Journal of Combinatorial Mathematics
and Combinatorial Computing
www.combinatorialpress.com/jcmcc

Combinatorial Press

Towards the linear arboricity conjecture: graph products

Yuina Tanaka[✉]

ABSTRACT

For a graph G, let $la(G)$ denote the linear arboricity of G and $\Delta(G)$ denote the maximum degree of G. The famous linear arboricity conjecture was made by Akiyama, Exoo, and Harary [Covering and packing in graphs. IV. Linear arboricity] in 1981. It asserts that $la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$. In this paper, we prove the linear arboricity conjecture for products of a path and a complete graph, and for products of a path and a tree.

Keywords: linear arboricity, linear arboricity conjecture, graph products

2020 Mathematics Subject Classification: 05C15, 05C70.

1. Introduction

In graph theory, a *linear forest* is a forest in which every component is a path. For a graph G, the *linear arboricity* of G (defined by Harary [8] in 1970) is denoted by $la(G)$ and is the minimum number of edge-disjoint linear forests whose union is $E(G)$. The maximum degree of G is denoted by $\Delta(G)$. We can observe

$$la(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil. \tag{1}$$

Also, it is well-known that, for regular graphs,

$$la(G) \geq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil. \tag{2}$$

In 1981, Akiyama, Exoo, and Harary [2] made the famous linear arboricity conjecture:

* Corresponding author.

Received 27 Oct 2025; Revised 01 Dec 2025; Accepted 30 Jan 2026; Published Online 13 Feb 2026.

DOI: 10.61091/jcmcc130-05

Conjecture 1.1 (Akiyama, Exoo, and Harary, [2]). *For every graph G,*

$$la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil.$$

This conjecture is still open, but it was proven for some graphs:

Theorem 1.2 (Akiyama, Exoo, and Harary, [1]). *For a complete graph K_n,*

$$la(K_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Theorem 1.3 (Akiyama, Exoo, and Harary, [1]). *For a tree T,*

$$la(T) = \left\lceil \frac{\Delta(T)}{2} \right\rceil.$$

Also, see: [3, 5, 7, 10, 11].

In 1992, Alon and Spencer [4] proved the following upper bound:

Theorem 1.4 (Alon and Spencer, [4]). *For every graph G,*

$$la(G) \leq \frac{\Delta(G)}{2} + O\Big(\Delta(G)^{\frac{2}{3}} log(\Delta(G))^{\frac{1}{3}}\Big).$$

In 2020, Ferber, Fox, and Jain [6] proved the following improved upper bound:

Theorem 1.5 (Ferber, Fox, and Jain, [6]). *For every graph G,*

$$la(G) \leq \frac{\Delta(G)}{2} + C\Delta(G)^{\frac{2}{3}-\alpha},$$

for constants C and α.

In this paper, we prove the linear arboricity conjecture for products of a path and a complete graph, and for products of a path and a tree.

For two graphs G_1 and G_2, the *Cartesian product* of them is denoted by $G_1 \Box G_2$, whose vertex set is

$$V(G_1 \Box G_2) = \{(v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2)\},$$

and edge set is

$$E(G_1 \Box G_2) = \{(v_1, v_2)(v_1', v_2') \mid v_1 = v_1' \text{ and } v_2 v_2' \in E(G_2),\ \text{or } v_1 v_1' \in E(G_1) \text{ and } v_2 = v_2'\}.$$

For two graphs G_1 and G_2, the *direct product* of them is denoted by $G_1 \times G_2$, whose vertex set is

$$V(G_1 \times G_2) = \{(v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2)\},$$

and edge set is

$$E(G_1 \times G_2) = \{(v_1, v_2)(v_1', v_2') \mid v_1v_1' \in E(G_1) \text{ and } v_2v_2' \in E(G_2)\}.$$

For two graphs G_1 and G_2, the *strong product* of them is denoted by $G_1 \boxtimes G_2$, whose vertex set is

$$V(G_1 \boxtimes G_2) = \{(v_1, v_2) \mid v_1 \in V(G_1), v_2 \in V(G_2)\},$$

and edge set is

$$E(G_1 \boxtimes G_2) = E(G_1 \square G_2) \cup E(G_1 \times G_2).$$

In the following theorems, we assume $n, m \geq 2$, because if $n = 1$, then the linear arboricity conjecture is proven by Theorem 1.2. If $m = 1$, then $P_n \square K_1$ is only one path and $P_n \times K_1$ does not have an edge.

