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A study on the weighted efficient domination problem for C_4-free bipartite graphs

Andreas Brandstädt* and Raffaele Mosca

ABSTRACT

A vertex set D in a finite undirected graph G is an *efficient dominating set* (*e.d.s.* for short) of G if every vertex of G is dominated by exactly one vertex of D. The *Efficient Domination* (ED) problem asks for the existence of an e.d.s. in G. The *Weighted Efficient Dominating Set* (WED for short) problem further asks for an e.d.s. of minimum/maximum weight in a given graph G. The ED problem is known to be NP-complete, even for claw-free graphs, for P_7-free graphs, for chordal bipartite graphs, for planar bipartite graphs of maximum degree 3 and girth at least g for every fixed g, and thus for C_4-free bipartite graphs. This manuscript reports a study on the WED problem for C_4-free bipartite graphs (in the context of a study for bipartite graphs) and shows that the WED problem can be solved in polynomial time for $(S_{1,2,5}, C_4)$-free bipartite graphs, for (P_{10}, C_4)-free bipartite graphs, and for some related graphs classes.

Keywords: weighted efficient domination, C_4-free bipartite graphs, P_{10}-free bipartite graphs, $S_{1,2,5}$-free bipartite graphs, polynomial time

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1. Introduction

1.1. Preliminaries

For any missing concept or notation let us refer to [10].

Let G be a finite undirected graph, let $V(G)$ denote its vertex set with $|V(G)|= n$, and let $E(G)$ denote its edge set with $|E(G)|= m$. For any $U \subseteq V(G)$, let $N(U) :=$

* Corresponding author.

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$\{v \in V(G) \setminus U : v$ is adjacent to some vertex of $U\}$ denote the *neighborhood of* U, and let $N[U] := N(U) \cup U$ denote the *closed neighborhood of* U; in particular, if $U = \{u\}$, then let us simply write $N(u)$ and $N[u]$. For any $U, W \subseteq V(G)$, if every vertex in U is adjacent (is nonadjacent) to every vertex in W, we denote it by $U \textcircled{1} W$ (by $U \textcircled{0} W$); in particular, for $U = \{u\}$, we simply denote $U \textcircled{1} W$ (denote $U \textcircled{0} W$) by $u \textcircled{1} W$ (by $u \textcircled{0} W$). For any $u \in V(G)$ and for any $W \subseteq V(G)$, with $u \notin W$, let us say that u *contacts* W if u is adjacent to some vertex of W. For any $u, v \in V(G)$, $dist_G(u, v)$ denotes the minimum distance between u and v in G, i.e., the minimum length of a path between u and v in G; similarly, for any $v \in V(G)$ and for any $U \subseteq V(G)$, $dist_G(U, v) = min\{dist_G(u, v) : u \in U\}$. Furthermore, for any $u \in V(G)$ and for any natural i, let us write $N_i(u) = \{v \in V(G) : dist_G(u, v) = i\}$; similarly, for any $U \subseteq V$ and for any natural i, let us write $N_i(U) = \bigcup_{u \in U} N_i(u) \setminus U$. Finally, for any $U \subseteq V(G)$, let $G[U]$ denote the subgraph of G induced by vertex set U.

For a set $\mathcal{F}$ of graphs, a graph G is called $\mathcal{F}$*-free* if G contains no induced subgraph isomorphic to a member of $\mathcal{F}$; in particular, we say that G is H*-free* if G is $\{H\}$-free.

For $h \geq 1$, let P_h denote the chordless path with h vertices, and let K_h denote the complete graph with h vertices (clearly, $P_2 = K_2$). For $h \geq 4$, let C_h denote the chordless cycle with h vertices. The *girth* of a graph G is the length of a shortest [and thus induced] cycle contained in G. For $i, j, k \geq 0$, let $S_{i,j,k}$ denote the tree formed by three chordless paths, namely $P_{i+1}, P_{j+1}, P_{k+1}$, which share a unique endpoint. Thus, the *claw* is $S_{1,1,1}$, the *chair* (also called the *fork*) is $S_{1,1,2}$, and P_k is isomorphic to $S_{0,0,k-1}$. Finally, we denote by $G + H$ the disjoint union of graphs G and H, in particular let $2H$ denote $H + H$.

1.2. *The weighted efficient domination (WED) problem*

Let $G = (V(G), E)$ be a finite undirected graph. A vertex v *dominates* itself and its neighbors. A vertex subset $D \subseteq V(G)$ is an *efficient dominating set* (*e.d.s.* for short) of G if every vertex of G is dominated by exactly one vertex in D. Then, for any e.d.s. D of G, one has $|D \cap N[v]| = 1$ for every $v \in V(G)$. The notion of efficient domination was introduced by Biggs [4] under the name *perfect code*. Note that a graph may have no e.d.s., e.g., a C_4 or a C_5 (in particular, in [28], it is considered the problem to compute the minimum number of vertices in a set S such that the graph $G - S$ has an e.d.s.).

Let us mention the following theorem shown in [40]:

Theorem 1.1. **[40]** *If a graph has two or more e.d.s., then all such e.d.s. have the same cardinality.*

The *Efficient Dominating Set* (*ED* for short) problem can be defined as follows:

- Given a graph G, check if G has an e.d.s.

The *Weighted Efficient Dominating Set* (*WED* for short) problem can be defined as follows:

- Given a graph G and a weight function $w : V(G) \to \mathbb{R}$, compute $D^* \in$ ARG MIN$\{w(D) : D$ is an e.d.s. of $G\}$, where $w(D) = \sum_{d \in D} w(d)$.

Note that, since a graph may have no e.d.s., the WED problem may have no feasible solution.

In particular, if $w(v) = 1$ for every $v \in V(G)$, then the WED problem is the ED problem.

Finally, according to [7], let us point out that, by Theorem 1.1 and since negative weights are allowed, the *Weighted Efficient Dominating Set* could be equivalently defined with respect to maximum weight (instead of minimum weight).

Let us mention [6] and [27] for a detailed study on this topic. The WED problem is motivated by various applications, including coding theory and resource allocation in parallel computer networks; see e.g [3, 2, 4, 20, 32, 33, 34, 39, 40, 44, 45]. Efficient dominating sets are also called *perfect dominating sets* and *independent perfect dominating sets* in various papers.

The Exact Cover problem asks for a subset $\mathcal{F}'$ of a set family $\mathcal{F}$ over a ground set, say V, containing every vertex in V exactly once, i.e., $\mathcal{F}'$ forms a partition of V. As shown by Karp [29], this problem is NP-complete even for set families containing only 3-element subsets of V (see problem X3C [SP2] in [25]). Clearly, the ED problem is the Exact Cover problem for the closed neighborhood hypergraph of G, i.e., if $D = \{d_1, \ldots, d_k\}$ is an e.d.s. of G then $N[d_1] \cup \ldots \cup N[d_k]$ forms a partition of $V(G)$. In [3, 2], it was shown that the ED problem is NP-complete.

A set M of edges in a graph G is an *efficient edge dominating set* (*e.e.d.s.* for short) of G if and only if it is an e.d.s. in its line graph $L(G)$. The Efficient Edge Dominating Set (EED for short) problem asks for the existence of an e.e.d.s. in a given graph G. Thus, the EED problem for a graph G corresponds to the ED problem for its line graph $L(G)$. Again, note that not every graph has an e.e.d.s. Efficient edge dominating sets are also called *dominating induced matchings* in some papers (see e.g. [9, 19]). In [26], it was shown that the EED problem is NP-complete; see also [9, 19, 37, 38]. Thus, the ED problem is NP-complete for line graphs. Since line graphs are claw-free, the ED problem is NP-complete for claw-free graphs as well.

Many papers have studied the complexity of the ED problem on special graph classes (see [12, 40] for references). The ED problem remains NP-complete for bipartite graphs [45], $2P_3$-free chordal graphs [40, 45], chordal unipolar graphs [23, 45], line graphs of bipartite graphs [38], planar bipartite graphs [39], chordal bipartite graphs [39] (a bipartite graph G is *chordal bipartite* if G is C_{2k}-free for every $k \geq 3$. i.e., it is a hole-free bipartite graph), planar graphs with maximum degree at most 3 [24, 30]. Thus, for every $k \geq 3$, the ED problem is NP-complete for C_{2k}-free bipartite graphs. Moreover, the ED problem is NP-complete for planar bipartite graphs [39] and even for planar bipartite graphs of maximum degree 3 [12] and girth at least g for every fixed g [42]. Thus, the ED problem is NP-complete for C_4-free bipartite graphs.

A *linear forest* is the disjoint union of chordless paths, namely, it is a C_k-free (for any $k \geq 3$) and claw-free graph. The NP-completeness of the ED problem for $2P_3$-free chordal graphs, for bipartite graphs, and for claw-free graphs mentioned above implies:

Corollary 1.2. *For any graph F, different to a $2P_3$-free linear forest, the ED problem is NP-complete in the class of F-free graphs.*

Corollary 1.2 motivated the study of the (W)ED problem in the class of F-free graphs, when F is a $2P_3$-free linear forest. The last remaining open case was the complexity of the ED problem on P_6-free graphs (see [8]); actually, it was open even for P_6-free chordal graphs. Lokshtanov, Pilipczuk, and van Leeuwen [35], and independently Brandstädt and Mosca [13], showed that the ED problem is solvable in polynomial time for P_6-free graphs (the time bound in the direct approach of [13] is $O(n^5m)$ while in [35], it is more than n^{500}).

As mentioned above, in [23, 43, 45], it was shown that ED is NP-complete for $2P_3$-free chordal unipolar graphs and thus, in general, for P_7-free graphs. Then, by the above polynomial results on P_6-free graphs, one obtains a dichotomy.

Furthermore, if the ED problem is solvable in polynomial time for $(P_k+\ell P_2)$-free graphs, $\ell \geq 0$, then it is solvable in polynomial time for $(P_k + (\ell+1)P_2)$-free graphs [8]. Thus, the ED problem is solvable in polynomial time for $(P_6 + \ell P_2)$-free graphs for every fixed $\ell \geq 1$. Let us mention that the ED problem is solvable in linear time for P_5-free graphs [5]. Moreover the ED problem is solvable in polynomial for different subclasses of P_7-free graphs [11].

However, regardless of Corollary 1.2, the WED problem is solvable in polynomial time for distance-hereditary graphs [39], for AT-free graphs and for dually chordal graphs [7], and for some "comb product graphs" [31].

Let us conclude with a focus on *bipartite graphs*.

Lu and Tang [39] showed that the ED problem is solvable in linear time for bipartite permutation graphs (which are a subclass of convex bipartite graphs). In [11], it is shown that the ED problem can be solved in polynomial time for interval bipartite graphs (which include convex bipartite graphs, which are a subclass of chordal bipartite graphs). In [1], it is shown that the ED problem can be solved in polynomial time for diameter three bipartite graphs, while the ED problem is NP-complete for graphs of diameter $d = 3k$ or $d = 3k + 2$ for $k \geq 1$.

It is well known that for every graph class with bounded clique-width, the ED problem can be solved in polynomial time [21]; for instance, the clique-width of claw-free bipartite, i.e., $K_{1,3}$-free bipartite graphs is bounded. Dabrowski and Paulusma [22] published a dichotomy for boundedness of clique-width of H-free bipartite graphs (see also [36]). For instance, the clique-width of $S_{1,2,3}$-free bipartite graphs is bounded (which includes $K_{1,3}$-free bipartite graphs).

Since the ED problem remains NP-complete for bipartite graphs, Corollary 1.2 and the above results motivate the study of the (W)ED problem on *F-free bipartite graphs*, when F is not a $2P_3$-free linear forest and when F-free bipartite graphs have unbounded clique width.

In [14], it is shown that the ED problem can be solved in polynomial time for P_7-free bipartite graphs, for ℓP_4-free bipartite graphs for fixed ℓ, for $S_{2,2,4}$-free bipartite graphs as well as for P_9-free bipartite graphs with degree at most 3. Furthermore the WED problem can be solved in polynomial time for $S_{1,1,5}$-free bipartite graphs [15] for $S_{1,3,3}$-free bipartite graphs [16], and for P_8-free bipartite graphs [17, 18].

In particular one could consider the following open problems:

What is the complexity of the ED problem for

- P_k-free bipartite graphs, $k \geq 9$?
- $S_{1,3,4}$-free bipartite graphs ?
- $S_{1,1,6}$-free bipartite graphs ?
- $S_{1,2,5}$-free bipartite graphs ?
- $S_{2,2,k}$-free bipartite graphs for $k \geq 5$?
- chordal bipartite graphs with vertex degree at most 3 ?

1.3. This manuscript

This manuscript reports a study on the WED problem for C_4-free bipartite graphs, as motivated above, recalling that the ED problem remains NP-complete for C_4-free bipartite graphs.

In detail, this manuscript introduces the following results.

Theorem 1.3. *For $(S_{1,2,5}, C_4)$-free bipartite graphs, the WED problem can be solved in $O(n^{19})$ time.*

Theorem 1.4. *For (P_{10}, C_4)-free bipartite graphs, the WED problem can be solved in $O(n^6)$ time.*

The approach introduced for Theorem 1.4 can be extended for (P_t, C_4)-free bipartite graphs, for $t \geq 11$, as discussed later.

More generally along this study – mainly due to the first author – it turned out that many polynomial results could be introduced by a similar approach.

In particular the WED problem is solvable in polynomial time for the following graph classes:

- (P_{11}, C_4, C_6)-free bipartite graphs,
- (P_{13}, C_4, C_6, C_8)-free bipartite graphs,
- $(2P_7, C_4, C_6, C_8)$-free bipartite graphs,
- $(P_{16}, C_4, C_6, C_8, C_{10})$-free bipartite graphs,
- $(2P_9, C_4, C_6, C_8, C_{10})$-free bipartite graphs.

Let us conclude with a result from [42] which could be useful to present this topic.

Consider the following transformation [which recalls that introduced in [41] showing that the Maximum Independent Set problem remains NP-hard for graphs of girth at least g for every fixed g]: Let $G = (V(G), E(G))$ be a graph and let $G' = (V(G'), E(G'))$ be the graph obtained from G by replacing every edge $xy \in E$ with an induced P_5 of vertices x, a, b, c, y and edges xa, ab, bc, cy, namely $x - a - b - c - y$; in particular let us say that $x - a - b - c - y$ is a *blue* path of G' and that vertices a, b, c are the *interior* vertices of $x - a - b - c - y$.

Theorem 1.5. **[42]** *Let $G = (V(G), E(G))$ be any graph and let $G' = (V(G'), E(G'))$ be the graph obtained from G as above. Then G has an e.d.s. of cardinality k if and only if G' has an e.d.s. of cardinality $k + |E|$.*

Proof. First let us assume that G has an e.d.s. say D of cardinality k. Then one can define an e.d.s. D' of G' as follows with reference to each edge $xy \in E(G)$ (by definition of e.d.s. for each edge $xy \in E(G)$ at most one vertex of xy is in D):

if $xy \in E(G)$, $x \in D$, $y \notin D$, then with reference to the blue path $x - a - b - c - y$ of G', set x, c in D';

if $xy \in E(G)$, $x \notin D$, $y \notin D$, then with reference to the blue path $x - a - b - c - y$ of G', set b in D'.

By definition of e.d.s. and by construction one has that D' is an e.d.s. of G'; in particular, for each blue path P of G' there is exactly one interior vertex of P belonging to D', so that D' has cardinality $k + |E|$.

Then let us assume that G' has an e.d.s. say D' of cardinality $k + |E(G)|$. Note that by definition of e.d.s., for each blue path P of G', there is exactly one interior vertex of P belonging to D'. Then let us observe that:

if $a \in D'$, $b, c, \notin D'$, then by definition of e.d.s. one has $x \notin D'$ and $y \in D'$;

if $b \in D'$, $a, c, \notin D'$, then by definition of e.d.s. one has $x, y \notin D'$.

By definition of e.d.s and by construction one has that $D = D' \cap V(G)$ is an e.d.s. of G; in particular, for each blue path P of G' there is exactly one interior vertex of P belonging to D', so that D has cardinality k. □

Notice that, for any graph G, the graph G' obtained from G as above is bipartite. Furthermore the above theorem admits a repeated application: in particular it shows that the ED problem remains NP-complete for bipartite graphs of girth at least g for every fixed g.

2. The WED problem for $(S_{1,2,5}, C_4)$-free bipartite graphs

2.1. Preliminaries

Let $G = (V(G), E(G)) = (X, Y, E(G))$ be an $(S_{1,2,5}, C_4)$-free bipartite graph.

