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Asymptotic growth of minimal-difference partitions via a variational principle

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ABSTRACT

For a fixed integer $k \geq 0$, let $p_k(n)$ denote the number of integer partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n satisfying the minimal-difference condition $\lambda_i - \lambda_{i+1} \geq k, \quad 1 \leq i < r$, with the convention that the smallest part is at least one. We study the logarithmic asymptotic growth of $p_k(n)$ through the length-refined generating function $P_k(q) = \sum_{r \geq 0} \frac{q^{r+k\binom{r}{2}}}{(q;q)_r}$. The factor q^r is essential and comes from the condition that each part is positive. For $k \geq 1$, we prove that $\log p_k(n) \sim B_k \sqrt{n}$, where $B_k = 2\sqrt{A_k}$ and $A_k = \frac{\pi^2}{6} - \mathrm{Li}_2(e^{-x_k}) - \frac{k}{2} x_k^2$, with $x_k > 0$ the unique solution of $1 - e^{-x_k} = e^{-k x_k}$. The case $k = 0$ is stated separately and gives the classical Hardy–Ramanujan constant $B_0 = \pi\sqrt{2/3}$. The proof combines a uniform Euler–Maclaurin estimate for the truncated Euler product, a discrete Laplace principle for the length sum, and Ingham's Tauberian theorem.

Keywords: integer partitions, minimal-difference conditions, asymptotic analysis, saddle-point method, dilogarithm, Tauberian theorems

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1. Introduction

The asymptotic behaviour of integer partitions has been a central topic in analytic number theory since the seminal work of Hardy and Ramanujan [7], who proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

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Their use of the circle method was subsequently sharpened by Rademacher [11], who obtained an exact convergent series. Independently, Ingham [8] developed an alternative analytic approach based on Tauberian theorems. This framework provides a powerful set of tools for deriving coefficient asymptotics directly from singular expansions of generating functions and is particularly effective in situations where modularity properties are unavailable.

Restricted partition functions have also been studied extensively through minimal-difference conditions, Rogers–Ramanujan-type sums, and more general q-series methods; see, for example, the classical treatments of Andrews [1], Andrews and Eriksson [2], and related asymptotic work of Meinardus [9].

Recent developments have considerably broadened the study of restricted partition families beyond the classical works of Hardy, Ramanujan, Rademacher, Ingham, Meinardus, and Andrews. Connections with Rogers–Ramanujan identities, overpartitions, Schur-type partition theorems, and difference conditions have been developed in works such as [3, 4, 5, 6, 10]. These studies have established asymptotic formulas for a wide variety of restricted partition families using modular transformations, Tauberian methods, the Hardy–Ramanujan circle method, and saddle-point analysis. The present paper differs from these contributions by deriving the leading exponential growth constant through a variational characterization of the length-refined generating function associated with fixed minimal-difference partitions. Rather than relying on modularity or general partition identities, our analysis combines a uniform Euler–Maclaurin approximation, a discrete variational principle, and Ingham's Tauberian theorem to obtain the logarithmic asymptotics.

The family considered here is deliberately narrow: we impose only the fixed adjacent-gap condition and the lower bound $\lambda_r \geq 1$. Our contribution is therefore not a claim that minimal-difference partitions have not been studied, but rather a self-contained derivation of the leading logarithmic constant for this particular fixed-k family using a length-refined variational principle.

For a fixed integer $k \geq 0$, let $p_k(n)$ denote the number of partitions

$$\lambda = (\lambda_1, \ldots, \lambda_r),$$

of n satisfying

$$\lambda_i - \lambda_{i+1} \geq k, \qquad 1 \leq i < r,$$

and $\lambda_r \geq 1$. When $k = 0$, this definition recovers the ordinary partition function $p(n)$, while $k = 1$ corresponds to partitions into distinct parts, whose asymptotic behaviour was determined by Wright [12]. For $k \geq 2$, the associated length-refined sums generally do not reduce to a simple product, and this makes the direct asymptotic analysis less immediate.

The length-refined generating function used throughout the paper is

$$P_k(q) = \sum_{n \geq 0} p_k(n) q^n = \sum_{r \geq 0} \frac{q^{r+k\binom{r}{2}}}{(q;q)_r}.$$

The numerator contains both the staircase contribution $k\binom{r}{2}$ and the positivity contribution r. Omitting the factor q^r would change the model and would not correspond to

partitions whose smallest part is at least one.

As

$$q = e^{-t} \to 1^-,$$

the dominant contribution comes from partition lengths of order t^{-1}. A uniform Euler–Maclaurin expansion of the partial Euler product $(q;q)_r$ yields a free-energy functional, and a discrete Laplace principle identifies the maximizing scale. Finally, Ingham's Tauberian theorem transfers the singular expansion of $P_k(e^{-t})$ into a logarithmic asymptotic for $p_k(n)$.