In 2013, Tao and Lin [9] proved the linear arboricity conjecture for Cartesian product of a path and a complete graph.

Theorem 1.6 (Tao and Lin, [9])**.** *For Cartesian product of a path and a complete graph,*

$$la(P_n \square K_m) = \left\lceil \frac{m+1}{2} \right\rceil \leq \left\lceil \frac{\Delta(P_n \square K_m) + 1}{2} \right\rceil,$$

for $n, m \geq 2$.

In section 2, we first prove the following theorem, which proves the linear arboricity conjecture for direct product of a path and a complete graph.

Theorem 1.7. *For direct product of a path and a complete graph,*

$$la(P_n \times K_m) = m - 1 = \left\lceil \frac{\Delta(P_n \times K_m)}{2} \right\rceil,$$

for $n \geq 3$ *and* $m \geq 2$.

$$la(P_2 \times K_m) = \left\lceil \frac{m}{2} \right\rceil = \left\lceil \frac{\Delta(P_2 \times K_m) + 1}{2} \right\rceil,$$

for $m \geq 2$.

By combining Theorems 1.6 and 1.7, we further prove the following theorem for $n \geq 3$, which proves the linear arboricity conjecture for strong product of a path and a complete graph. The case $n = 2$ is an open problem.

Theorem 1.8. *For strong product of a path and a complete graph,*

$$la(P_n \boxtimes K_m) = \left\lceil \frac{3m-1}{2} \right\rceil = \left\lceil \frac{\Delta(P_n \boxtimes K_m)}{2} \right\rceil,$$

for $n \geq 3$ *and* $m \geq 2$.

In Section 3, we study Cartesian, direct, and strong products of a path P_n with $n \geq 2$ and a tree T. If $n = 1$, then the linear arboricity conjecture is proven by Theorem 1.3. In the following theorems, we assume $\Delta(T)$ is even. The case $\Delta(T)$ is odd is an open problem.

Theorem 1.9. *For a path and a tree,*
(a)

$$la(P_n \square T) = \left\lceil \frac{\Delta(P_n \square T)}{2} \right\rceil,$$

for $n \geq 2$ and even $\Delta(T)$.
(b)

$$la(P_n \times T) = \left\lceil \frac{\Delta(P_n \times T)}{2} \right\rceil,$$

for $n \geq 2$ and even $\Delta(T)$.
(c)

$$la(P_n \boxtimes T) = \left\lceil \frac{\Delta(P_n \boxtimes T)}{2} \right\rceil,$$

for $n \geq 2$ and even $\Delta(T)$.

2. Products of a path and a complete graph

We first prove Theorem 1.7:
Proof of Theorem 1.7. Let

$$V(P_n) = \{u_1, u_2, \dots, u_n\},$$

and

$$E(P_n) = \{u_1u_2, u_2u_3, \dots, u_{n-1}u_n\}.$$

Let

$$V(K_m) = \{v_1, v_2, \dots, v_m\}.$$

For convenience, we will also use notations $v_{m+1}, v_{m+2}, \dots, v_{2m-1}$, which are defined by $v_{m+1} = v_1$, $v_{m+2} = v_2$, $\dots$, $v_{2m-1} = v_{m-1}$.

We first assume $n \geq 3$. So, for $2 \leq k \leq n-1$ and $1 \leq l \leq m$, the degree of (u_k, v_l) is $2m-2$. For $k = 1$ or $k = n$ and $1 \leq l \leq m$, the degree of (u_k, v_l) is $m-1$. So, in $P_n \times K_m$, the maximum degree is $2m-2$ and $\left\lceil \frac{\Delta(P_n \times K_m)}{2} \right\rceil = m-1$.