For any vertex $u \in V(G)$, we say that u is *forced* if $u \in D$ for every e.d.s. D of G, and u is *excluded* if $u \notin D$ for every e.d.s. D of G.

By a forced vertex, G can be reduced to G' as follows:

Claim 1. If $u \in V(G)$ is forced, then G has an e.d.s. D with $u \in D$ if and only if the reduced graph $G' = G \setminus N[u]$ has an e.d.s. $D' = D \setminus \{u\}$ such that all vertices in $N_2(u)$ are excluded in G'.

Analogously, for any vertex $v \in V(G)$, we say that $u \in V(G)$ is *v-forced* if $u \in D$ for every e.d.s. D of G with $v \in D$, and u is *v-excluded* if $u \notin D$ for every e.d.s. D of G with $v \in D$. For checking whether G has an e.d.s. D with $v \in D$, we can clearly reduce G by forced vertices as well as by v-forced vertices when we assume that $v \in D$:

Claim 2. If $u \in V(G)$ is v-forced, for a vertex $v \in V(G)$, then G has an e.d.s. D with $v \in D$ if and only if the reduced graph $G' = G \setminus N[u]$ has an e.d.s. $D' = D \setminus \{u\}$ with $v \in D'$ such that all vertices in $N_2(u)$ are v-excluded in G'.

Similarly, for $k \geq 2$, $u \in V(G)$ is $(v_1, \ldots, v_k)$-*forced* if $u \in D$ for every e.d.s. D of G with $v_1, \ldots, v_k \in D$, and correspondingly, $u \in V(G)$ is $(v_1, \ldots, v_k)$-*excluded* if $u \notin D$ for such e.d.s. D, and G can be reduced by the same principle.

Clearly, for every component of G, the WED problem is independently solvable. Thus, we can assume that G is connected.

Recall that $dist_G(u, v)$ denotes the minimum distance, i.e., the minimum length of a path between u and v in G. By definition of e.d.s., the distance between two D-vertices is at least 3. Moreover, if $u, u' \in D \cap X$ or $u, u' \in D \cap Y$ then $dist_G(u, u') \geq 4$.

If for an e.d.s. D in the bipartite graph $G = (X, Y, E)$, $|D|= 1$, say without loss of generality, $D = \{x\}$ with $x \in X$ and $D \cap Y = \emptyset$, then every $y \in Y$ must have the D-neighbor $x \in D$, i.e., $x①Y$ and by the e.d.s. property, $|X|= 1$, i.e., $X = \{x\}$ (else there is no such e.d.s. in G), which is a trivial e.d.s. solution.

Now assume that $|D|\geq 2$, and without loss of generality, $D \cap X \neq \emptyset$. Since G is connected, every $x \in D \cap X$ must have at least distance 2 in G, say (x, y, x') induce a P_3 in G. Then by the e.d.s. property, $y, x' \notin D$ and x' must have a D-neighbor $y' \in D \cap Y$ (else there is no such e.d.s. in G). Thus, $D \cap X \neq \emptyset$ and $D \cap Y \neq \emptyset$.

Then by the above let us formalize the following obseravtion.

Observation 2.1. *If D is an e.d.s. of G, then there is an induced P_4 of G say P, such that D contains the two endpoints of P.*

2.2. *Distance levels*

Let us fix an induced P_4 of G, say P of vertices p_1, p_2, p_3, p_4, and edges p_1p_2, p_2p_3, p_3p_4.

Let us recall the notation $N_h(\{p_1, p_4\}) = \{v \in V(G) : dist(v, \{p_1, p_4\}) = h\}$, for $h \geq 1$, for the *distance levels* with respect to $\{p_1, p_4\}$.

NOTATION: for brevity let us write $\{p_1, p_4\} = N_0$ and $N_h(\{p_1, p_4\}) = N_h$ for $h \geq 1$.

Notice that $G[N_0 \cup N_1]$ is connected.

Then let us assume that $N_0 \subseteq D$, where D is a (possible) e.d.s. D of G.

By definition of e.d.s., we have:

$$D \cap (N_1 \cup N_2) = \emptyset. \tag{1}$$

If there is a vertex $x \in N_2$ with $N(x) \cap N_3 = \emptyset$, then G has no e.d.s. D with $N_0 \subseteq D$. Thus, we assume:

$$\text{Every vertex } x \in N_2 \text{ has a neighbor in } N_3. \tag{2}$$

If there is a vertex $x \in N_2$ with $|N(x) \cap N_3|= 1$, say $N(x) \cap N_3 = \{y\}$, then y is N_0-forced. Then one can update $N_0 := N_0 \cup \{y\}$ and re-define the distance levels with respect to N_0. Notice that $G[N_0 \cup N_1]$ is still connected. Thus, we assume:

$$\text{Every vertex } x \in N_2 \text{ has at least two neighbors in } N_3. \tag{3}$$

If there is a vertex $y \in N_3$ with $N(y) \cap N_4 = \emptyset$, then y is N_0-forced. Then one can update $N_0 := N_0 \cup \{y\}$ and re-define the distance levels with respect to N_0 as above. Notice that $G[N_0 \cup N_1]$ is still connected.

Thus, we assume:

$$\text{Every vertex } y \in N_3 \text{ has a neighbor in } N_4. \tag{4}$$

In particular, we have:

Lemma 2.2. *For every $r_2 \in N_2$, r_2 is no endpoint of a P_6 whose remaining vertices are in $N_0 \cup N_1$.*

Proof. Let $r_2 \in N_2$ be the endpoint of a P_k $(r_2, r_1, v_1, \dots, v_k)$, $k \geq 1$, with $r_1 \in N_1$ and $v_1, \dots, v_k \in N_0 \cup N_1$. Clearly, (r_2, r_1, r_0) induce a P_3 in G with $r_1 \in N_1$ and $r_0 \in N_0$.

Suppose to the contrary that r_2 is the endpoint of a P_6 $(r_2, r_1, v_1, v_2, v_3, v_4)$ with $r_1 \in N_1$ and $v_1, \dots, v_4 \in N_0 \cup N_1$. By (3), r_2 must have at least two neighbors in N_3, say $r_2r_3 \in E$, $r_2r_3' \in E$ with $r_3, r_3' \in N_3$ and without loss of generality, $r_3 \in D$. By defnition of e.d.s., $r_3' \notin D$ and r_3' must have a D-neighbor $s \in D \cap (N_3 \cup N_4)$. But then $(r_2, r_3, r_3', s, r_1, v_1, v_2, v_3, v_4)$ (with midpoint r_2) induce an $S_{1,2,5}$ in G, which is a contradiction.

Thus, r_2 is no endpoint of a P_6 whose remaining vertices are in $N_0 \cup N_1$, i.e., r_2 is the endpoint of a P_5 $(r_2, r_1, v_1, v_2, v_3)$ with $r_1 \in N_1$ and $v_1, v_2, v_3 \in N_0 \cup N_1$ or a P_4 (r_2, r_1, v_1, v_2) with $r_1 \in N_1$ and $v_1, v_2 \in N_0 \cup N_1$ or a P_3 (r_2, r_1, r_0) with $r_1 \in N_1$ and $r_0 \in N_0$, and Lemma 2.2 is shown. □

Lemma 2.3. *Every $r_2 \in N_2$ is no endpoint of a P_5 $P = (r_2, r_3, r_4, r_5, r_6)$ with $r_3 \in N_3$, $r_4 \in N_3 \cup N_4$, $r_5 \in N_3 \cup N_4 \cup N_5$, $r_6 \in N_3 \cup N_4 \cup N_5 \cup N_6$, i.e., $N_6 = \emptyset$, N_5 is independent, every edge in N_4 does not contact N_5, and for every edge r_3s_3 in N_3, there is no such P_4 (r_3, s_3, r_4, r_5) with $r_4 \in N_4$ and $r_5 \in N_4 \cup N_5$.*

Proof. Let (x, y_1, x_1, y) induce a P_4 in $G[N_0 \cup N_1]$ with $x, y \in N_0$ and $y_1, x_1 \in N_1$.

Suppose to the contrary that $r_2 \in N_2$ is the endpoint of a P_5 $P = (r_2, r_3, r_4, r_5, r_6)$ with $r_3 \in N_3$, $r_4 \in N_3 \cup N_4$, $r_5 \in N_3 \cup N_4 \cup N_5$, $r_6 \in N_3 \cup N_4 \cup N_5 \cup N_6$. Without loss of generality, assume that $r_2 \in N_2 \cap X$.

If $r_2y_1 \in E$ then $(y_1, x, x_1, y, r_2, r_3, r_4, r_5, r_6)$ (with midpoint y_1) induce an $S_{1,2,5}$ in G, which is a contradiction. Thus, $r_2y_1 \notin E$ and $r_1r_2 \in E$ with $r_1 \in N_1$, $r_1 \neq y_1$.

First assume that $r_1x \in E$. Since there is no such C_4 (r_1, x, y_1, x_1) in G, $r_1x_1 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, r_1, x, y_1, x_1, y)$ with $r_1, x_1, y_1 \in N_1$ and $x, y \in N_0$, which is a contradiction by Lemma 2.2. Thus, $r_1x \notin E$, i.e., $r_0r_1 \in E$ with $r_0 \in N_0$, $r_0 \neq x$. If $r_1x_1' \in E$ with $yx_1' \in E$ (possibly $x_1 = x_1'$) then $(r_1, r_0, x_1', y, r_2, r_3, r_4, r_5, r_6)$ (with midpoint r_1) induce an $S_{1,2,5}$ in G, which is a contradiction. Thus, $r_1x_1' \notin E$, i.e., $r_1 ⓪ N(y)$.

Since $G[N_0 \cup N_1]$ is connected, r_0 and y must have distance at least 3; without loss of generality, (r_0, r_1', x_1', y) induce a P_4 with $r_1', x_1' \in N_1$ (possibly $x_1 = x_1'$ but $r_1 \neq r_1'$). Since $r_1'x_1' \in E$ then, since there is no such C_4 (r_0, r_1, r_2, r_1') in G, $r_1'r_2 \notin E$. But then r_2

is the endpoint of a P_6 $(r_2, r_1, r_0, r'_1, x'_1, y)$ with $r_1, r'_1, x'_1 \in N_1$ and $r_0, y \in N_0$, which is a contradiction by Lemma 2.2. Thus, $r'_1 x'_1 \notin E$, i.e., $r'_1 \textcircled{0} N(y)$, which is a contradiction.

Thus, $r_2 \in N_2$ is no endpoint of a P_5 $P = (r_2, r_3, r_4, r_5, r_6)$ with $r_3 \in N_3$, $r_4 \in N_3 \cup N_4$, $r_5 \in N_3 \cup N_4 \cup N_5$, $r_6 \in N_3 \cup N_4 \cup N_5 \cup N_6$, i.e., $N_6 = \emptyset$, N_5 is independent, every edge in N_4 does not contact N_5, and for every edge $r_3 s_3$ in N_3, there is no such P_4 (r_3, s_3, r_4, r_5) with $r_4 \in N_4$, $r_5 \in N_4 \cup N_5$, and Lemma 2.3 is shown. □

Corollary 2.4. *For every $r_2 \in N_2$ and $r_3 \in D \cap N_3$ with $r_2 r_3 \in E$, $r_2 \in N_2$ is no endpoint of P_4 (r_2, r_3, r_4, r_5) with $r_3 \in D \cap N_3$, $r_4 \in N_3 \cup N_4$, $r_5 \in N_3 \cup N_4 \cup N_5$.*

2.3. When no D-vertex is midpoint of a P_5 in G

Let us show that an e.d.s. of G in ARG MIN$\{w(D) : D$ is an e.d.s. of G such that no D-vertex is midpoint of an induced $P_5\}$ can be computed in polynomial time.

Let D denote such a possible e.d.s.

Lemma 2.5. *For every P_8 $P = (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$ in G, $x_1, x_2, y_2, x_3, y_3, y_4$ are excluded.*

Proof. Let $P = (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$ induce a P_8 in G, and in this subsection, every D-vertex is no midpoint of a P_5 in G.

Since x_2 is midpoint of a P_5 $(x_1, y_1, x_2, y_2, x_3)$ in G, $x_2 \notin D$. Analogously, since y_2 is midpoint of a P_5 $(y_1, x_2, y_2, x_3, y_3)$ in G, $y_2 \notin D$, since x_3 is midpoint of a P_5 $(x_2, y_2, x_3, y_3, x_3)$ in G, $x_3 \notin D$, and since y_3 is midpoint of a P_5 $(y_2, x_3, y_3, x_4, y_4)$ in G, $y_3 \notin D$, i.e., x_2, y_2, x_3, y_3 are excluded.

Finally suppose to the contrary that either $x_1 \in D$ or $y_4 \in D$; without loss of generality, assume that $x_1 \in D$; recall that x_2, y_2 are excluded. Then by definition of e.d.s., $y_1, x_2 \notin D$ and x_2 must have a D-neighbor $y \in D$ with $x_2 y \in E$, $y \neq y_1, y_2$. Clearly, by the e.d.s. property, $x_1 y \notin E$. Since in this subsection, y is no midpoint of a P_5 in G, $y x_3 \notin E$ and $y x_4 \notin E$ (else $y \in D$ is midpoint of a P_5 (y_1, x_2, y, x_3, y_3) or (y_1, x_2, y, x_4, y_4)).

But then $(x_2, y, y_1, x_1, y_2, x_3, y_3, x_4, y_4)$ (with midpoint x_2) induce an $S_{1,2,5}$ in G, which is a contradiction. Thus, $x_1 \notin D$. Analogously, $y_4 \notin D$, i.e., x_1, y_4 are excluded, and Lemma 2.5 is shown. □

Lemma 2.6. *For every P_8 $P = (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$ in G, there are exactly two D-vertices $y_1 \in D$ and $x_4 \in D$ in P which are forced.*

Proof. Let $P = (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$ induce a P_8 in G, and in this subsection, every D-vertex is no midpoint of a P_5 in G. Recall that by Lemma 2.5, $x_1, x_2, y_2, x_3, y_3, y_4$ are excluded.

Suppose to the contrary that either $y_1 \notin D$ or $x_4 \notin D$; without loss of generality, assume that $y_1 \notin D$. By Lemma 2.5, $x_1, x_2 \notin D$ and y_1 must have a D-neighbor $x \in D$ with $x y_1 \in E$, $x \neq x_1, x_2$. Moreover, by the e.d.s. property, x_1 must have a D-neighbor

$y \in D$ with $x_1 y \in E$.

Since in this subsection, $x, y \in D$ are no midpoints of a P_5 in G, $xy_2 \notin E$, $xy_3 \notin E$, and $xy_4 \notin E$ (else $x \in D$ is midpoint of a P_5 (x_1, y_1, x, y_2, x_3) or (x_1, y_1, x, y_3, x_4) or (x_1, y_1, x, y_4, x_4)), as well as $yx_3 \notin E$ and $yx_4 \notin E$ (else $y \in D$ is midpoint of a P_5 (y_1, x_1, y, x_3, y_3) or (y_1, x_1, y, x_4, y_4)).

Since there is no such C_4 (y, x_1, y_1, x_2) in G, we have $yx_2 \notin E$. But then $(y_1, x, x_1, y, x_2, y_2, x_3, y_3, x_4)$ (with midpoint y_1) induce an $S_{1,2,5}$ in G, which is a contradiction. Thus, $y_1 \in D$ is forced.

Analogously, $x_4 \in D$ is forced, i.e., y_1, x_4 are forced, and Lemma 2.6 is shown. □

Summarizing one can proceed as follows. Set $Y := \emptyset$; then, for each induced P_8 $P = (x_1, y_1, \dots, x_4, y_4)$ of G, set $Y := Y \cup \{y_1, x_4\}$. Thus by Lemma 2.6 and by construction, all vertices in Y are forced, and $G[V(G) \setminus Y]$ is P_8-free. Then by Claim 1, the problem is reduced to [the connected components of] $G[V(G) \setminus N[Y]]$, with the assumption that vertices in $N_2(Y)$ are excluded; that can be done in polynomial time, since WED is solvable in polynomial time for P_8-free bipartite graphs [17, 18]; in particular, the WED problem for (P_8, C_4)-free bipartite graphs can be solved in $O(n^3)$ time, by Corollary 8 introduced in the next section concerning the WED problem in (P_{10}, C_4)-free bipartite graphs; thus the running time bound of the above procedure can be estimated as $O(n^8 \cdot n^3)$ time.

