^2).\end{aligned}$$

Since $1 \le x \le t-1$, we can move $t-x$ additional pebbles to u. Thus,

$$\mu_t(C_n^2) \le t2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}+\left\lceil \frac{n}{3} \right\rceil.$$

Case 3. $n \equiv 2 \pmod 3$.
Let

$$p(V(C_n^2))=t2^{n-\left(\frac{n}{3}+2\right)}+2.$$

By placing $t2^{n-\left(\frac{n}{3}+2\right)}-1$ pebbles on v_{n-2} and one pebble each on v_{n-1}, v_n, v_{n-3}, we cannot place t pebbles on the vertex v_1 using a monophonic path. Thus,

$$\mu_t(C_n^2) \ge t2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}+3.$$

Now we prove the sufficient condition. Consider any distribution of

$$t2^{n-\left(\lfloor \frac{n}{3} \rfloor+2\right)}+3$$

pebbles on $V(C_n^2)$. If $t=1$, then the result follows from Theorem 4.1. The rest of the proof is similar to that of Case 1. □

Theorem 5.3. *The monophonic t-pebbling number of the lollipop graph is*

$$t2^{n+1} + m - 2.$$

Proof. Suppose

$$p(V(L(m,n))) = t2^{n+1} + m - 3.$$

By placing $t2^{n+1} - 1$ pebbles on u_n and one pebble each on $v_2, v_3, \cdots, v_{m-1}$, we cannot place t pebbles on v_m using a monophonic path. Thus,

$$\mu_t(L(m,n)) \geq t2^{n+1} + m - 2.$$

Now let us prove the sufficient condition. If $t = 1$, then the result follows from Theorem 4.2. Assume that the result is true for $2 \leq t \leq t-1$. Let u be the target.

Case 1. Suppose $p(u) = 0$.

Clearly, the graph $L(m,n)$ has at least $2^{n+2} + m - 3$ pebbles. Note that

$$d_\mu(u,v) \leq n+1,$$

where $v \in V(L(m,n))$, and we can move a pebble to u at a cost of 2^{n+1} pebbles. Hence, the number of pebbles remaining on $V(L(m,n))$ other than u is at least

$$\mu_t(L(m,n)) - 2^{n+1} = t2^{n+1} + m - 2 - 2^{n+1} = (t-1)2^{n+1} + m - 2 = \mu_{t-1}(L(m,n)).$$

Thus, by induction, we can move $t-1$ additional pebbles to u.

Case 2. Suppose $p(u) = y$, where $1 \leq y \leq t-1$.

Thus, the total number of pebbles distributed on $V(L(m,n)) - \{u\}$ is

$$\begin{aligned}\mu_t(L(m,n)) - p(u) &= t2^{n+1} + m - 2 - y\\ &\geq (t-y)t2^{n+1} + m - 2\\ &= \mu_{t-y}(L(m,n)).\end{aligned}$$

Since $1 \leq y \leq t-1$, we can move $t-y$ additional pebbles to u. Hence, the result follows. □

Theorem 5.4. *The monophonic t-pebbling number of the tadpole graph is*

$$t2^{m+n-2} + 1.$$

Proof. Let

$$p(V(T_{m,n})) = t2^{m+n-2}.$$

By placing $t2^{n+m-2} - 1$ pebbles on u_n and one pebble on v_m, we cannot place t pebbles on v_{m-1} using a monophonic path. Thus,

$$\mu_t(T_{m,n}) \geq t2^{n+m-2} + 1.$$

Now let us prove the sufficient condition. If $t = 1$, then the result follows from Theorem 4.3. Assume that the result is true for $2 \leq t \leq t-1$. Let u be the target.

Clearly, the graph $T_{m,n}$ has at least $2^{n+m-1}+1$ pebbles. Also,

$$d_\mu(u, v) \leq n + m - 2,$$

where $v \in V(T_{m,n})$, and we can move one pebble to any target vertex at a cost of 2^{n+m-2} pebbles. Hence, the number of pebbles remaining on $V(T_{m,n}) - \{u\}$ is at least

$$\begin{aligned} \mu_t(T_{m,n}) - 2^{n+m-2} &= t2^{n+m-2} + 1 - 2^{n+m-1} \\ &= (t-1)2^{n+m-1} + 1 \\ &= \mu_{t-1}(T_{m,n}). \end{aligned}$$

Thus, by induction, we can move $t-1$ additional pebbles to u. Suppose $p(u) = x$, where $1 \leq x \leq t-1$. Thus, the total number of pebbles distributed on the vertices of the graph is

$$\begin{aligned} \mu_t(T_{m,n}) - p(u) &= t2^{n+m-2} + 1 - x \\ &\geq (t-x)t2^{n+m-2} + 1 \\ &= \mu_{t-x}(T_{m,n}). \end{aligned}$$