We construct $m-1$ linear forests by defining their edge sets as follows:

For $1 \leq t \leq m-1$, let

$$E_t = \{(u_i, v_j)(u_{i+1}, v_{j+t}) \mid 1 \leq i \leq n-1, i \text{ is odd}, 1 \leq j \leq m\}$$

$$\cup\{(u_k, v_{l+t})(u_{k+1}, v_l) \mid 1 \leq k \leq n-1, k \text{ is even}, 1 \leq l \leq m\}.$$

Then, the edges in E_t form a linear forest, which has m disjoint paths. If n is even, then these m disjoint paths are

$$\begin{aligned}
&\{(u_1, v_1), (u_2, v_{1+t}), (u_3, v_1), (u_4, v_{1+t}), \dots, (u_n, v_{1+t})\},\\
&\{(u_1, v_2), (u_2, v_{2+t}), (u_3, v_2), (u_4, v_{2+t}), \dots, (u_n, v_{2+t})\},\\
&\dots\\
&\{(u_1, v_m), (u_2, v_{m+t}), (u_3, v_m), (u_4, v_{m+t}), \dots, (u_n, v_{m+t})\}.
\end{aligned}$$

If n is odd, then these m disjoint paths are

$$\begin{aligned}
&\{(u_1, v_1), (u_2, v_{1+t}), (u_3, v_1), (u_4, v_{1+t}), \dots, (u_n, v_1)\},\\
&\{(u_1, v_2), (u_2, v_{2+t}), (u_3, v_2), (u_4, v_{2+t}), \dots, (u_n, v_2)\},\\
&\dots\\
&\{(u_1, v_m), (u_2, v_{m+t}), (u_3, v_m), (u_4, v_{m+t}), \dots, (u_n, v_m)\}.
\end{aligned}$$

The $m-1$ linear forests cover all $m(m-1)(n-1)$ edges in $P_n \times K_m$, and each edge belongs to exactly one linear forest. So, $la(P_n \times K_m) \leq m-1 = \left\lceil \frac{\Delta(P_n \times K_m)}{2} \right\rceil$. By combining this result with (1), we get $la(P_n \times K_m) = m-1 = \left\lceil \frac{\Delta(P_n \times K_m)}{2} \right\rceil$.

We then assume $n = 2$. So, $\Delta(P_2 \times K_m) = m-1 = \Delta(K_m)$. We need to prove $la(P_2 \times K_m) = \lceil \frac{m}{2} \rceil = \left\lceil \frac{\Delta(P_2 \times K_m)+1}{2} \right\rceil$.

For K_m, assume the $la(K_m) = \lceil \frac{m}{2} \rceil$ linear forests are $F_1, F_2, \dots, F_{la(K_m)}$. For $P_2 \times K_m$, we construct $la(K_m)$ linear forests by defining their edge sets as follows:

For $1 \leq t \leq la(K_m)$, let

$$E_t = \{(u_i, v_j)(u_k, v_l) \mid u_i u_k \in E(P_2), v_j v_l \in E(F_t)\}.$$

Then, the edges in E_t form a linear forest, which has twice as many paths as F_t. So, $la(P_2 \times K_m) \leq la(K_m) = \lceil \frac{m}{2} \rceil$. $P_2 \times K_m$ is a regular graph. By combining this result with (2), we get $la(P_2 \times K_m) = la(K_m) = \lceil \frac{m}{2} \rceil = \left\lceil \frac{\Delta(P_2 \times K_m)+1}{2} \right\rceil$. □

We combine Theorems 1.6 and 1.7 to prove Theorem 1.8:

Proof of Theorem 1.8. We assumed $n \geq 3$. So, $\Delta(P_n \boxtimes K_m) = 3m-1$. By $E(P_n \boxtimes K_m) = E(P_n \square K_m) \cup E(P_n \times K_m)$, we have

$$\begin{aligned}
la(P_n \boxtimes K_m) &\leq la(P_n \square K_m) + la(P_n \times K_m)\\
&= \left\lceil \frac{m+1}{2} \right\rceil + m - 1\\
&= \left\lceil \frac{3m-1}{2} \right\rceil\\
&= \left\lceil \frac{\Delta(P_n \boxtimes K_m)}{2} \right\rceil.
\end{aligned}$$

By combining this result with (1), we get $la(P_n \boxtimes K_m) = \left\lceil \frac{\Delta(P_n \boxtimes K_m)}{2} \right\rceil$. □

3. Products of a path and a tree

In this section, we prove Theorem 1.9:

Proof of Theorem 1.9.

Let

$$V(P_n) = \{u_1, u_2, \ldots, u_n\},$$

and

$$E(P_n) = \{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n\}.$$

Let

$$V(T) = \{v_1, v_2, \ldots, v_m\}.$$

(a) We first prove

$$la(P_n \Box T) = \left\lceil \frac{\Delta(P_n \Box T)}{2} \right\rceil.$$

If $n \geq 3$, then $\Delta(P_n \Box T) = \Delta(T)+2$. We need to prove $la(P_n \Box T) = \lceil \frac{\Delta(T)+2}{2} \rceil = \frac{\Delta(T)}{2}+1$. (We assumed $\Delta(T)$ is even.) If $n = 2$, then $\Delta(P_n \Box T) = \Delta(T) + 1$. We still need to prove $la(P_n \Box T) = \lceil \frac{\Delta(T)+1}{2} \rceil = \frac{\Delta(T)}{2} + 1$.

In Theorem 1.3, we have $la(T) = \lceil \frac{\Delta(T)}{2} \rceil = \frac{\Delta(T)}{2}$. Assume the $la(T)$ linear forests are $F_1, F_2, \ldots, F_{la(T)}$. For $P_n \Box T$, we construct $\frac{\Delta(T)}{2} + 1$ linear forests by defining their edge sets as follows:

For $1 \leq t \leq la(T) = \frac{\Delta(T)}{2}$, let

$$E_t = \{(u_i, v_j)(u_i, v_k) \mid 1 \leq i \leq n, 1 \leq j, k \leq m, v_jv_k \in E(F_t)\}.$$

Let

$$E_{\frac{\Delta(T)}{2}+1} = \{(u_i, v_j)(u_{i+1}v_j) \mid 1 \leq i \leq n-1, 1 \leq j \leq m\}.$$

Our idea is: For $1 \leq t \leq la(T) = \frac{\Delta(T)}{2}$, we let E_t be the set of edges in the n copies of F_t. Then, we let $E_{\frac{\Delta(T)}{2}+1}$ be the set of edges in the m copies of P_n. So, the edges in each of $E_1, E_2, \ldots, E_{\frac{\Delta(T)}{2}+1}$ form a linear forest.

So, $la(P_n \Box T) \leq \frac{\Delta(T)}{2} + 1 = \lceil \frac{\Delta(P_n \Box T)}{2} \rceil$. By combining this result with (1), we get $la(P_n \Box T) = \lceil \frac{\Delta(P_n \Box T)}{2} \rceil$.

(b) We prove

$$la(P_n \times T) = \left\lceil \frac{\Delta(P_n \times T)}{2} \right\rceil.$$

We first assume $n \geq 3$. So, $\Delta(P_n \times T) = 2\Delta(T)$. We need to prove $la(P_n \times T) = \Delta(T)$.

In Theorem 1.3, we have $la(T) = \lceil \frac{\Delta(T)}{2} \rceil = \frac{\Delta(T)}{2}$. (We assumed $\Delta(T)$ is even.) Assume the $la(T)$ linear forests are $F_1, F_2, \ldots, F_{la(T)}$. We divide the vertices in T into two parts V^+ and V^-:

We choose a vertex v and put it in V^+. Then, every vertex with even distance from v is put in V^+ and every vertex with odd distance from v is put in V^-. Because T is a tree, every vertex in V^+ is only adjacent to vertices in V^- and every vertex in V^- is only adjacent to vertices in V^+.