Thus, in this subsection, the WED problem can be solved in polynomial time.

2.4. *When D-vertices are midpoints of a P_5 in G*

Let us show that an e.d.s. of G in $\text{ARG MIN}\{w(D) : D$ is an e.d.s. of G such that some D-vertex is midpoint of an induced $P_5\}$ can be computed in polynomial time.

Let D denote such a possible e.d.s.

Now assume that there is at least one D-vertex which is midpoint of an induced P_5 in G, say (u_1, v_1, u, v_2, u_2), with midpoint $u \in D$. Then by definition of e.d.s., $u_1, u_2 \notin D$ and u_1, u_2 must have D-neighbors in G.

Thus, we can assume that:

— either $(v, u_1, v_1, u, v_2, u_2)$ induce a C_6, with $u, v \in D$, which we denote as $2D - C_6$; in this case one can set $N_0 := \{u, v\}$;

— or $(v, u_1, v_1, u, v_2, u_2, v')$ induce a P_7, with $u, v, v' \in D$, which we denote as $3D - P_7$; in this case one can set $N_0 := \{u, v, v'\}$.

In other words there is either a $2D - C_6$ in $G[N_0 \cup N_1]$ or a $3D - P_7$ in $G[N_0 \cup N_1]$.

Then the problem is to compute an e.d.s. D [of minimum weight] of G, such that $N_0 \subseteq D$, according to the above two cases.

Notice that in both cases one has:

$$G[N_0 \cup N_1] \text{ is connected.} \tag{5}$$

2.4.1. When there is a $2D - C_6$ in $G[N_0 \cup N_1]$. Let $C = (x, y_1, x_1, y, x_1', y_1')$ induce a $2D - C_6$ in $G[N_0 \cup N_1]$ with $x, y \in N_0$, $x_1, x_1', y_1, y_1' \in N_1$. Then for every D-vertex in

$D \cap N_3$ which is (x, y)-forced, it can be in N_0.

Lemma 2.7. *For every $2D - C_6$ in $G[N_0 \cup N_1]$, $N_2 = \emptyset$.*

Proof. Let $C = (x, y_1, x_1, y, x'_1, y'_1)$ induce a $2D - C_6$ in $G[N_0 \cup N_1]$ with $x, y \in N_0$, $x_1, x'_1, y_1, y'_1 \in N_1$.

Suppose to the contrary that $N_2 \neq \emptyset$.

First assume that $r_2 \in N_2$ contacts $2D - C_6$ $C \in G[N_0 \cup N_1]$; without loss of generality, $r_2 \in N_2 \cap X$. By (1), $r_2 \notin D$ and r_2 must have a D-neighbor in N_3. By (3), r_2 must have two neighbors $r_3, r'_3 \in N_3$, say $r_3 \in D \cap N_3$, and by the e.d.s. property, $r'_3 \notin D$ and r'_3 must have a D-neighbor $s \in D \cap (N_3 \cup N_4)$.

If $y_1 r_2 \in E$ then, since there is no such C_4 (r_2, y_1, x, y'_1) in G, $y'_1 r_2 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, y_1, x_1, y, x'_1, y'_1)$ with $x_1, y_1, x'_1, y'_1 \in N_1$ and $y \in N_0$, which is a contradiction by Lemma 2.2. Thus, $y_1 r_2 \notin E$. Moreover, if $y'_1 r_2 \in E$ then, since there is no such C_4 (r_2, y_1, x, y'_1) in G, $y_1 r_2 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, y'_1, x'_1, y, x_1, y_1)$ with $x_1, y_1, x'_1, y'_1 \in N_1$ and $y \in N_0$, which is a contradiction by Lemma 2.2. Thus, $y'_1 r_2 \notin E$.

If $v r_2 \in E$ with $xv \in E$, $v \neq y_1, y'_1$, then, since there is no such C_4 (v, x, y_1, x_1) in G, $v x_1 \notin E$ and since there is no such C_4 (v, x, y'_1, x'_1) in G, $v x'_1 \notin E$.

But then r_2 is the endpoint of a P_6 (r_2, v, x, y_1, x_1, y) with $v, x_1, y_1 \in N_1$ and $x, y \in N_0$, which is a contradiction by Lemma 2.2.

Thus, $r_2 ⓪ N(x)$, say $r_1 r_2 \in E$ with $r_1 \in N_1$, $r_1 x \notin E$. Analogously, for $r_2 \in N_2 \cap Y$, $r_2 ⓪ N(y)$, say $r_1 r_2 \in E$ with $r_1 \in N_1$, $r_1 y \notin E$. Without loss of generality, $r_2 \in N_2 \cap X$. Then $r_1 \in N_1 \cap Y$ and $r_1 x \notin E$, say $r_0 r_1 \in E$ with $r_0 \in N_0$, $r_0 \neq x$.

If $r_1 x_1 \in E$ then r_2 is the endpoint of a P_6 $(r_2, r_1, x_1, y_1, x, y'_1)$ with $r_1, x_1, y_1, y'_1 \in N_1$ and $x \in N_0$, which is a contradiction by Lemma 2.2. Thus, $r_1 x_1 \notin E$ and analogously, $r_1 x'_1 \notin E$.

If $r_1 u \in E$ with $yu \in E$, $u \neq x_1, x'_1$, then, since there is no such C_4 (u, y, x_1, y_1) in G, $u y_1 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, r_1, u, y, x_1, y_1)$ with $r_1, u, x_1, y_1 \in N_1$ and $y \in N_0$, which is a contradiction by Lemma 2.2. Thus, $r_1 ⓪ N(y)$.

Assume that $dist_G(y, r_0) = 3$, say (r_0, r'_1, u, y) induce a P_4 in G (possibly $u = x_1$ or $u = x'_1$). Since there is no such C_4 (r_0, r_1, r_2, r'_1) in G, $r'_1 r_2 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, r_1, r_0, r'_1, u, y)$ with $r_1, r'_1, u \in N_1$ and $r_0, y \in N_0$, which is a contradiction by Lemma 2.2. Thus, $N_2 \cap X = \emptyset$. Analogously, if there is an $r_2 \in N_2 \cap Y$ then there is a contradiction, and $N_2 \cap Y = \emptyset$.

Thus, $N_2 = \emptyset$, and Lemma 2.7 is shown. □

Then for every $2D - C_6$ in $G[N_0 \cup N_1]$, it can be done in polynomial time. The running time bound can be estimated as $O(n^6)$ time. Now assume that there are no such $2D - C_6$'s in G.

2.4.2. When there is no $2D - C_6$ in $G[N_0 \cup N_1]$ but a $3D - P_7$ in $G[N_0 \cup N_1]$.

Lemma 2.8. *For $3D - P_7$ in $G[N_0 \cup N_1]$: if the midpoint is in $N_0 \cap X$, then $N_2 \cap X = \emptyset$;*

if the midpoint is in $N_0 \cap Y$, then $N_2 \cap Y = \emptyset$.

Proof. Without loss of generality, let $(x_1, y_1, x_2, y_2, x_3, y_3, x_4)$ induce a $3D-P_7$ in $G[N_0 \cup N_1]$ with midpoint $y_2 \in N_0 \cap Y$, and with $x_1, x_4 \in N_0$, $y_1, x_2, x_3, y_3 \in N_1$.

Suppose to the contrary that $N_2 \cap Y \neq \emptyset$.

First assume that $r_2 \in N_2 \cap Y$ contacts $N(y_2)$. If $r_2x_2 \in E$ then, since there is no such C_4 (r_2, x_2, y_2, x_3) in G, $r_2x_3 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, x_2, y_2, x_3, y_3, x_4)$, which is a contradiction by Lemma 2.2. Thus, $r_2x_2 \notin E$.

If $r_2x_3 \in E$ then, since there is no such C_4 (r_2, x_2, y_2, x_3) in G, $r_2x_2 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, x_3, y_2, x_2, y_1, x_1)$, which is a contradiction by Lemma 2.2. Thus, $r_2x_3 \notin E$.

If $r_2u \in E$ with $y_2u \in E$, $u \neq x_2, x_3$, then, since there is no such C_4 (r_2, u, y_2, x_2) in G, $r_2x_2 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, u, y_2, x_2, y_1, x_1)$, which is a contradiction by Lemma 2.2. Thus, $r_2 \in N_2 \cap Y$ does not contact $N(y_2)$, i.e., $r_2 ⓪ N(y_2)$, say $r_1r_2 \in E$ with $r_1 \in N_1 \cap X$, $r_1y_2 \notin E$.

If $r_1y_1 \in E$ then r_2 is the endpoint of a P_6 $(r_2, r_1, y_1, x_2, y_2, x_3)$, which is a contradiction by Lemma 2.2. Thus, $r_1y_1 \notin E$ and analogously, $r_1y_3 \notin E$.

If $r_1v \in E$ with $x_1v \in E$, $v \neq y_1$, then, since there is no such C_4 (v, x_1, y_1, x_2) in G, $vx_2 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, r_1, v, x_1, y_1, x_2)$, which is a contradiction by Lemma 2.2. Thus, r_1 does not contact $N(x_1)$ and analogously, r_1 does not contact $N(x_4)$, i.e., $r_1 ⓪ N(x_1) \cup N(x_4)$.

Let $r_0r_1 \in E$ with $r_0 \in N_0$, $r_0 \neq y_2$. Assume that $G[N_0 \cup N_1]$ is connected; without loss of generality, $dist_G(x_1, r_0) = 3$, say (x_1, v, r'_1, r_0) induce a P_4 in G, $r'_1 \neq r_1$. Since there is no such C_4 (r_0, r_1, r_2, r'_1) in G, $r'_1r_2 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, r_1, r_0, r'_1, v, x_1)$, which is a contradiction by Lemma 2.2. Thus, there is no such distance between x_1 and r_0, and analogously, there is no such distance between x_4 and r_0.

Thus, for midpoint $y_2 \in N_0 \cap Y$, $N_2 \cap Y = \emptyset$. Analogously, if $(y_1, x_1, y_2, x_2, y_3, x_3, y_4)$ induce a $3D-P_7$ in $G[N_0 \cup N_1]$ with midpoint $x_2 \in N_0 \cap X$, $N_2 \cap X = \emptyset$, and Lemma 2.8 is shown. □

Corollary 2.9. *For $3D-P_7$ in $G[N_0 \cup N_1]$ with midpoint in $N_0 \cap Y$, $N_3 \cap X = \emptyset$, $N_4 \cap Y = \emptyset$, $N_5 \cap X = \emptyset$, i.e., N_3, N_4, N_5 are independent.*

Corollary 2.10. *For $3D-P_7$ in $G[N_0 \cup N_1]$ with midpoint in $N_0 \cap X$, $N_3 \cap Y = \emptyset$, $N_4 \cap X = \emptyset$, $N_5 \cap Y = \emptyset$, i.e., N_3, N_4, N_5 are independent.*

Corollary 2.11. *For two $3D-P_7$'s in $G[N_0 \cup N_1]$ with midpoints in $N_0 \cap X$ and in $N_0 \cap Y$, $N_2 = \emptyset$.*

Lemma 2.12. *For $3D-P_7$ $(x_1, y_1, x_2, y_2, x_3, y_3, x_4)$ in $G[N_0 \cup N_1]$, if the midpoint $y_2 \in N_0 \cap Y$, then for $r_2 \in N_2 \cap X$, $r_2y_1 \in E$ and $r_2y_3 \in E$.*

Proof. Let $(x_1, y_1, x_2, y_2, x_3, y_3, x_4)$ induce a $3D-P_7$ in $G[N_0 \cup N_1]$ with midpoint $y_2 \in$

$N_0 \cap Y$, and with $x_1, x_4 \in N_0$, $y_1, x_2, x_3, y_3 \in N_1$.

Suppose to the contrary that for $r_2 \in N_2 \cap X$, $r_2y_1 \notin E$ or $r_2y_3 \notin E$; without loss of generality, $r_2y_1 \notin E$.

If $r_2y_3 \in E$ then r_2 is the endpoint of a P_6 $(r_2, y_3, x_3, y_2, x_2, y_1)$, which is a contradiction by Lemma 2.2. Thus, $r_2y_3 \notin E$, say $r_1r_2 \in E$ with $r_1 \in N_1$, $r_1 \neq y_1, y_3$.

If $r_1x_1 \in E$ then, since there is no such C_4 (r_1, x_1, y_1, x_2) in G, $r_1x_2 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, r_1, x_1, y_1, x_2, y_2)$, which is a contradiction by Lemma 2.2. Thus, $r_1x_1 \notin E$, and analogously, $r_1x_4 \notin E$, say $r_0r_1 \in E$ with $r_0 \in N_0$, $r_0 \neq x_1, x_4$.

If $r_1x_2 \in E$ then, since there is no such C_4 (r_1, x_2, y_2, x_3) in G, $r_1x_3 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, r_1, x_2, y_2, x_3, y_3)$, which is a contradiction by Lemma 2.2. Thus, $r_1x_2 \notin E$ and analogously, $r_1x_3 \notin E$. If $r_1u \in E$ with $y_2u \in E$, $u \neq x_2, x_3$, then, since there is no such C_4 (u, y_2, x_2, y_1) in G, $uy_1 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, r_1, u, y_2, x_2, y_1)$, which is a contradiction by Lemma 2.2. Thus, $r_1u \notin E$, i.e., $r_1 ⓪ N(y_2)$.

Assume that $dist_G(r_0, y_2) = 3$, say (r_0, r_1', u, y_2) with $r_1' \neq r_1$ (possibly $u = x_2$ or $u = x_3$). Since there is no such C_4 (r_2, r_1, r_0, r_1') in G, $r_1'r_2 \notin E$. But then r_2 is the endpoint of a P_6 $(r_2, r_1, r_0, r_1', u, y_2)$, which is a contradiction by Lemma 2.2.

Thus, $r_2y_1 \in E$. Analogously, $r_2y_3 \in E$, and Lemma 2.12 is shown. □

Corollary 2.13. *For $3D - P_7$ $(y_1, x_1, y_2, x_2, y_3, x_3, y_4)$ in $G[N_0 \cup N_1]$, if the midpoint $x_2 \in N_0 \cap X$, then for $r_2 \in N_2 \cap Y$, $r_2x_1 \in E$ and $r_2x_3 \in E$.*

Corollary 2.14. *For $3D - P_7$ in $G[N_0 \cup N_1]$, if the midpoint is in $N_0 \cap Y$, then $|N_2 \cap X| = 1$. Analogously, for $3D - P_7$ in $G[N_0 \cup N_1]$, if the midpoint is in $N_0 \cap X$, then $|N_2 \cap Y| = 1$.*

Proof. Without loss of generality, let $(x_1, y_1, x_2, y_2, x_3, y_3, x_4)$ induce a $3D - P_7$ in $G[N_0 \cup N_1]$ with midpoint $y_2 \in N_0 \cap Y$, and with $x_1, x_4 \in N_0$, $y_1, x_2, x_3, y_3 \in N_1$. Then by Lemma 2.12, for $r_2 \in N_2 \cap X$, $r_2y_1 \in E$ and $r_2y_3 \in E$.

Suppose to the contrary that $|N_2 \cap X| \geq 2$, say $r_2, s_2 \in N_2 \cap X$ with $r_2 \neq s_2$. Then by Lemma 2.12, also $s_2y_1 \in E$ and $s_2y_3 \in E$. But then there is a C_4 (y_1, r_2, y_3, s_2) in G, which is a contradiction. Thus, $|N_2 \cap X| = 1$, and analogously, for $3D - P_7$ in $G[N_0 \cup N_1]$ with midpoint in $N_0 \cap X$, $|N_2 \cap Y| = 1$. □

Then by the above: $|N_2| = 1$, N_i is independent for $i = 3, 4, 5$. On the other hand by Lemma 2.3: $N_6 = \emptyset$, N_5 is independent, every edge in N_4 does not contact N_5, and for every edge r_3s_3 in N_3, there is no such P_4 (r_3, s_3, r_4, r_5) with $r_4 \in N_4$ and $r_5 \in N_4 \cup N_5$. It follows that there is only one D-vertex in N_3, and there is no such D-vertex in N_3 with midpoint P_5. Moreover, there is no such D-vertex in N_4 with midpoint P_5, and there is no such D-vertex in N_5 with midpoint P_5.

Then one can split the problem in $|N_3|$ instances, one instance for each $r_3 \in N_3$ with r_3 fixed in D, of the WED problem for $G[N_3 \cup N_4 \cup N_5]$ by referring to the case when no D-vertex is midpoint of a P_5 in G. That can be done in polynomial time. Since the above procedure is repeated for each induced P_7 of G, according to Subsection 2.3, the running time bound can be estimated as $O(n^7 \cdot n \cdot n^{11})$ time.