Since $1 \leq x \leq t-1$, we can move $t-x$ additional pebbles to u. Thus,

$$\mu_t(T_{m,n}) \leq t2^{n+m-1}.$$

Hence, the proof is complete. □

Theorem 5.5. *The monophonic t-pebbling number of the double star path graph is*

$$t2^{n+3} + l + m - 2.$$

Proof. Let

$$p(V(P_n(l,m))) = t2^{n+3} + l + m - 3.$$

By placing one pebble each on

$$x_2, x_3, \cdots, x_l, \; y_2, y_3, \cdots, y_m$$

and $t2^{n+3} - 1$ pebbles on x_1, we cannot place t pebbles on y_1 using a monophonic path. Thus,

$$\mu_t(P_n(l,m)) \geq t2^{n+3} + l + m - 2.$$

To prove the sufficient condition, consider a distribution of

$$t2^{n+3} + l + m - 2$$

pebbles on $V(P_n(l,m))$. If $t = 1$, then the result follows from Theorem 4.4. Assume that the result is true for $2 \leq t \leq t-1$. Let u be the target.

We notice that the graph has at least

$$2^{n+4} + l + m - 2$$

pebbles. Clearly,

$$d_\mu(u, v) \leq n + 3,$$

where $v \in V(P_n(l, m))$. Hence, using at most 2^{n+3} pebbles, we can place one pebble on u. Now

$$\begin{aligned} \mu_t(P_n(l, m)) - 2^{n+3} &= t2^{n+3} + l + m - 2 - 2^{n+3} \\ &= (t-1)2^{n+3} + l + m - 2 \\ &= \mu_{t-1}(P_n(l, m)). \end{aligned}$$

Thus, by induction, we can move $t-1$ additional pebbles to u. Suppose we have y pebbles on u, where $y < t$. Then the total number of pebbles on the remaining vertices of the graph is

$$\begin{aligned} \mu_t(P_n(l, m)) - p(u) &= t2^{n+3} + l + m - 2 - y \\ &\geq (t-y)2^{n+3} + l + m - 2 \\ &= \mu_{t-y}(P_n(l, m)). \end{aligned}$$

Thus, we can move $t - y$ pebbles to u. Therefore,

$$\mu_t(P_n(l, m)) \leq t2^{n+3} + l + m - 2.$$

□

Theorem 5.6. *The monophonic t-pebbling number of the class of fuse graphs is*

$$t2^{l+1} + k - 1.$$

Proof. Let

$$p(V(F_l(k))) = t2^{l+1} + k - 2.$$

By placing one pebble on each vertex

$$v_{l+2}, v_{l+3}, \cdots, v_{n-1}$$

and $t2^{l+1} - 1$ pebbles on v_{l+1}, we cannot place t pebbles on the vertex v_0 using a monophonic path. Thus,

$$\mu_t(F_l(k)) \geq t2^{l+1} + k - 1.$$

Now let us prove the sufficient condition. If $t = 1$, then the result follows from Theorem 4.5. Assume that the result is true for $2 \leq t \leq t - 1$. Let us fix any vertex u as the target.

Case 1. Suppose $p(u) = 0$.

Clearly, the graph $F_l(k)$ has at least

$$2^{l+2} + k - 1$$

pebbles. The distance satisfies

$$d_\mu(u, v) \leq l + 1,$$

where $v \in V(F_l(k))$, and we can move one pebble to any target vertex at a cost of 2^{l+1} pebbles. Hence, the number of pebbles remaining on the vertices of the graph $F_l(k)$ other than the vertex u is at least

$$\begin{aligned} \mu_t(F_l(k)) - 2^{l+1} &= t2^{l+1} + k - 1 - 2^{l+1} \\ &= (t-1)2^{l+1} + k - 1 \\ &= \mu_{t-1}(F_l(k)). \end{aligned}$$

Thus, by induction, we can move $t - 1$ additional pebbles to u.

Case 2. Suppose $p(u) = y$, where $1 \leq y \leq t - 1$.

Thus, the total number of pebbles on the vertices of the graph is

$$\begin{aligned} \mu_t(F_l(k)) - p(u) &= t2^{l+1} + k - 1 - y \\ &\geq (t-y)2^{l+1} + k - 1 \\ &= \mu_{t-y}(F_l(k)). \end{aligned}$$

Since $1 \leq y \leq t - 1$, we can move $t - y$ additional pebbles to u. Thus,

$$\mu_t(F_l(k)) \leq t2^{l+1} + k - 1.$$

□

6. Conclusion

In this article, we computed the MPN and $MtPN$ of several cycle and path graphs. This approach has been developed based on the limitations of the methodology used in the literature. The concept of the monophonic pebbling number helps in minimizing or optimizing the flow of information through a network. Our future scope will be to develop an algorithm that can be implemented on a computer.

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