For $P_n \times T$, we construct $2la(T) = \Delta(T)$ linear forests by defining their edge sets as follows:

For $1 \leq t \leq la(T) = \frac{\Delta(T)}{2}$, let

$$E_t^1 = \{(u_i, v_j)(u_{i+1}, v_k) \mid 1 \leq i \leq n-1, v_j v_k \in E(F_t), v_j \in V^+, v_k \in V^-\},$$

and

$$E_t^2 = \{(u_i, v_j)(u_{i+1}, v_k) \mid 1 \leq i \leq n-1, v_j v_k \in E(F_t), v_j \in V^-, v_k \in V^+\}.$$

The edges in E_t^1 form a linear forest, because, for every $1 \leq i \leq n-1$, $\{(u_i, v_j)(u_{i+1}, v_k) \mid v_j v_k \in E(F_t), v_j \in V^-, v_k \in V^+\}$ form a linear forest which is isomorphic to F_t. If we take different $i' \neq i$, then $\{(u_{i'}, v_j)(u_{i'+1}, v_k) \mid v_j v_k \in E(F_t), v_j \in V^-, v_k \in V^+\}$ is disjoint with $\{(u_i, v_j)(u_{i+1}, v_k) \mid v_j v_k \in E(F_t), v_j \in V^-, v_k \in V^+\}$. So, E_t^1 is a linear forest, because it is a disjoint union of $n-1$ linear forests.

Similarly, the edges in E_t^2 also form a linear forest. So, $la(P_n \times T) \leq 2la(T) = \Delta(T)$. By combining this result with (1), we get $la(P_n \times T) = \Delta(T) = \lceil \frac{\Delta(P_n \times T)}{2} \rceil$.

We then assume $n = 2$. So, $\Delta(P_n \times T) = \Delta(T)$. We need to prove $la(P_n \times T) = \frac{\Delta(T)}{2}$, which can be proven by a similar construction: We construct E_t^1 and E_t^2 in the same way, and we can combine E_t^1 and E_t^2, because $\{(u_1, v_j)(u_2, v_k) \mid v_j v_k \in E(F_t), v_j \in V^+, v_k \in V^-\}$ and $\{(u_1, v_j)(u_2, v_k) \mid v_j v_k \in E(F_t), v_j \in V^-, v_k \in V^+\}$ are disjoint. So, $E_t^1 \cup E_t^2$ form a linear forest when $n = 2$.

So, we get $\frac{\Delta(T)}{2}$ linear forests and $la(P_n \times T) \leq \frac{\Delta(T)}{2}$. By combining this result with (1), we get $la(P_n \times T) = \frac{\Delta(T)}{2}$.

(c) We prove

$$la(P_n \boxtimes T) = \left\lceil \frac{\Delta(P_n \boxtimes T)}{2} \right\rceil.$$

We first assume $n \geq 3$. So, $\Delta(P_n \boxtimes T) = 3\Delta(T) + 2$. By $E(P_n \boxtimes T) = E(P_n \square T) \cup E(P_n \times T)$, we have

$$\begin{aligned} la(P_n \boxtimes T) &\leq la(P_n \square T) + la(P_n \times T) \\ &= \frac{\Delta(T)}{2} + 1 + \Delta(T) \\ &= \frac{3\Delta(T) + 2}{2} \\ &= \left\lceil \frac{\Delta(P_n \boxtimes T)}{2} \right\rceil. \end{aligned}$$

By combining this result with (1), we get $la(P_n \boxtimes T) = \left\lceil \frac{\Delta(P_n \boxtimes T)}{2} \right\rceil$.