Thus, in this subsection, the WED problem can be solved in polynomial time.

Thus, Theorem 1.3 is shown.

3. The WED problem for (P_{10}, C_4)-free bipartite graphs

3.1. *Preliminary*

Let $G = (V(G), E(G)$ be a graph, with $|V(G)|= n$ and $|E|= m$.

NOTATION: For any subset $C \subseteq V(G)$ let us write:

$\mathcal{F}_C = \{(C', \mathcal{D}(C')) : C \subseteq C' \subseteq C \cup N(C)$; $\mathcal{D}(C')$ is the family of e.d.s. D' of $G[C']$ such that $C' \setminus C \subseteq D'$ and $\mathcal{D}(C')$ is non-empty$\}$.

In other words: for any possible e.d.s. D of G, some vertices of D should dominate vertices of C, in particular $D \cap C$ may be non-empty; that defines a local vertex set C' which is formed by vertices of C and by those vertices of $D \setminus C$ which dominate some vertex of C; then $\mathcal{F}_C$ collects all such possible local vertex sets C'.

Observation 3.1. *Let G be a graph. For any subset $C \subseteq V(G)$, such that $|C|\leq t$ for some $t \in \mathbb{N}$, the following statements hold:*

(i) *for any $(C', \mathcal{D}(C')) \in \mathcal{F}_C$, one has $|C' \setminus C|\leq t$;*

(ii) *family $\mathcal{F}_C$ contains $O(n^t)$ members;*

(iii) *for any $(C', \mathcal{D}(C')) \in \mathcal{F}_C$, family $\mathcal{D}(C')$ contains at most 2^t members;*

(iv) *if G has an e.d.s. say D, then there is a member $(C', \mathcal{D}(C')) \in \mathcal{F}_C$, such that $D \cap C' = D'$ for some $D' \in \mathcal{D}(C')$.*

Proof. Statement (i) follows since $|C|\leq t$ and by definition of e.d.s. Statement (ii) follows by statement (i) and by construction; in fact, for each $(C', \mathcal{D}(C')) \in \mathcal{F}_C$, $C' \setminus C$ is a subset of $V(G) \setminus C$ of at most t elements. Statement (iii) follows by statement (i) and by construction; in fact, for each $(C', \mathcal{D}(C')) \in \mathcal{F}_C$, $C' \setminus C$ if fixed in the possible e.d.s. D' of $G[C']$ while the other possible elements of D' are a subset of C. Statement (iv) follows by construction and by definition of e.d.s. □

Then let us report the following result.

Theorem 3.2. **[44, 45]** *The WED problem can be solved for trees in $O(n + m)$ time.*

NOTATION: Let us denote any rooted tree, with root say a, as $T(a)$; furthermore let us denote as $T_j(a)$ the set of nodes of $T(a)$ at distance j from node a (for $j = 1, 2, \ldots$).

3.2. *The solution method*

Let us show that the WED problem can be solved for (P_{10}, C_4)-free bipartite graphs in polynomial time.

Since the WED problem can be solved for trees in $O(n+m)$ time, by Theorem 3.2, and since a tree can recognized in $O(n+m)$ time, let us just show that the WED problem can be solved for (P_{10}, C_4)-free bipartite graphs which are not trees in polynomial time.

For any bipartite graph $G = (V(G), E(G))$, let us write $V(G) = V_1(G) \cup V_2(G)$, where $V_1(G)$ and $V_2(G)$ are the two sides of G [i.e. they are independent sets].

Let us introduce the following algorithm to solve the WED problem for (P_{10}, C_4)-free bipartite graphs *which are not trees.*

Algorithm WED-P10

Input: a connected (P_{10}, C_4)-free bipartite graph G which is not a tree.

Output: an e.d.s. of G of minimum weight or a proof that G has no e.d.s.

begin

:: set $D^* := \emptyset$ and set $w(D^*) := \infty$;

:: for $k = 6, 8, 10$, do

:: begin

:: :: if G contains some induced C_k, then:

:: :: :: $(k.1)$ take *any* such induced C_k's, say, of vertex-set C;

:: :: :: $(k.2)$ compute family $\mathcal{F}_C$ [according to Observation 3.1];

:: :: :: $(k.3)$ for each member $(C', \mathcal{D}(C'))$ of $\mathcal{F}_C$ do

:: :: :: :: $(k.3.1)$ for each e.d.s. $D' \in \mathcal{D}(C')$ do

:: :: :: :: begin

:: :: :: :: :: $(k.3.1.1)$ compute $D^{**} \in \text{ARG MIN}\{w(D) : D$ is an e.d.s. of G, with $D \cap C' = D'\}$;

:: :: :: :: :: $(k.3.1.2)$ if such an e.d.s D^{**} exists and if $w(D^{**}) < w(D^*)$, then set $D^* := D^{**}$;

:: :: :: :: end

:: :: :: $(k.4)$ if $D^* = \emptyset$, then return "G has no e.d.s.", and STOP;

:: :: :: $(k.5)$ if $D^* \neq \emptyset$, then return D^*, and STOP;

:: end

end.

Lemma 3.3. *Algorithm WED-P10 is correct. Furthermore Algorithm WED-P10 can be executed in max*$\{O(n^3 \cdot T_6), O(n^3 \cdot T_8), O(n^3 \cdot T_{10})\}$ *time, where* $O(T_k)$ *is the running time bound of step (k.3.1.1) for* $k = 6, 8, 10$*; thus, it can be executed in polynomial time, provided that step (6.3.1.1), step (8.3.1.1), and step (10.3.1.1) can be executed in polynomial time.*

Proof. As a preliminary let us remark that, since G is (P_{10}, C_4)-free bipartite, the only possible induced cycles in G are C_6's, C_8's, C_{10}'s.

Let us show that Algorithm WED-P10 is correct, by the following exhaustive cases.

Case 1. G has an e.d.s., say D^*, of minimum weight.

By Observation 3.1-(iv), for *any* subset $C \subseteq V(G)$ [with $|C| \leq t$ for some $t \in \mathbb{N}$], there is a member $(C', \mathcal{D}(C')) \in \mathcal{F}_C$, such that $D^* \cap C' = D'$ for some $D' \in \mathcal{D}(C')$.

If G contains some induced C_6's, then for *any* such induced C_6's, say, of vertex-set C, there is a member $(C', \mathcal{D}(C')) \in \mathcal{F}_C$, such that $D \cap C' = D'$ for some $D' \in \mathcal{D}(C')$: in this case, D^* [or another e.d.s. of the same weight] will be detected at the corresponding step (6.3.1.1), and then D^* [or another e.d.s. of the same weight] will be fixed at the corresponding step (6.3.1.2); finally D^* [or another e.d.s. of the same weight] will be returned at step (6.5), since at steps (6.3.1.1) the algorithm detects just certain e.d.s. of G (if one), and at steps (6.3.1.2) the algorithm possibly updates the current optimal solution with respect to these certain e.d.s. of G.

If G is C_6-free and contains some induced C_8's, then for *any* such induced C_8's, say, of vertex-set C, there is a member $(C', \mathcal{D}(C')) \in \mathcal{F}_C$, such that $D^* \cap C' = D'$ for some $D' \in \mathcal{D}(C')$: in this case, D^* [or another e.d.s. of the same weight] will be detected at the corresponding step (8.3.1.1), and then D^* [or another e.d.s. of the same weight] will be fixed at the corresponding step (8.3.1.2); finally D^* [or another e.d.s. of the same weight] will be returned at step (8.5), since at steps (8.3.1.1) the algorithm detects just certain e.d.s. of G (if one), and at steps (8.3.1.2) the algorithm possibly updates the current optimal solution with respect to these certain e.d.s. of G.

If G is (C_6, C_8)-free and contains some induced C_{10}'s, then for *any* such induced C_{10}'s, say, of vertex-set C, there is a member $(C', \mathcal{D}(C')) \in \mathcal{F}_C$, such that $D^* \cap C' = D'$ for some $D' \in \mathcal{D}(C')$: in this case, D^* [or another e.d.s. of the same weight] will be detected at the corresponding step (10.3.1.1), and then D^* [or another e.d.s. of the same weight] will be fixed at the corresponding step (10.3.1.2); finally D^* [or another e.d.s. of the same weight] will be returned at step (10.5), since at steps (10.3.1.1) the algorithm detects just certain e.d.s. of G (if one), and at steps (10.3.1.2) the algorithm possibly updates the current optimal solution with respect to these certain e.d.s. of G.

No other case is possible, since G is (P_{10}, C_4)-bipartite and is not a tree.

Therefore, in Case 1, the algorithm is correct.

Case 2. G has no e.d.s.

Algorithm WED-P10 starts by setting $D^* = \emptyset$ and by setting $w(D^*) = \infty$. Then, at step $(k.4)$, the algorithm returns "G has no e.d.s.", provided that $D^* = \emptyset$. As shown for Case 1, if G should have an e.d.s., then D^* would be updated [with $D^* \neq \emptyset$] at some step $(k.3.1.1)$, for some $k \in \{6, 8, 10\}$.

Therefore, in Case 2, the algorithm is correct.

Now let us focus on the computational complexity of Algorithm WED-P10.

First let us observe that: step (8.3.1.1) is executed only if G is C_6-free; step (10.3.1.1) is executed only if G is (C_6, C_8)-free.

For $k \in \{6, 8, 10\}$: step $(k.1)$ can be executed in $O(nm)$ time (in fact e.g., for $k = 6$, G has an induced C_6 if and only if for some vertex v of G there is a vertex $y \in N_3(v)$ with two neighbors in $N_2(v)$); step $(k.2)$ can be executed in $O(n^k)$ time [according to Observation 3.1-(i)]: in fact, one should consider just subsets U of $N(C)$ of cardinality at most k, and for each such subsets U check if the graph $G[C \cup U]$ has an e.d.s. D' with $U \subseteq D'$]; step $(k.3)$ consists of $O(n^k)$ calls of step $(k.3.1.1)$ [according to to Observation 3.1-(ii)-(iii)], so that it can be executed in $O(n^k \cdot T)$ time, where $O(T)$ is the running time bound of step $(k.3.1.1)$.

This completes the proof of Lemma 3.3. □

Lemma 3.4. *Step (6.3.1.1) can be executed in $O(n^3)$ time.*

Proof. By definition of Algorithm WED-P10, let us fix any induced C_6 of G of vertex-set C, say of vertices $c_1, \ldots, c_6$ and of edges $c_1c_2, \ldots, c_6c_1$, and let us fix any member $(C', \mathcal{D}(C'))$ of $\mathcal{F}_C$ and any e.d.s. $D' \in \mathcal{D}(C')$.

Then let us show that $D^{**} \in \text{ARG MIN}\{w(D) : D \text{ is an e.d.s. of } G, \text{ with } D \cap C' = D'\}$ can be computed in polynomial time. For convenience let us denote D^{**} as D.

Let us recall the notation $N_h(D') = \{v \in V(G) : dist(v, D') = h\}$, for $h \geq 1$, for the *distance levels* with respect to D'.

NOTATION: for brevity let us write $D' = N_0$ and $N_h(D') = N_h$ for $h \geq 1$.

Let us observe that, by construction, one has $C \subseteq N_0 \cup N_1$.

Let us observe that $D \cap (N_1 \cup N_2) = \emptyset$ by definition of e.d.s., so that each vertex of N_2 has to be dominated by some vertex of N_3, that is, of $D \cap N_3$: in particular each vertex of N_3 which is isolated in $G[N_3 \cup N_4 \cup \ldots]$ is forced to be in D.

Let $\mathcal{H}$ be the family of connected components of $G[N_3 \cup N_4 \cup \ldots]$.

For any vertex $x \in N_2$, let us say that x is: of *Type 1* if x is adjacent to some vertex of C; of *Type 2* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \cap C$ [that is of $D' \cap C$]; of *Type 3* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \setminus C$ [that is of $D' \setminus C$]. Let us observe that, by definition of e.d.s., x is of [exactly] one of the above types.

Then let us consider the following properties.

P1. Each vertex of N_2 is the endpoint of an induced P_6 of G formed together with five vertices of $N_0 \cup N_1$.

Proof. Let $x \in N_2$. If x is of Type 1, then x is adjacent to exactly one vertex of C [since G is bipartite C_4-free], and then P1 holds. If x is of Type 2 or of Type 3, then P1 holds by definition of Type 2 or of Type 3, and by construction [since G is bipartite C_4-free]. $\square$

P2. Each member of $\mathcal{H}$ is a tree, rooted at some vertex of N_3, of depth at most two.

Proof. Let H be a member of $\mathcal{H}$.

First let us show that H is a tree. By contradiction, assume that H contains an induced C_k for some $k = 6, 8, 10$ [recall that G is bipartite (P_{10}, C_4)-free]. Now let $x \in N_2$ be a vertex whose distance from the C_k is minimum over all vertices of N_2: let P be a shortest path from x to C [thus all vertices of P are in $N_3 \cup N_4 \cup \ldots$]. Notice that the vertex of P, which is adjacent to some vertex of the C_k, is the endpoint of an induced P_5 together with four vertices of the C_k [since G is bipartite C_4-free]; on the other hand, by P1, x is the endpoint of an induced P_6 of G formed together with five vertices of $N_0 \cup N_1$; thus, G contains an induced P_{10}, a contradiction.

Then let us show that H, rooted at some vertex of N_3, is of depth at most two. Let a be any vertex of $V(H) \cap N_3$ and let us fix a as the root of H, namely, $H = T(a)$ according to the notation introduced above. Then let $x \in N_2$ be a neighbor of a; notice that x is nonadjacent to any vertex of $T_j(a)$, for $j = 1, 2, 3$, since G is bipartite C_4-free. Thus $T_3(a) = \emptyset$, else otherwise by P1 an induced P_{10} arises, involving vertices x and a. $\square$

In what follows, by P2 [if necessary] let us write any member H of $\mathcal{H}$ as $H = T(a)$ for some $a \in N_3$ [that is H is a tree, rooted at vertex $a \in N_3$, of depth at most two].

P3. For each member $H = T(a)$ of $\mathcal{H}$, with $|T_1(a)| \geq 2$ and $|T_2(a)| > 0$, the following statements hold:

(i) vertex a would be forced to be *not* in D;

(ii) there are at most $|T_1(a)|$ possibilities for $D \cap V(H)$, namely, $D(H,t) = \{t\} \cup (T_2(a) \setminus N(t))$ for each $t \in T_1(a)$ provided that $D(H,t)$ is an e.d.s. of H; in particular, in this case, $D(H,t) \cap N_3 = \emptyset$;

(iii) if none of the above sets $D(H,t)$ is an e.d.s. of H, then G has not the sought e.d.s. D; else, one can choose one of them, of minimum weight.

Proof. Concerning (i). Since $|T_2(a)| > 0$ and since $T_3(a) = \emptyset$ [by P1], vertex a would be forced to be *not* in D, by definition of e.d.s.

Concerning (ii). By statement (i), exactly one vertex t of $T_1(a)$ should be in D [by definition of e.d.s.], and in this case vertex t should be contained in exactly one e.d.s. of H, namely $D(H,t) = \{t\} \cup (T_2(a) \setminus N(t))$ recalling that $T_2(a)$ is an independent set.

Then let us show, if $D(H,t)$ is an e.d.s. of H, then $D(H,t) \cap N_3 = \emptyset$. Notice that every vertex x of N_2 is adjacent to no vertex in $D(H,t) \cap T_1(a) = \{t\}$; in fact otherwise x is the endpoint of an induced P_5 [since G is bipartite C_4-free and by definition of e.d.s.] together with t, a, another $\bar{t} \in T_1(a)$ [which does exist as $|T_1(a)| \geq 2$], and the vertex in $D(H,t) \cap N(\bar{t}) \cap T_2(a)$ [which does exist as $D(H,t)$ is an e.d.s. of H]; thus, by P1, an induced P_{10} arises. Furthermore notice that every vertex x of N_2 is adjacent to no vertex say z of $D(H,t) \cap T_2(a)$: in fact otherwise x is the endpoint of an induced P_5 [since G is bipartite C_4-free and by definition of e.d.s.] together with z, the neighbor of z in $T_1(a)$, a, and t; thus, by P1, an induced P_{10} arises. Therefore $D(H,t) \cap N_3 = \emptyset$.

Concerning (iii). According to statement (ii), for each $t \in T_1(a)$, one can easily check whether $D(H,t)$ is an e.d.s. of H. If, for any $t \in T_1(a)$, $D(H,t)$ is not an e.d.s. of H, then G has not the sought e.d.s. D. Else, one can choose the set $D(H,t)$ [which is an e.d.s. of H] of minimum weight, for $t \in T_1(a)$. □

P4. For each member $H = T(a)$ of $\mathcal{H}$, such that:

(i) $|T_1(a)| = 1$ and $|T_2(a)| > 0$, or

(ii) $|T_1(a)| \neq 1$ and $|T_2(a)| = 0$,

vertex a would be forced to be in D while the other vertices of H to be *not* in D.

Proof. Concerning case (i): since $|T_2(a)| > 0$ and since $T_3(a) = \emptyset$ [by P1], vertex a would be forced to be *not* in D and should be dominated by the vertex in $T_1(a)$ by definition of e.d.s.

Concerning case (ii): the proof directly follows by construction and by definition of e.d.s.. □

Let us observe that the only members $H = T(a)$ of $\mathcal{H}$, which enjoy [the assumptions of] neither P3 nor P4, are those such that $|T_1(a)| = 1$ and $|T_2(a)| = 0$, i.e., are *edges* of G.

Then let $\{\mathcal{H}_1, \mathcal{H}_2\}$ be the partition of $\mathcal{H}$ such that: $\mathcal{H}_1$ is formed by those members of $\mathcal{H}$ which enjoy [the assumptions of] either of P3 or of P4, while $\mathcal{H}_2$ is formed by those members of $\mathcal{H}$ with $|T_1(a)|= 1$ and $|T_2(a)|= 0$, i.e., by those members of $\mathcal{H}$ which are *edges* of G.

ASSUMPTION 1: Let us assume that, for each member H of $\mathcal{H}_1$ which enjoys P3, one can determine $D \cap H$ according to P3 [else G has not the sought e.d.s. D by P3-(iii)].

Then by the above properties one can simplify the problem as follows.

Let us focus on $G' = G[N_2 \cup N_3 \cup N_4 \cup \ldots]$.

Let us assume that all vertices of G' are uncolored: the goal is to check if vertices of G' can be colored/partitioned into *white* vertices, i.e. those in D, and *black* vertices, i.e. those in $V(G) \setminus D$, with the constraint that all vertices of N_2 have to be black.

In particular the following forcing rules hold:

(R1) if vertex w is colored white, then all vertices of $N(w)$ should be colored black;

(R2) if vertex b is colored black and if b has a neighbor w colored white, then all vertices of $N(b) \setminus \{w\}$ should be colored black;

(R3) if h_1, h_2 are the two vertices of a member of $\mathcal{H}_2$ and if h_1 is colored black (respectively, white), then h_2 should be colored white (respectively, black).

Then one can execute the following *Pre-procedure* whose running time bound can be estimated as $O(n^2)$ time: **(1)** assign color black to all vertices of N_2; **(2)** for each member H of $\mathcal{H}_1$, assign to each vertex of H the forced color according to P3 and to P4 [by Assumption 1]; **(3)** apply iteratively rules (R1), (R2), (R3).

Notice that: for any $H = T(a)$ in $\mathcal{H}_1$ enjoying P3, vertex a is black [by P3-(i)], and no vertex of N_2 is adjacent to any white vertex of $H = T(a)$ [by P3-(ii)]; for any $H = T(a)$ in $\mathcal{H}_1$ enjoying P4, vertex a is white, while the other vertices of $T(a)$ are black [by P4].

ASSUMPTION 2: Let us assume that the above pre-procedure does not lead to a contradiction [else G has not the sought e.d.s. D].

At this point one can just focus on the following set and family:

$N_2' = \{x \in N_2 : x$ is adjacent to no white vertex [by the pre-procedure]$\}$,

$\mathcal{H}_2' = \{H \in \mathcal{H}_2 :$ the two vertices of H are still uncolored [by the pre-procedure]$\}$.

Let $V(\mathcal{H}_2')$ denote the vertex set of members of $\mathcal{H}_2'$.

P5. Graph G has the sought e.d.s. D if and only if $G[N_2' \cup V(\mathcal{H}_2')]$ has an e.d.s. say $D_{\text{peripheral}}$ with $N_2' \cap D_{\text{peripheral}} = \emptyset$.

Proof. In fact so far we showed that, if G has the sought e.d.s. D, then some forcing conditions involve N_2 (namely, $D \cap N_2 = \emptyset$, and each vertex of N_2 should be dominated by some vertex of N_3) and all members of $\mathcal{H} \setminus \mathcal{H}_2'$ (according to the pre-procedure). Notice that, by construction and by definition of the pre-procedure, no vertex of $N_2 \setminus N_2'$ contacts any member of $\mathcal{H}_2'$. Summarizing, N_2' is a cut set of G, and each vertex of N_2' should be dominated by some vertex of $V(\mathcal{H}_2')$. Then, according to Assumption 1 and to Assumption 2, P5 is shown with $D = D' \cup W \cup D_{\text{peripheral}}$ where W is the set of vertices colored by white [by the pre-procedure]. $\square$

The following property P6 is just a technical parenthesis which will be useful later.

P6 Let Q be a connected component of $G[N_2' \cup V(\mathcal{H}_2')]$, such that Q contains exactly one vertex say x of N_2' [in particular, x is a cutset by construction, and is colored black by the pre-procedure]. Then one can easily determine an "optimal" coloring of Q (i.e. of

$Q \setminus \{x\}$).

Proof. By assumption Q is formed by vertex x and by say ℓ members of $\mathcal{H}_2'$ [namely ℓ edges] contacted by x. Thus, Q admits ℓ e.d.s. *not containing* x, in particular one can easily compute them and choose anyone of minimum weight [which will be part of $D_{\text{peripheral}}$]. □

P7. One can compute $D^*_{\text{peripheral}} \in \text{ARG MIN}\{w(D_{\text{peripheral}}) : D_{\text{peripheral}}$ is an e.d.s. of $G[N_2' \cup V(\mathcal{H}_2')]$ with $N_2' \cap D_{\text{peripheral}} = \emptyset\}$ in $O(n^3)$ time.

Proof. First let us recall that (by construction and by definition of e.d.s.) each vertex of N_2' has to be covered by some vertex of $V(\mathcal{H}_2')$. Without loss of generality, to our aim, let us assume that $G[N_2' \cup V(\mathcal{H}_2')]$ is connected.

Then let us recall the aforementioned induced C_6 in $G[N_0 \cup N_1]$, of vertex-set C, say of vertices $c_1, \dots, c_6$, and say of edges $c_1c_2, \dots, c_6c_1$.

Claim 1. One can assume (without loss of generality) that:

(i) each vertex of N_2' contacts at least two members of $\mathcal{H}_2'$;

(ii) each vertex of N_2' is of Type 1 [i.e. is adjacent to some vertex of C].

Proof. Proof of (i): In fact let $x \in N_2'$: if x contacts no member of $\mathcal{H}_2'$, then the sought e.d.s. D^* does not exist; if x contacts exactly one member of $\mathcal{H}_2'$, then the neighbor of x (in such a member) would be forced to be in D^*, so that one could iteratively apply rules (R1), (R2), (R3), and then reduce (again) the problem to a subset of N_2' and to a subfamily of $\mathcal{H}_2'$.

Proof of (ii): Let $x \in N_2'$. Let us recall that x can be of Type 1, or of Type 2, or of Type 3.

First let us observe that x can not be of Type 3. By contradiction assume that x is of Type 3. Then x is adjacent to a vertex $y \in N_1 \setminus C$, which is adjacent to a vertex $z \in N_0 \setminus C$, which is adjacent to a vertex of C; furthermore, by statement (i), x contacts at least one [actually more than one] member say H of $\mathcal{H}_2'$; then, since G is bipartite C_4-free, the two vertices of H, x, y, z, and five vertices of C induce a P_{10}, a contradiction.

Then let us assume that x is of Type 2. Then let $u \in N_1$ be adjacent to x and be adjacent to $c_1 \in D' \cap C$ (without loss of generality by symmetry). Then let us consider the following two exhaustive occurrences.

:: Assume that c_3 or c_5 has a neighbor in $D' \setminus C$. Without loss of generality by symmetry, assume that c_5 has a neighbor in $z \in D' \setminus C$, i.e. $z \in N_0$. Then [by construction and since G is bipartite C_4-free] vertices $z, c_4, c_3, c_2, c_1, u, x$ induce a P_8. On the other hand, by statement (i), vertex x contacts at least one [actually more than one] member say H of $\mathcal{H}_2'$. Then an induced P_{10} would arise together the two vertices of H, i.e., this occurrence is not possible.

:: Assume that c_3 and c_5 have no neighbor in $D' \setminus C$. Then, by definition of e.d.s., $c_4 \in D' \cap C$. By statement (i), vertex x contacts at least two members of $\mathcal{H}_2'$, say of vertices h_1, h_2 and of vertices k_1, k_2, respectively, with x adjacent to h_1 and to k_1 (without loss of generality by symmetry).

Let us show that h_1 and h_2 are adjacent to no vertex of $N_2' \setminus \{x\}$. In fact:

if h_1 is adjacent to a vertex $x_1 \in N_2'$, then: either x_1 is of Type 1, and thus an induced

P_{10} arises involving vertices k_2, k_1, x, h_1, x_1, and five vertices of C; or x_1 is of Type 2 with a neighbor $u_1 \in N_2'$ adjacent to c_1 or to c_4 [with $u_1 \neq u$ since G is C_4-free], and thus an induced P_{10} arises involving vertices $k_2, k_1, x, h_1, x_1, u_1$, and four vertices of C;

if h_2 is adjacent to a vertex $x_2 \in N_2'$, then x_2 is adjacent to c_3 or to c_5 [else $x_2, h_2, h_1, x, u, c_1, c_2, c_3, c_4, c_5$ induce a P_{10}], say to c_5 without loss of generality by symmetry, and thus $k_1, x, h_1, h_2, x_2, c_5, c_4, c_3, c_2, c_1$ induce a P_{10}.

Then h_1 and h_2 are adjacent to no vertex of $N_2' \setminus \{x\}$. It follows [by a similar argument] that every member H of $\mathcal{H}_2'$, which is contacted by x, is contacted by no vertex of $N_2' \setminus \{x\}$. Then one can easily determine an "optimal" coloring of each such members H according to P6.

This completes the proof of statement (ii). □

Claim 2. For any member H of $\mathcal{H}_2'$, of vertices say h_1, h_2, one has that: h_1 is adjacent to at most one vertex of N_2', and h_2 is adjacent to at most one vertex of N_2'.

Proof. By symmetry let us just prove that h_1 is adjacent to exactly one vertex of N_2': in particular, let us assume that h_1 has a neighbor $x \in N_2'$ (else the claim is proved), and by Claim 1 let us assume that x is adjacent to c_1 (without loss of generality by symmetry).

Then let us show that no vertex $x' \in N_2'$, different to x, is adjacent to h_1: in fact otherwise, since G is bipartite C_4-free, x' is nonadjacent to x and is adjacent to c_3 (without loss of generality by symmetry) since G is bipartite C_4-free; then by Claim 1 and since G is bipartite C_4-free, there exist two distinct members of $\mathcal{H}_2'$, say H' and H'', such that x' contacts H' while x contacts H''; then $V(H'), x', c_3, c_4, c_5, c_6, c_1, x, V(H'')$ induce a P_{11}, that is not possible since G is P_{10}-free. □

NOTATION. Let us say that a pair $\{x, y\} \subseteq N_2'$ is a *complementary pair of* N_2' if there is a member H of $\mathcal{H}_2'$, of vertices say h_1, h_2, such that x is adjacent to h_1 while y is adjacent to h_2; in particular, in this case, let us say that $\{x, y\}$ *covers* H, and that H is *covered by* $\{x, y\}$. Notice that: x is nonadjacent to h_2 (by Claim 2), y is nonadjacent to h_1 (by Claim 2), x is nonadjacent to y (since G is C_4-free). Furthermore in what follows, as an *agreement*, let us assume that any complementary pair $\{x, y\}$ of N_2' is an ordered pair with $x \in V_1(G)$ and $y \in V_2(G)$.

Claim 3. Let $\{x, y\}$ be a complementary pair of N_2'. Then $\{x, y\}$ covers at least two members $\mathcal{H}_2'$.

Proof. By assumption, $\{x, y\}$ is a complementary pair of N_2', that is, there is a member H of $\mathcal{H}_2'$, of vertices say h_1, h_2, such that x is adjacent to h_1 while y is adjacent to h_2.

Without loss of generality by symmetry, let us assume that x is adjacent to c_1, and that y is adjacent either to c_2 or to c_4.

First let us show that there are no distinct members of $\mathcal{H}_2'$, say H_x and H_y, such that: x contacts H_x and does not contact H_y, while y contacts H_y and does not contact H_x. In fact, by contradiction, assume that such distinct members exist. Then: if y is adjacent to c_2, then $V(H_y), y, h_2, h_1, x, c_1, c_6, c_5, c_4$ induce a P_{10}, that is not possible; if y is adjacent to c_4, then $V(H_y), y, c_4, c_5, c_6, c_1, x, V(H_x)$ induce a P_{10}, that is not possible. Then the assertion is shown.

Then, by Claim 1, there is a member of $\mathcal{H}_2'$, say K of vertices k_1, k_2, such that both x

and y contact K, i.e., x is adjacent to k_1 while y is adjacent to k_2 (by Claim 2).

This completes the proof of Claim 3. □

Claim 4. Let $\{x, y\}$ be a complementary pair of N'_2. Then x and y are adjacent respectively to opposite vertices of $G[C]$, i.e., x is adjacent to c_i while y is adjacent to c_{i+3} for some $i \in \{1, \ldots, 3\}$.

Proof. By Claim 3, $\{x, y\}$ covers at least two two members say H, K of $\mathcal{H}'_2$, of vertices say respectively h_1, h_2, and k_1, k_2, such that x is adjacent to h_1 and k_1 while y is adjacent to h_2 and k_2.

Let us assume that x is adjacent to c_1 without loss of generality by symmetry. Then y is nonadjacent to c_2 [else $h_2, y, k_2, k_1, x, c_1, c_6, c_5, c_4, c_3$ induce a P_{10}], y is nonadjacent to c_6 [similarly by symmetry], so that y is adjacent to c_4. □

Claim 5. Let $\{x, y\}$ be a complementary pair of N'_2 and let H be a member of $\mathcal{H}'_2$, of vertices say h_1, h_2, such that x is adjacent to h_1 while y is adjacent to h_2. Then: if one should fix the color of h_1 [or of h_2] as white, then for any member H^* of $\mathcal{H}'_2$ which is contacted by x or by y, the color of each vertex of H^* would be determined by the forcing rules.

Proof. By Claim 3, $\{x, y\}$ covers at least two two members say H, K of $\mathcal{H}'_2$, of vertices say respectively h_1, h_2, and k_1, k_2, such that x is adjacent to h_1 and k_1 while y is adjacent to h_2 and k_2.

Now let us assume that one fixes the color of h_1 as white [so that, by rule (R3), the color of h_2 is forced to be black]. Then, by rule (R2), the color of k_1 is forced to be black; so that, by rule (R3), the color of k_2 is forced to be white; let us recall that y is adjacent to k_2; at this point, by rule (R2), each vertex of any member H^* of $\mathcal{H}'_2$ [different to H and to K] which is contacted by x or y, is forced to be black, so that by rule (R3) the other vertex of H^* is forced to be white. This shows Claim 3. □

For any two complementary pairs $\{x, y\}$ and $\{x', y'\}$ of N'_2, let us say that $\{x, y\}$ and $\{x', y'\}$ are *disjoint* if $x \neq x'$ and $y \neq y'$.

Claim 6. For any two disjoint complementary pairs $\{x, y\}$ and $\{x', y'\}$ of N'_2, one has that: x is adjacent to y', while y is adjacent to x'.

Proof. By Claim 3, $\{x, y\}$ covers at least two two members say H, K of $\mathcal{H}'_2$, of vertices say respectively h_1, h_2, and k_1, k_2, such that x is adjacent to h_1 and k_1 while y is adjacent to h_2 and k_2. By Claim 3, $\{x', y'\}$ covers at least two two members say H', K' of $\mathcal{H}'_2$, of vertices say respectively h'_1, h'_2, and k'_1, k'_2, such that x' is adjacent to h'_1 and k'_1 while y' is adjacent to h'_2 and k'_2.