We then assume $n = 2$. So, $\Delta(P_n \boxtimes T) = 2\Delta(T) + 1$. By $E(P_n \boxtimes T) = E(P_n \Box T) \cup E(P_n \times T)$, we have

$$
\begin{aligned}
la(P_n \boxtimes T) &\le la(P_n \Box T) + la(P_n \times T) \\
&= \frac{\Delta(T)}{2} + 1 + \frac{\Delta(T)}{2} \\
&= \frac{2\Delta(T) + 2}{2} \\
&= \left\lceil \frac{2\Delta(T) + 1}{2} \right\rceil \\
&= \left\lceil \frac{\Delta(P_n \boxtimes T)}{2} \right\rceil .
\end{aligned}
$$

By combining this result with (1), we get $la(P_n \boxtimes T) = \left\lceil \frac{\Delta(P_n \boxtimes T)}{2} \right\rceil$. □

4. Open problem

In this paper, we proved the linear arboricity conjecture for products of a path and a complete graph, and for products of a path and a tree. However, in Theorem 1.8, we assumed $n \ge 3$. In Theorem 1.9, we assumed $\Delta(T)$ is even. So, there are two open problems.

Problem 4.1 (about Theorem 1.8). *Prove the linear arboricity conjecture for $P_n \boxtimes K_m$ when $n = 2$.*

Problem 4.2 (about Theorem 1.9). *Prove the linear arboricity conjecture for $P_n \Box T$, $P_n \times T$, and $P_n \boxtimes T$ when $\Delta(T)$ is odd.*

References

[1] J. Akiyama, G. Exoo, and F. Harary. Covering and packing in graphs. III. Cyclic and acyclic invariants. *Mathematica Slovaca*, 30(4):405–417, 1980.

[2] J. Akiyama, G. Exoo, and F. Harary. Covering and packing in graphs. IV. Linear arboricity. *Networks. An International Journal*, 11(1):69–72, 1981. https://doi.org/10.1002/net.3230110108.

[3] N. Alon. The linear arboricity of graphs. *Israel Journal of Mathematics*, 62(3):311–325, 1988. https://doi.org/10.1007/BF02783300.

[4] N. Alon and J. H. Spencer. *The probabilistic method.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1992, pages xvi+254. With an appendix by Paul Erdős, A Wiley-Interscience Publication.

[5] H. Enomoto and B. Péroche. The linear arboricity of some regular graphs. *Journal of Graph Theory*, 8(2):309–324, 1984. https://doi.org/10.1002/jgt.3190080211.

[6] A. Ferber, J. Fox, and V. Jain. Towards the linear arboricity conjecture. *Journal of Combinatorial Theory. Series B*, 142:56–79, 2020. https://doi.org/10.1016/j.jctb.2019.08.009.

[7] F. Guldan. Some results on linear arboricity. *Journal of Graph Theory*, 10(4):505–509, 1986. https://doi.org/10.1002/jgt.3190100408.

[8] F. Harary. Covering and packing in graphs. I. *Annals of the New York Academy of Sciences*, 175:198–205, 1970.

[9] F. Tao and W. Lin. Linear arboricity of Cartesian products of graphs. *Journal of Southeast University. English Edition. Dongnan Daxue Xuebao. Yingwen Ban*, 29(2):222–225, 2013.

[10] J.-L. Wu. On the linear arboricity of planar graphs. *Journal of Graph Theory*, 31(2):129–134, 1999. https://doi.org/10.1002/(sici)1097-0118(199906)31:2<129::aid-jgt5>3.0.co;2-a.

[11] J.-L. Wu and Y.-W. Wu. The linear arboricity of planar graphs of maximum degree seven is four. *ournal of Graph Theory*, 58(3):210–220, 2008. https://doi.org/10.1002/jgt.20305.

Yuina Tanaka
Hosei University, 3-7-2 Kajino-cho, Koganei-shi, Tokyo, 184-8584, Japan
Liberty University, Lynchburg, VA 24515, USA
E-mail tanakayuina@outlook.com