Let us recall that, by the above agreement, one has that $x, x' \in V_1(G)$ and $y, y' \in V_2(G)$; thus, x is nonadjacent to x', and y is nonadjacent to y'.

Note that, by Claim 2 and since $\{x, y\}$ and $\{x', y'\}$ are disjoint, one has that: x and y do not contact $V(H')$ and $V(K')$; x' and y' do not contact $V(H)$ and $V(K)$.

Let us assume, without loss of generality by symmetry, that x is adjacent to c_1; thus, by Claim 4, y is adjacent to c_4.

Now, by contradiction, assume that either x is nonadjacent to y', or y is nonadjacent to x'.

First let us assume that x is nonadjacent to y', and y is nonadjacent to x': if x' is adjacent to c_1, then $h_2, y, k_2, k_1, x, c_1, x', h_1', h_2', y'$ induce a P_{10}; if x' is adjacent to c_3, then $h_2, y, k_2, k_1, x, c_1, c_2, c_3, x', h_1'$, induce a P_{10}; if x' is adjacent to c_5, then similarly by symmetry there are vertices which induce a P_{10}.

Then let us assume that x is adjacent to y', and y is nonadjacent to x': then $h_2, y, k_2, k_1, x, y', k_2', k_1', x', h_1'$ induce a P_{10}.

Finally let us assume that x is nonadjacent to y', and y is adjacent to x': such an occurrence can be treated similarly to the previous one by symmetry.

This completes the proof of Claim 6. □

Claim 7. There are at most three mutually disjoint complementary pairs of N_2'.

Proof. By contradiction assume that there are four mutually disjoint complementary pairs of N_2', say $\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_4, y_4\}$. By the above agreement, one has $x_1, x_2, x_3, x_4 \in V_1(G)$ and $y_1, y_2, y_3, y_4 \in V_2(G)$. Then by Claim 4 and by Claim 6 one has that, for $i = 1, 2, 3, 4$, x_i is nonadjacent to y_i and is adjacent to y_j for every $j \in \{1, 2, 3, 4\} \setminus \{i\}$. Then x_1, y_2, x_3, y_4 induce a C_4, a contradiction. □

Now let us conclude the proof of P7.

For that let us describe the following possible solution method.

Step INITIALIZATION: set $D^*_{\text{peripheral}} := \emptyset$ and set $w(D^*_{\text{peripheral}}) := \infty$.

Step A. Compute a maximal family, say $\mathcal{P}$, of mutually disjoint complementary pairs of N_2' [note that, by Claim 7, family $\mathcal{P}$ has at most three members]:

:: A1. if $\mathcal{P} = \{\emptyset\}$, then proceed as follows: note that, by construction and since $\mathcal{P} = \{\emptyset\}$, each member of $\mathcal{H}_2'$ is contacted by exactly one vertex of N_2'; then, by Claim 2, each connected component say Q of $G[N_2' \cup V(\mathcal{H}_2')]$ contains exactly one vertex of N_2'; then one can easily determine an "optimal" coloring of Q according to P6; then return the sought e.d.s. $D^*_{\text{peripheral}}$, and STOP;

:: A2. if $\mathcal{P} \neq \{\emptyset\}$, then let us write $\mathcal{P} = \{\{x, y\}, \{x', y'\}, \{x'', y''\}\}$, without loss of generality for this description; then go to Step B.

Step B. Select any three members of $\mathcal{H}_2'$, namely three edges $(h_1, h_2), (h_1', h_2'), (h_1'', h_2'')$, which are respectively covered by $\{x, y\}, \{x', y'\}, \{x'', y''\}$. Note that, by rule (R3), each such edges can be colored in *two ways* [e.g., concerning edge (h_1, h_2), one can fix either "h_1 black and h_2 white" or " h_1 white and h_2 black"]: thus there are *eight ways* to color edges $(h_1, h_2), (h_1', h_2'), (h_1'', h_2'')$.

Step C. For each such eight ways, namely for $\alpha \in \{1, \ldots, 8\}$, fix the corresponding coloring of edges $(h_1, h_2), (h_1', h_2'), (h_1'', h_2'')$, and determine the color of the remaining vertices [by the forcing conditions] as follows:

— color vertices of those members of $\mathcal{H}_2'$ which are contacted by at least one vertex in $\{x, y, x', y', x'', y''\}$ according to Claim 5;

— color vertices of the remaining members of $\mathcal{H}_2'$ – let $\mathcal{H}_2''$ denote the subfamily of $\mathcal{H}_2'$ formed by these remaining members – as follows: note that, by Claim 2, by Claim 7, and by step C1, each member of $\mathcal{H}_2''$ is contacted by exactly one vertex of N_2';

thus, each connected component say Q of $G[N_2'' \cup V(\mathcal{H}_2'')]$ contains exactly one vertex say x_Q of N'_2, so that one can easily determine an "optimal" coloring of Q according to P6;

— if the above procedure has not led to any contradiction, then: (i) denote the set of white vertices as $D^{\alpha}_{\text{peripheral}}$; (ii) if $w(D^{\alpha}_{\text{peripheral}}) < w(D^{*}_{\text{peripheral}})$, then set $D^{*}_{\text{peripheral}} := D^{\alpha}_{\text{peripheral}}$.

Step D.

:: If $D^{*}_{\text{peripheral}} = \emptyset$, then return "$G$ has no e.d.s.", and STOP.

:: If $D^{*}_{\text{peripheral}} \neq \emptyset$, then return $D^{*}_{\text{peripheral}}$, and STOP.

The above solution method is correct by Claim 1, Claim 2, Claim 5, and Claim 7. Furthermore the running time bound can be estimated as $O(n^3)$ time.

This completes the proof of P7. □

This completes the proof of Lemma 3.4. □

Lemma 3.5. *Step (8.3.1.1) can be executed in polynomial time.*

Proof. The proof is similar (but shorter) to that of Lemma 3.4.

As a preliminary let us remark that Step (8.3.1.1) is executed only if G is C_6-free: thus let us assume that G is C_6-free.

By definition of Algorithm WED-P10, let us fix any induced C_8 of G of vertex-set C, say of vertices $c_1, \ldots, c_8$ and of edges $c_1c_2, \ldots, c_8c_1$, and let us fix any member $(C', \mathcal{D}(C'))$ of $\mathcal{F}_C$ and any e.d.s. $D' \in \mathcal{D}(C')$.

Then let us show that $D^{**} \in \text{ARG MIN}\{w(D) : D \text{ is an e.d.s. of } G, \text{ with } D \cap C' = D'\}$ can be computed in polynomial time. For convenience let us denote D^{**} as D.

Let us recall the notation $N_h(D') = \{v \in V(G) : dist(v, D') = h\}$, for $h \geq 1$, for the *distance levels* with respect to D'.

NOTATION: for brevity let us write $D' = N_0$ and $N_h(D') = N_h$ for $h \geq 1$.

Let us observe that by construction one has $C \subseteq N_0 \cup N_1$.

Let us observe that $D \cap (N_1 \cup N_2) = \emptyset$ by definition of e.d.s., so that each vertex of N_2 has to be covered by some vertex of N_3, that is, of $D \cap N_3$: in particular each vertex of N_3 which is isolated in $G[N_3 \cup N_4 \cup \ldots]$ is forced to be in D.

Let $\mathcal{H}$ be the family of connected components of $G[N_3 \cup N_4 \cup \ldots]$.

For any vertex $x \in N_2$, let us say that x is: of *Type 1* if x is adjacent to some vertex of C; of *Type 2* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \cap C$ [that is of $D' \cap C$]; of *Type 3* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \setminus C$ [that is of $D' \setminus C$]. Let us observe that, by definition of e.d.s., x is of [exactly] one of the above types.

Then let us consider the following properties.

P1. Each vertex of N_2 is the endpoint of an induced P_8 of G formed together with seven vertices of $N_0 \cup N_1$.

Proof. Let $x \in N_2$. If x is of Type 1, then x is adjacent to exactly one vertex of C [since G is bipartite (C_4, C_6)-free], and then P1 holds true. If x is of Type 2 or of Type 3, then P1 holds true by definition of Type 2 or of Type 3, and by construction [since G

is bipartite (C_4, C_6)-free]. □

P2. Each member of $\mathcal{H}$ is a singleton.

Proof. Let H be a member of $\mathcal{H}$. Let us show that H has exactly one vertex. By contradiction, recalling that H is connected, assume that H contains at least two vertices. Now, let $x \in N_2$ be a vertex adjacent to a vertex h of H, and let h' be a neighbor of h in H. By P1, x is the endpoint of an induced P_8 of G formed together with seven vertices of $N_0 \cup N_1$; thus, G contains an induced P_{10}, a contradiction. □

At this point, by construction and by the above observations, one has that the sought e.d.s. D exists if and only if $D = D' \cup V(\mathcal{H})$ where $V(\mathcal{H})$ is the vertex set of all members of $\mathcal{H}$.

The running time bound for this final check can be estimated as $O(n^2)$.

This completes the proof of Lemma 3.5. □

Lemma 3.6. *Step (10.3.1.1) can be executed in constant time.*

Proof. The proof is similar (but shorter) to that of Lemma 3.4.

As a preliminary let us remark that Step (10.3.1.1) is executed only if G is (C_6, C_8)-free: thus let us assume that G is (C_6, C_8)-free.

By definition of Algorithm WED-P10, let us fix any induced C_{10} of G of vertex-set C, say of vertices $c_1, \ldots, c_8$ and of edges $c_1c_2, \ldots, c_8c_1$, and let us fix any member $(C', \mathcal{D}(C'))$ of $\mathcal{F}_C$ and any e.d.s. $D' \in \mathcal{D}(C')$.

Then let us show that $D^{**} \in \text{ARG MIN}\{w(D) : D \text{ is an e.d.s. of } G, \text{ with } D \cap C' = D'\}$ can be computed in constant time. For convenience let us denote D^{**} as D.

Let us recall the notation $N_h(D') = \{v \in V(G) : dist(v, D') = h\}$, for $h \geq 1$, for the *distance levels* with respect to D'.

NOTATION: for brevity let us write $D' = N_0$ and $N_h(D') = N_h$ for $h \geq 1$.

Let us observe that by construction one has $C \subseteq N_0 \cup N_1$.

Let us observe that $D \cap (N_1 \cup N_2) = \emptyset$ by definition of e.d.s., so that each vertex of N_2 has to be covered by some vertex of N_3, that is, of $D \cap N_3$: in particular each vertex of N_3 which is isolated in $G[N_3 \cup N_4 \cup \ldots]$ is forced to be in D.

Let $\mathcal{H}$ be the family of connected components of $G[N_3 \cup N_4 \cup \ldots]$.

For any vertex $x \in N_2$, let us say that x is: of *Type 1* if x is adjacent to some vertex of C; of *Type 2* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \cap C$ [that is of $D' \cap C$]; of *Type 3* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \setminus C$ [that is of $D' \setminus C$]. Let us observe that, by definition of e.d.s., x is of [exactly] one of the above types.

Then let us consider the following properties.

P1. $N_2 = \emptyset$.

Proof. Let $x \in N_2$. If x is of Type 1, then x is adjacent to exactly one vertex of C [since G is bipartite (C_4, C_6, C_8)-free], and then an induced P_{10} arises. If x is of Type 2 or of Type 3, then then an induced P_{10} arises by definition of Type 2 or of Type 3, and by construction [since G is bipartite (C_4, C_6, C_8)-free]. □

At this point, by construction and by the above observations, one has that the sought e.d.s. D always exists and $D = D'$.

This completes the proof of Lemma 3.6. □

Thus, Theorem 1.4 is shown.

The following corollary is useful for the running time bound introduced in Subsection 2.3.

Corollary 3.7. *For (P_8, C_4)-free bipartite graphs, then WED problem can be solved in $O(n^3)$ time.*

Proof. The running time bound can be derived by the proof of Theorem 1.4, since for any induced C_6 or C_8 of vertex set say C, one has that: for any fixed member $(C', \mathcal{D}(C'))$ of $\mathcal{F}_C$ and for any $D' \in \mathcal{D}(C')$, $N_3(D')$ is formed by singletons while $N_4(D') = \emptyset$. □

4. The WED problem for subclasses of (P_t, C_4)-free bipartite graphs

This section can be view as first attempt of extension of the results in Section 3, in particular, let us adopt the preliminary introduced in Section 3.

For any bipartite graph $G = (V(G), E(G))$, let us write $V(G) = V_1(G) \cup V_2(G)$, where $V_1(G)$ and $V_2(G)$ are the two sides of G [i.e. they are independent sets].

Let us introduce the following algorithm – that generalizes Algorithm WED-P10 – to solve WED for (P_t, C_4)-free bipartite graphs, for $t \geq 11$, *which are not trees.*

Algorithm WED

Input: a connected (P_t, C_4)-free bipartite graph G, for $t \geq 11$, which is not a tree.

Output: an e.d.s. of G of minimum weight or a proof that G has no e.d.s.

begin

:: set $D^* := \emptyset$ and set $w(D^*) := \infty$;

:: for $k = 6, 8, 10, \ldots, 2\lfloor t/2 \rfloor$, do

:: begin

:: :: if G contains some induced C_k, then:

:: :: :: $(k.1)$ take *any* such induced C_k's, say, of vertex-set C;

:: :: :: $(k.2)$ compute family $\mathcal{F}_C$ [according to Observation 3.1];

:: :: :: $(k.3)$ for each member $(C', \mathcal{D}(C'))$ of $\mathcal{F}_C$ do

:: :: :: :: $(k.3.1)$ for each e.d.s. $D' \in \mathcal{D}(C')$ do

:: :: :: :: begin

:: :: :: :: :: $(k.3.1.1)$ compute $D^{**} \in \text{ARG MIN}\{w(D) : D$ is an e.d.s. of G, with $D \cap C' = D'\}$;

:: :: :: :: :: $(k.3.1.2)$ if such an e.d.s D^{**} exists and if $w(D^{**}) < w(D^*)$, then set $D^* := D^{**}$;

:: :: :: :: end

:: :: :: $(k.4)$ if $D^* = \emptyset$, then return "G has no e.d.s.", and STOP;

:: :: :: (k.5) if $D^* \neq \emptyset$, then return D^*, and STOP;
:: end
end.

Lemma 4.1. *Algorithm WED is correct. Furthermore Algorithm WED can be executed in max*$\{O(n^3 \cdot T_k) : k = 6, 8, 10, \ldots, 2\lfloor t/2 \rfloor\}$ *time, where* $O(T_k)$ *is the running time bound of step (k.3.1.1) for* $k = 6, 8, 10, \ldots, 2\lfloor t/2 \rfloor$*; thus, it can be executed in polynomial time, provided that each step (k.3.1.1), for* $k = 6, 8, 10, \ldots, 2\lfloor t/2 \rfloor$*, can be executed in polynomial time.*

Proof. As a preliminary let us remark that, since G is (P_t, C_4)-free bipartite, the only possible induced cycle in G are C_6's, C_8's, …, C_t's.

Then the proof is very similar to that of Lemma 3.3 and is not reported for brevity. □

Then for completeness – with reference to the polynomial results claimed in Subsection 1.3 – let us show, by the above approach, that WED can be solved for a subclass of bipartite (P_t, C_4)-free graphs (for $t \geq 11$) in polynomial time.

4.1. The WED problem for bipartite (P_{13}, C_4, C_6, C_8)-free graphs

In this subsection let us show that WED can be solved for bipartite (P_{13}, C_4, C_6, C_8)-free graphs in polynomial time.

Then let G be a bipartite (P_{13}, C_4, C_6, C_8)-free graph.

Then let us apply Algorithm WED with input graph G.

Then the following lemma [which is just an adaptation of the lemma above] holds.

Lemma 4.2. *Algorithm WED is correct. Furthermore Algorithm WED-P10 can be executed in max*$\{O(n^3 \cdot T_k) : k = 10, 12\}$ *time, where* $O(T_k)$ *is the running time bound of step (k.3.1.1) for* $k = 10, 12$*. Thus Algorithm WED can be executed in polynomial time provided that each step (k.3.1.1), for* $k = 10, 12$*, can be executed in polynomial time.*

Proof. The lemma directly follows by Lemma 4.2 since G is (P_{13}, C_4, C_6, C_8)-free. □

The following lemma is just an adaptation of Lemma 3.4 with a slight modification.

Lemma 4.3. *Step (10.3.1.1) can be executed in* $O(n^2)$ *time.*

Proof. By definition of Algorithm WED, let us fix any induced C_{10} of G of vertex-set C, say of vertices $c_1, \ldots, c_{10}$ and of edges $c_1c_2, \ldots, c_{10}c_1$, and let us fix any member $(C', \mathcal{D}(C'))$ of $\mathcal{F}_C$ and any e.d.s. $D' \in \mathcal{D}(C')$.

Then let us show that $D^{**} \in$ ARG MIN$\{w(D) : D$ is an e.d.s. of G, with $D \cap C' = D'\}$ can be computed in polynomial time. For convenience let us denote D^{**} as D.

Let us recall the notation $N_h(D') = \{v \in V(G) : dist(v, D') = h\}$, for $h \geq 1$, for the *distance levels* with respect to D'.

NOTATION: for brevity let us write $D' = N_0$ and $N_h(D') = N_h$ for $h \geq 1$.

Let us observe that by construction one has $C \subseteq N_0 \cup N_1$.

Let us observe that $D \cap (N_1 \cup N_2) = \emptyset$ by definition of e.d.s., so that each vertex of N_2 has to be covered by some vertex of N_3, that is, of $D \cap N_3$: in particular each vertex of N_3 which is isolated in $G[N_3 \cup N_4 \cup \ldots]$ is forced to be in D.

Let $\mathcal{H}$ be the family of connected components of $G[N_3 \cup N_4 \cup \ldots]$.

For any vertex $x \in N_2$, let us say that x is: of *Type 1* if x is adjacent to some vertex of C; of *Type 2* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \cap C$ [that is of $D' \cap C$]; of *Type 3* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \setminus C$ [that is of $D' \setminus C$]. Let us observe that, by definition of e.d.s., x is of [exactly] one of the above types.

Let us remark that one can assume that x is either of Type 1 or of Type 2: in fact, if x is of Type 3, then x is adjacent to no vertex of N_3 [else an induced P_{13} arises] so that the sought e.d.s. D would not exist.

Then let us consider the following properties.

P1. Each vertex of N_2 is the endpoint of an induced P_{10} of G formed together with nine vertices of $N_0 \cup N_1$.

Proof. Let $x \in N_2$. If x is of Type 1, then x is adjacent to exactly one vertex of C [since G is bipartite (C_4, C_6, C_8)-free], and then P1 holds. If x is of Type 2, then P1 holds by definition of Type 2, and by construction [since G is bipartite (C_4, C_6, C_8)-free]. $\square$

P2. Each member of $\mathcal{H}$ is a tree, rooted at some vertex of N_3, of depth at most one.

Proof. Let H be a member of $\mathcal{H}$.

First let us show that H is a tree. By contradiction, assume that H contains an induced C_k for some $k = 10, 12$ [recall that G is bipartite (C_4, C_6, C_8)-free]. Now let $x \in N_2$ be a vertex whose distance from the C_k is minimum over all vertices of N_2: let P be a shortest path from x to C [thus all vertices of P are in $N_3 \cup N_4 \cup \ldots$]. Notice that the vertex of P is adjacent to no vertex of the C_k [since G is bipartite (C_4, C_6, C_8)-free]; on the other hand, by P1, x is the endpoint of an induced P_{10} of G formed together with nine vertices of $N_0 \cup N_1$; thus, G contains an induced P_{13}, a contradiction.

Then let us show that H, rooted at some vertex of N_3, is of depth at most one. Let a be any vertex of $V(H) \cap N_3$ and let us fix a as the root of H, namely, $H = T(a)$ according to the notation introduced above. Then let $x \in N_2$ be a neighbor of a; notice that x is nonadjacent to any vertex of $T_j(a)$, for $j = 1, 2$, since G is bipartite C_4-free. Thus $T_2(a) = \emptyset$, else otherwise by P1 and by the above an induced P_{10} arises, involving vertices x and a. $\square$

In what follows, by P2 [if necessary] let us write any member H of $\mathcal{H}$ as $H = T(a)$ for some $a \in N_3$ [that is H is a tree, rooted at vertex $a \in N_3$, of depth at most one].

P3. For each member $H = T(a)$ of $\mathcal{H}$, with $|T_1(a)| \neq 1$, vertex a would be forced to be in D while the other vertices of H to be *not* in D.

Proof. The proof follows by construction, by definition of e.d.s., and by P2. $\square$

Let us observe that the only members $H = T(a)$ of $\mathcal{H}$, which do not enjoy [the assumptions of] P3, are those such that $|T_1(a)| = 1$ [and $|T_2(a)| = 0$ by P2], i.e., are *edges* of

G.

Then let $\{\mathcal{H}_1, \mathcal{H}_2\}$ be the partition of $\mathcal{H}$ such that: $\mathcal{H}_1$ is formed by those members of $\mathcal{H}$ which enjoy [the assumptions of] of P3, while $\mathcal{H}_2$ is formed by those members of $\mathcal{H}$ with $|T_1(a)| = 1$ [and $|T_2(a)| = 0$ by P2], i.e., by those members of $\mathcal{H}$ which are *edges* of G.

Let us focus on $G' = G[N_2 \cup N_3 \cup N_4 \cup \ldots]$.

Let us assume that all vertices of G' are uncolored: the goal is to check if vertices of G' can be colored/partitioned into *white* vertices, i.e. those in D, and *black* vertices, i.e. those in $V(G) \setminus D$, with the constraint that all vertices of N_2 have to be black.

In particular the following forcing rules hold:

(R1) if vertex w is colored white, then all vertices of $N(w)$ should be colored black;

(R2) if vertex b is colored black and if b has a neighbor w colored white, then all vertices of $N(b) \setminus \{w\}$ should be colored black;

(R3) if h_1, h_2 are the vertices of a member of $\mathcal{H}_2$ and if h_1 is colored black (respectively, white), then h_2 should be colored white (respectively, black).

Then one can execute the following *Pre-procedure* whose running time bound can be estimated as $O(n^2)$ time: (1) assign color black to all vertices of N_2; (2) for each member H of $\mathcal{H}_1$, assign to each vertex of H the forced color according to P3 ; (3) apply iteratively rules (R1), (R2), (R3).

ASSUMPTION 1: Let us assume that the above pre-procedure does not lead to a contradiction [else G has not the sought e.d.s. D].

At this point one can just focus on the following set and family:

$N_2' = \{x \in N_2 : x$ is adjacent to no white vertex [by the pre-procedure]$\}$,

$\mathcal{H}_2' = \{H \in \mathcal{H}_2 :$ vertices of H are still uncolored [by the pre-procedure]$\}$.

Let $V(\mathcal{H}_2')$ denote the vertex set of members of $\mathcal{H}_2'$.

P4. Graph G has the sought e.d.s. D if and only if $G[N_2' \cup V(\mathcal{H}_2')]$ has an e.d.s. say $D_{\text{peripheral}}$ with $N_2' \cap D_{\text{peripheral}} = \emptyset$.

Proof. The proof is similar to that of P5 of Lemma 3.4. □

The following property P5 is just a technical parenthesis which will be useful later.

P5 Let Q be a connected component of $G[N_2' \cup V(\mathcal{H}_2')]$, such that Q contains exactly one vertex say x of N_2' [in particular, x is a cutset by construction, and is colored black by the pre-procedure]. Then one can easily determine an "optimal" coloring of Q (i.e. of $Q \setminus \{x\}$).

Proof. The proof is similar to that of P6 of Lemma 3.4. □

P6. One can compute $D^*_{\text{peripheral}} \in \text{ARG MIN}\{w(D_{\text{peripheral}}) : D_{\text{peripheral}}$ is an e.d.s. of $G[N_2' \cup V(\mathcal{H}_2')]$ with $N_2' \cap D_{\text{peripheral}} = \emptyset\}$ in $O(n^2)$ time.

Proof. First let us recall that (by construction and by definition of e.d.s.) each vertex of N_2' has to be covered by some vertex of $V(\mathcal{H}_2')$. Without loss of generality, to our aim, let us assume that $G[N_2' \cup V(\mathcal{H}_2')]$ is connected.

Then let us recall the aforementioned induced C_{10} in $G[N_0 \cup N_1]$, of vertex-set C, say of vertices $c_1, \ldots, c_{10}$, and say of edges $c_1c_2, \ldots, c_{10}c_1$.

Claim 1. One can assume (without loss of generality) that:

(i) each vertex of N_2' contacts at least two members of $\mathcal{H}_2'$;

(ii) each vertex of N_2' is of Type 1 [i.e. is adjacent to some vertex of C].

Proof. Proof of (i): In fact let $x \in N_2'$: if x contacts no member of $\mathcal{H}_2'$, then the sought e.d.s. D^* does not exist; if x contacts exactly one member of $\mathcal{H}_2'$, then the neighbor of x (in such a member) would be forced to be in D^*, so that one could iteratively apply rules (R1), (R2), (R3), and then reduce (again) the problem to a subset of N_2' and to a subfamily of $\mathcal{H}_2'$.

Proof of (ii): Let $x \in N_2'$. Let us recall that, according to the above, x can be either of Type 1 or of Type 2. By statement (i) vertex x contacts at least one [actually more than one] member say H of $\mathcal{H}_2'$.

Let us observe that x can not be of Type 2. By contradiction assume that x is of Type 2. Then x is adjacent to a vertex $y \in N_1 \setminus C$, which is adjacent to a vertex $z \in N_0 \cap C$; then, since G is bipartite (C_4, C_6, C_8)-free, the two vertices of H, x, y and nine vertices of C induce a P_{13}, a contradiction. □

Claim 2. For any member H of $\mathcal{H}_2'$, of vertices say h_1, h_2, one has that: h_1 is adjacent to at most one vertex of N_2', and h_2 is adjacent to at most one vertex of N_2'.

Proof. By symmetry let us just prove that h_1 is adjacent to exactly one vertex of N_2': in particular, let us assume that h_1 has a neighbor $x \in N_2'$ (else the claim is proved), and by Claim 1 let us assume that x is adjacent to c_1 (without loss of generality by symmetry).

Then let us show that no vertex $x' \in N_2'$, different to x, is adjacent to h_1: in fact if x' is adjacent to h_1, then, since G is bipartite (C_4, C_6, C_8)-free, x' is nonadjacent to x and is nonadjacent to any vertex of C; but this is not possible since, by Claim 1-(ii), x' is of Type 1. □

Notation. Let us say that a pair $\{x, y\} \subseteq N_2'$ is a *complementary pair of* N_2' if there is a member H of $\mathcal{H}_2'$, of vertices say h_1, h_2, such that x is adjacent to h_1 while y is adjacent to h_2; in particular, in this case, let us say that $\{x, y\}$ *covers* H, and that H is *covered by* $\{x, y\}$. Notice that: x is nonadjacent to h_2 (by Claim 2), y is nonadjacent to h_1 (by Claim 2), x is nonadjacent to y (since G is C_4-free). Furthermore in what follows, as an *agreement*, let us assume that any complementary pair $\{x, y\}$ of N_2' is an ordered pair with $x \in V_1(G)$ and $y \in V_2(G)$.

Claim 3. Let $\{x, y\}$ form a complementary pair of N_2'. Then x and y are adjacent respectively to opposite vertices of $G[C]$, i.e., x is adjacent to c_i while y is adjacent to c_{i+5} for some $i \in \{1, \ldots, 5\}$.

Proof. Let us assume that x is adjacent to c_1 without loss of generality by symmetry. Then y is nonadjacent to c_2 [else h_1, h_2, y, c_2, c_1, x induce a C_6], y is nonadjacent to c_{10} [similarly by symmetry], y is nonadjacent to c_4 [else $h_1, h_2, y, c_4, c_3, c_2, c_1, x$ induce a C_8], y is nonadjacent to c_8 [similarly by symmetry], so that y is adjacent to c_6. □

Claim 4. Let $\{x, y\}$ form a complementary pair of N_2'. Let Q be the connected component of $G[N_2' \cup V(\mathcal{H}_2')]$ containing x and y. Then $V(Q) \cap N_2' = \{x, y\}$.

Proof. Since $\{x, y\}$ forms a complementary pair of N_2', let H be a member of $\mathcal{H}_2'$, of vertices say h_1, h_2, such that x is adjacent to h_1 while y is adjacent to h_2. Let us assume that x is adjacent to c_1 without loss of generality by symmetry.

By contradiction assume that $V(Q) \cap N_2' \subset \{x, y\}$. Then by construction there is a vertex $z \in N_2'$ and a member K of $\mathcal{H}_2'$, of vertices say k_1, k_2, such that (without loss of generality by symmetry) x is adjacent to k_1 while z is adjacent to k_2. Note that by Claim 2: y is nonadjacent to k_2; z is nonadjacent to h_2. Furthermore, by Claim 3, y and z are adjacent to c_6. Then $x, h_1, h_2, y, c_6, z, k_2, k_1$ induce a C_8, a contradiction. □

Claim 5. Let Q be a connected component of $G[N_2' \cup V(\mathcal{H}_2')]$. Then Q contains at most two vertices of N_2'; in particular, if Q contains two vertices say x, y of N_2', then $\{x, y\}$ is a complementary pair of N_2'.

Proof. If Q contains two vertices say x, y of N_2', then by Claim 2 and since Q is connected, $\{x, y\}$ is a complementary pair of N_2'. Thus, by Claim 4, one obtains Claim 5. □

Now let us conclude the proof of P6.

For that let us describe the following possible solution method based on Claim 5.

For every connected component Q of $G[N_2' \cup V(\mathcal{H}_2')]$:

:: if Q contains just one vertex of N_2', say x, then one can easily determine an "optimal" coloring of Q according to P5;

:: if Q contains just two vertices of N_2', say x and y, then $\{x, y\}$ is a complementary pair of N_2'; then Q contains say ℓ edges, in particular, say ℓ^* edges are covered by $\{x, y\}$; then:

:: :: if $\ell^* \geq 3$, then G has not the sought $D_{\text{peripheral}}$; in fact according to (R3), in this case either x or y should be adjacent to two white vertices, and thus a contradiction would arise to the definition of e.d.s.;

:: :: if $\ell^* = 2$, then there are two edges (h_1, h_2) and (k_1, k_2), such that x is adjacent to h_1 and k_1 while y is adjacent to h_2 and k_2; then there just two possible colorings of Q: if the color of h_1 is fixed as white, then one can determinate the color of all other vertices of Q by forcing rules (R1)-(R2)-(R3) [in fact, vertex h_2 and k_1 would be black, then k_2 would be white; further, for each of the remaining $(\ell - 2)$ edges, the vertex which is adjacent to x or to y will be black while the other vertex will be white]; if the color of h_1 is fixed as black, then h_2 would be white, and then one can refer to the previous case by symmetry; summarizing one can easily determine an "optimal" coloring of Q;

:: :: if $\ell^* = 1$, then there is just one edge (h_1, h_2), such that x is adjacent to h_1 while y is adjacent to h_2; then each of the remaining $(\ell - 1)$ edges is contacted by exactly one vertex in $\{x, y\}$; let $\mathcal{E}_x$ the family of such edges contacted by x and let $\mathcal{E}_y$ the family of such edges contacted by y; if the color of h_1 is fixed as white, then: vertex h_2 would be black, for each edge in $\mathcal{E}_x$ the vertex adjacent to x would be black (and thus the other vertex would be white), finally at this point there are just $|\mathcal{E}_y|$ possible colorings for the edges in $\mathcal{E}_y$ (in fact exactly one neighbor of y should be white); if the color of h_1 is fixed as black, then h_2 would be white, and then one can refer to the previous case by symmetry; summarizing one can easily determine an "optimal" coloring of Q.

Then – apart from the case in which a contradiction arises, namely, in which a connected component Q contains a complementary pair of N_2' that covers at least three members of $\mathcal{H}_2'$ – the sought e.d.s. $D^*_{\text{peripheral}}$ is the union of all such "local" minimum e.d.s. for each connected component Q of $G[N_2' \cup V(\mathcal{H}_2')]$.

The above solution method is correct by Claim 1, Claim 2, and Claim 4. The running time bound can be estimated as $O(n^2)$.

This completes the proof of P5. $\square$

This completes the proof of Lemma 4.3. $\square$

Lemma 4.4. *Step (12.3.1.1) can be executed in constant time.*

Proof. The proof is similar (but shorter) to that of Lemma 4.3.

As a preliminary let us remark that Step (12.3.1.1) is executed only if G is (C_6, C_8, C_{10})-free: thus let us assume that G is (C_6, C_8, C_{10})-free.

By definition of Algorithm WED, let us fix any induced C_{12} of G of vertex-set C, say of vertices $c_1, \ldots, c_{12}$ and of edges $c_1c_2, \ldots, c_{12}c_1$, and let us fix any member $(C', \mathcal{D}(C'))$ of $\mathcal{F}_C$ and any e.d.s. $D' \in \mathcal{D}(C')$.

Then let us show that $D^{**} \in \text{ARG MIN}\{w(D) : D \text{ is an e.d.s. of } G, \text{ with } D \cap C' = D'\}$ can be computed in polynomial time. For convenience let us denote D^{**} as D.

Let us recall the notation $N_h(D') = \{v \in V(G) : dist(v, D') = h\}$, for $h \geq 1$, for the *distance levels* with respect to D'.

NOTATION: for brevity let us write $D' = N_0$ and $N_h(D') = N_h$ for $h \geq 1$.

Let us observe that by construction one has $C \subseteq N_0 \cup N_1$.

Let us observe that $D \cap (N_1 \cup N_2) = \emptyset$ by definition of e.d.s., so that each vertex of N_2 has to be covered by some vertex of N_3, that is, of $D \cap N_3$: in particular each vertex of N_3 which is isolated in $G[N_3 \cup N_4 \cup \ldots]$ is forced to be in D.

For any vertex $x \in N_2$, let us say that x is: of *Type 1* if x is adjacent to some vertex of C; of *Type 2* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \cap C$ [that is of $D' \cap C$]; of *Type 3* if x is not of Type 1 and is adjacent to a vertex of N_1 which is adjacent to a vertex of $N_0 \setminus C$ [that is of $D' \setminus C$]. Let us observe that, by definition of e.d.s., x is of [exactly] one of the above types.

Then let us consider the following property.

P1. Each vertex of N_2 is the endpoint of an induced P_{12} of G formed together with eleven vertices of $N_0 \cup N_1$.

Proof. Let us remark that, if a vertex v not in C is adjacent to some vertex of C, then v is adjacent to exactly one vertex of C [since G is bipartite (C_4, C_6, C_8, C_{10})-free]. Now let $x \in N_2$. If x is of Type 1, then x is adjacent to exactly one vertex of C, by definition of Type 1 and by the above remark. If x is of Type 2 nor of Type 3, then an induced P_{13} arises, by definition of such types and by the above remarks (i.e. no vertex of N_2 is of Type 2 nor of Type 3). $\square$

Then P1 implies that $N_3 = \emptyset$ [else, a vertex of N_2 should be adjacent to a vertex of N_3, so that an induced P_{13} would arise].

Then, by construction and by the above observations, one has that the sought e.d.s. D always exists and $D = D'$.

This completes the proof of Lemma 4.4. □

Then let us formalize and summarize the above as follows.

Theorem 4.5. *For bipartite (P_{13}, C_4, C_6, C_8)-free graphs, the WED problem can be solved in $O(n^5)$ time.*

5. Conclusions

The Weighted Efficient Set (WED) problem is a NP-hard problem and remains difficult for very restricted graph classes, such as, e.g., for claw-free graphs, for $2P_3$-free chordal graphs, and for bipartite planar graphs of maximum degree three and of fixed girth.

This manuscript reports a study on the WED problem for subclasses of *C_4-free bipartite graphs* – by the above the WED problem remains NP-hard for C_4-free bipartite graphs, even better, for C_4-free bipartite graphs which are planar and of maximum degree three.

Such subclasses concerned open directions of study according to the aforementioned results and to results on the boundedness of clique-width for bipartite graphs.

This manuscript shows that:

:: the WED problem can be solved in polynomial time for $(S_{1,2,5}, C_4)$-free bipartite graphs;

:: the WED problem can be solved in polynomial time for (P_{10}, C_4)-free bipartite graphs;

:: the approach introduced for for (P_{10}, C_4)-free bipartite graphs can be generalized, for subclasses of (P_t, C_4)-free bipartite graphs, for $t \geq 11$; in particular the WED problem can be solved in polynomial time for the following graph classes:

- (P_{11}, C_4, C_6)-free bipartite graphs,
- (P_{13}, C_4, C_6, C_8)-free bipartite graphs,
- $(2P_7, C_4, C_6, C_8)$-free bipartite graphs,
- $(P_{16}, C_4, C_6, C_8, C_{10})$-free bipartite graphs,
- $(2P_9, C_4, C_6, C_8, C_{10})$-free bipartite graphs.

The true technical bottleneck seems to be based on the fact that an efficient dominating set of a graph intersects "somehow" each vertex (cf. e.g. [13] and [14]), each induced P_4 (cf. e.g. [15]), each induced path/cycle of bounded length (cf. e.g. the results of this manuscript), and more generally each vertex subset of the graph. In particular, according to Observation 2, this fact can be a suitable starting point to define a solution method for the WED problem, once one fixes a bounded family of vertex subsets (such as, e.g., the vertices, the induced path/cycle of bounded length).

Let us observe that, with reference to the open cases introduced in Section 1 concerning *bipartite graphs*, the auxiliary assumption of C_4-freeness has not provided a deep progress. In particular, in the context of *C_4-free bipartite graphs*, the cases which remain open are the following graph classes:

– (P_k, C_4)-free bipartite graphs, $k \geq 11$,

– $(S_{2,2,5}, C_4)$-free bipartite graphs,

– $(S_{1,3,5}, C_4)$-free bipartite graphs,
– $(S_{1,2,6}, C_4)$-free bipartite graphs.

However this auxiliary assumption has allowed to introduce new possible insights and methods which could be useful for further studies as described above.

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References

[1] G. Abrishami and F. Rahbarnia. Polynomial-time algorithm for weighted efficient domination problem on diameter three planar graphs. *Information Processing Letters*, 140:25–29, 2018. https://doi.org/10.1016/j.ipl.2018.08.002.

[2] D. W. Bange, A. E. Barkauskas, L. H. Host, and P. J. Slater. Generalized domination and efficient domination in graphs. *Discrete Mathematics*, 159:1–11, 1996. https://doi.org/10.1016/0012-365X(95)00094-D.

[3] D. W. Bange, A. E. Barkauskas, and P. J. Slater. Efficient dominating sets in graphs. In R. D. Ringeisen and F. S. Roberts, editors, *Applications of Discrete Mathematics*, pages 189–199. SIAM, Philadelphia, PA, 1988.

[4] N. Biggs. Perfect codes in graphs. *Journal of Combinatorial Theory, Series B*, 15(3):289–296, 1973. https://doi.org/10.1016/0095-8956(73)90042-7.

[5] A. Brandstädt. Weighted efficient domination for P_5-free graphs in linear time. *CoRR*, abs/1507.06765, 2015. https://doi.org/10.48550/arXiv.1507.06765. arXiv: 1507.06765 [cs.DM].

[6] A. Brandstädt. Efficient domination and efficient edge domination: a brief survey. In *Algorithms and Discrete Applied Mathematics*, volume 10743 of *Lecture Notes in Computer Science*, pages 1–14. Springer, 2018. https://doi.org/10.1007/978-3-319-74180-2_1.

[7] A. Brandstädt, P. Fičur, A. Leitert, and M. Milanič. Polynomial-time algorithms for weighted efficient domination problems in AT-free graphs and dually chordal graphs. *Information Processing Letters*, 115:256–262, 2015. https://doi.org/10.1016/j.ipl.2014.09.024.

[8] A. Brandstädt and V. Giakoumakis. Weighted efficient domination for $(P_5 + kP_2)$-free graphs in polynomial time. *CoRR*, abs/1407.4593, 2014. https://doi.org/10.48550/arXiv.1407.4593. arXiv: 1407.4593 [cs.DM].

[9] A. Brandstädt, C. Hundt, and R. Nevries. Efficient edge domination on hole-free graphs in polynomial time. In *LATIN 2010: Theoretical Informatics*, volume 6034 of *Lecture Notes in Computer Science*, pages 650–661. Springer, 2010. https://doi.org/10.1007/978-3-642-12200-2_56.

[10] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: A Survey*, volume 3 of *SIAM Monographs on Discrete Mathematics and Applications*. SIAM, Philadelphia, PA, 1999. https://doi.org/10.1137/1.9780898719796.

[11] A. Brandstädt, A. Leitert, and D. Rautenbach. Efficient dominating and edge dominating sets for graphs and hypergraphs. In *Algorithms and Computation*, volume 7676 of *Lecture Notes in Computer Science*, pages 267–277. Springer, 2012. https://doi.org/10.1007/978-3-642-35261-4_30.

[12] A. Brandstädt, M. Milanič, and R. Nevries. New polynomial cases of the weighted efficient domination problem. In *Mathematical Foundations of Computer Science 2013*, volume 8087 of *Lecture Notes in Computer Science*, pages 195–206. Springer, 2013. https://doi.org/10.1007/978-3-642-40313-2_19.

[13] A. Brandstädt and R. Mosca. Weighted efficient domination for P_5-free and P_6-free graphs. *SIAM Journal on Discrete Mathematics*, 30(4):2288–2303, 2016. https://doi.org/10.1137/15M1039821.

[14] A. Brandstädt and R. Mosca. On efficient domination for some classes of H-free bipartite graphs. *Discrete Applied Mathematics*, 270:58–67, 2019. https://doi.org/10.1016/j.dam.2019.06.004.

[15] A. Brandstädt and R. Mosca. Finding efficient domination for $S_{1,1,5}$-free bipartite graphs in polynomial time. *CoRR*, abs/2010.16076, 2020. https://doi.org/10.48550/arXiv.2010.16076. arXiv: 2010.16076 [cs.DM].

[16] A. Brandstädt and R. Mosca. Finding efficient domination for $S_{1,3,3}$-free bipartite graphs in polynomial time. *CoRR*, abs/2008.04046, 2020. https://doi.org/10.48550/arXiv.2008.04046. arXiv: 2008.04046 [cs.DM].

[17] A. Brandstädt and R. Mosca. Finding efficient domination for P_8-free bipartite graphs in polynomial time. *CoRR*, abs/2101.01772, 2021. https://doi.org/10.48550/arXiv.2101.01772. arXiv: 2101.01772 [cs.DM].

[18] A. Brandstädt and R. Mosca. Weighted efficient domination for P_8-free bipartite graphs in polynomial time. *Discrete Applied Mathematics*, 380:326–347, 2026. https://doi.org/10.1016/j.dam.2025.10.023.

[19] D. M. Cardoso, N. Korpelainen, and V. V. Lozin. On the complexity of the dominating induced matching problem in hereditary classes of graphs. *Discrete Applied Mathematics*, 159(7):521–531, 2011. https://doi.org/10.1016/j.dam.2010.03.011.

[20] G. J. Chang, C. Pandu Rangan, and S. R. Coorg. Weighted independent perfect domination on cocomparability graphs. *Discrete Applied Mathematics*, 63(3):215–222, 1995. https://doi.org/10.1016/0166-218X(94)00067-3.

[21] B. Courcelle, J. A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory of Computing Systems*, 33(2):125–150, 2000. https://doi.org/10.1007/s002249910009.

[22] K. K. Dabrowski and D. Paulusma. Classifying the clique-width of H-free bipartite graphs. *Discrete Applied Mathematics*, 200:43–51, 2016. https://doi.org/10.1016/j.dam.2015.06.030.

[23] E. M. Eschen and X. Wang. Algorithms for unipolar and generalized split graphs. *Discrete Applied Mathematics*, 162:195–201, 2014. https://doi.org/10.1016/j.dam.2013.08.011.

[24] M. R. Fellows and M. N. Hoover. Perfect domination. *Australasian Journal of Combinatorics*, 3:141–150, 1991.

[25] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, San Francisco, 1979.

[26] D. L. Grinstead, P. J. Slater, N. A. Sherwani, and N. D. Holmes. Efficient edge domination problems in graphs. *Information Processing Letters*, 48(5):221–228, 1993. https://doi.org/10.1016/0020-0190(93)90084-M.

[27] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning. Efficient domination in graphs. In *Domination in Graphs: Core Concepts*, Springer Monographs in Mathematics, pages 259–289. Springer, Cham, 2023. https://doi.org/10.1007/978-3-031-09496-5_9.

[28] K. R. Hutson and S. T. Hedetniemi. Efficient domination in grid graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 122:325–342, 2024. https://doi.org/10.61091/jcmcc122-27.

[29] R. M. Karp. Reducibility among combinatorial problems. In R. E. Miller, J. W. Thatcher, and J. D. Bohlinger, editors, *Complexity of Computer Computations*, pages 85–103. Plenum Press, New York, 1972. https://doi.org/10.1007/978-1-4684-2001-2_9.

[30] J. Kratochvíl. *Perfect Codes in General Graphs*, number 7 in Rozpravy Československé Akademie Věd, Řada Matematických a Přírodních Věd. Academia, Praha, 1991.

[31] I. Kusumawardani, Dafik, E. Y. Kurniawati, I. H. Agustin, and R. Alfarisi. On resolving efficient domination number of path and comb product of special graph. *Journal of Physics: Conference Series*, 2157(1):012012, 2022. https://doi.org/10.1088/1742-6596/2157/1/012012.

[32] Y. D. Liang, C. L. Lu, and C. Y. Tang. Efficient domination on permutation graphs and trapezoid graphs. In T. Jiang and D. T. Lee, editors, *Computing and Combinatorics*, volume 1276 of *Lecture Notes in Computer Science*, pages 232–241. Springer, 1997. https://doi.org/10.1007/BFb0045090.

[33] Y.-L. Lin. Fast algorithms for independent domination and efficient domination in trapezoid graphs. In K.-Y. Chwa and O. H. Ibarra, editors, *Algorithms and Computation*, volume 1533 of *Lecture Notes in Computer Science*, pages 267–276. Springer, 1998.

[34] M. Livingston and Q. F. Stout. Distributing resources in hypercube computers. In *Proceedings of the Third Conference on Hypercube Concurrent Computers and Applications*, pages 222–231. ACM, 1988. https://doi.org/10.1145/62297.62324.

[35] D. Lokshtanov, M. Pilipczuk, and E. J. van Leeuwen. Independence and efficient domination on P_6-free graphs. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1784–1803. SIAM, 2016. https://doi.org/10.1137/1.9781611974331.ch124.

[36] V. V. Lozin and J. Volz. The clique-width of bipartite graphs in monogenic classes. *International Journal of Foundations of Computer Science*, 19(2):477–494, 2008. https://doi.org/10.1142/S0129054108005772.

[37] C. L. Lu, M.-T. Ko, and C. Y. Tang. Perfect edge domination and efficient edge domination in graphs. *Discrete Applied Mathematics*, 119:227–250, 2002. https://doi.org/10.1016/S0166-218X(01)00198-6.

[38] C. L. Lu and C. Y. Tang. Solving the weighted efficient edge domination problem on bipartite permutation graphs. *Discrete Applied Mathematics*, 87:203–211, 1998. https://doi.org/10.1016/S0166-218X(98)00057-2.

[39] C. L. Lu and C. Y. Tang. Weighted efficient domination problem on some perfect graphs. *Discrete Applied Mathematics*, 117:163–182, 2002. https://doi.org/10.1016/S0166-218X(01)00184-6.

[40] M. Milanič. Hereditary efficiently dominatable graphs. *Journal of Graph Theory*, 73(4):400–424, 2013. https://doi.org/10.1002/jgt.21685.

[41] O. J. Murphy. Computing independent sets in graphs with large girth. *Discrete Applied Mathematics*, 35(2):167–170, 1992. https://doi.org/10.1016/0166-218X(92)90041-8.

[42] R. C. Nevries. *Efficient Domination and Polarity*. PhD thesis, University of Rostock, Rostock, Germany, 2014. URN: urn:nbn:de:gbv:28-diss2014-0183-3.

[43] C. B. Smart and P. J. Slater. Complexity results for closed neighborhood order parameters. *Congressus Numerantium*, 112:83–96, 1995.

[44] C.-C. Yen. *Algorithmic Aspects of Perfect Domination*. PhD thesis, National Tsing Hua University, Taiwan, 1992.

[45] C.-C. Yen and R. C. T. Lee. The weighted perfect domination problem and its variants. *Discrete Applied Mathematics*, 66(2):147–160, 1996. https://doi.org/10.1016/0166-218X(94)00138-4.

Andreas Brandstädt
Institut für Informatik, Universität Rostock, D-18051 Rostock, Germany
E-mail andreas.brandstaedt@uni-rostock.de

Raffaele Mosca
Dipartimento di Economia, Universitá degli Studi “G. D’Annunzio”, Pescara 65121, Italy
E-mail r.mosca@unich.it