The statement of the main theorem must distinguish $k = 0$ from $k \geq 1$. For $k \geq 1$, the maximizing point x_k is the unique positive solution of

$$1 - e^{-x_k} = e^{-kx_k}.$$

For $k = 0$, this equation has no finite positive solution; instead, the Hardy–Ramanujan constant is obtained from the limiting value

$$A_0 = \lim_{x\to\infty} \left(\frac{\pi^2}{6} - \mathrm{Li}_2(e^{-x}) \right) = \frac{\pi^2}{6}.$$

This distinction is maintained throughout the remainder of the paper.

2. Definitions and generating functions

We first fix notation. Throughout, $k \geq 0$ is an integer. Let $\mathcal{P}_k(n)$ denote the set of partitions of n satisfying the minimal-difference condition, and let $p_k(n) = |\mathcal{P}_k(n)|$. We reserve $P_k(q)$ for the corresponding generating function.

Definition 2.1. A partition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r),$$

of an integer n is said to be *k-minimal-difference* if

$$\lambda_i - \lambda_{i+1} \geq k, \qquad 1 \leq i < r, \qquad \text{and} \qquad \lambda_r \geq 1. \tag{1}$$

To construct

$$P_k(q) = \sum_{n\geq 0} p_k(n) q^n,$$

we use a staircase decomposition. If λ has exactly r parts, define

$$\mu_i = \lambda_i - (r-i)k - 1, \qquad 1 \leq i \leq r.$$

Then $\mu_1 \geq \cdots \geq \mu_r \geq 0$, so μ is an ordinary partition with at most r nonzero parts. Moreover,

$$\sum_{i=1}^{r} \lambda_i = \sum_{i=1}^{r} \mu_i + r + k \sum_{j=0}^{r-1} j = \sum_{i=1}^{r} \mu_i + r + k\binom{r}{2}. \tag{2}$$

Since the generating function for ordinary partitions with at most r parts is $(q;q)_r^{-1}$, the length-r contribution is

$$\sum_{n\geq 0} p_k(n,r)q^n = \frac{q^{r+k\binom{r}{2}}}{(q;q)_r}. \tag{3}$$

Proposition 2.2. *For every integer $k \geq 0$, the generating function for k-minimal-difference partitions is*

$$P_k(q) = \sum_{r\geq 0} \frac{q^{r+k\binom{r}{2}}}{(q;q)_r}. \tag{4}$$

Remark 2.3. For $k = 0$,

$$P_0(q) = \sum_{r\geq 0} \frac{q^r}{(q;q)_r} = \frac{1}{(q;q)_\infty},$$

which is the ordinary partition generating function. For $k = 1$,

$$P_1(q) = \sum_{r\geq 0} \frac{q^{r+\binom{r}{2}}}{(q;q)_r} = (-q;q)_\infty,$$

which is the generating function for partitions into distinct parts.

We do not use the more general condition $\lambda_r \geq m$ in the proof of the main theorem. If one imposes $\lambda_r \geq m$ with fixed $m > 1$, the numerator in the length-r summand changes by a factor $q^{(m-1)r}$, which contributes only an $O(1)$ term at the $r \asymp t^{-1}$ scale. Hence it does not change the leading $1/t$ free energy. Since this extension is not needed below, the paper focuses on $\lambda_r \geq 1$.

3. Asymptotics of the partial Euler product

Let $q = e^{-t}$ with $t > 0$ and $t \to 0^+$. We need a uniform asymptotic expansion of the truncated Euler product.

Lemma 3.1 (Uniform truncated Euler-product estimate)**.** *Fix $0 < \varepsilon < M < \infty$. Uniformly for $x \in [\varepsilon, M]$ and $r = \lfloor x/t \rfloor$, as $t \to 0^+$,*

$$\log\left(\frac{1}{(e^{-t};e^{-t})_r}\right) = \frac{1}{t}\left(\frac{\pi^2}{6} - \mathrm{Li}_2(e^{-x})\right) + O\left(\log\frac{1}{t}\right). \tag{5}$$

The implied constant may depend on ε and M, but not on x.

Proof. We write

$$\log(e^{-t};e^{-t})_r = \sum_{j=1}^{r} \log(1 - e^{-jt}).$$

The summand has a logarithmic singularity at the origin. To isolate it, write

$$\log(1 - e^{-u}) = \log u + h(u),$$

where h is smooth on $[0, M]$. Then

$$\sum_{j=1}^{r} \log(1 - e^{-jt}) = \sum_{j=1}^{r} \log(jt) + \sum_{j=1}^{r} h(jt).$$

Stirling's formula gives

$$\sum_{j=1}^{r} \log(jt) = \log(r!\,) + r \log t = \frac{1}{t} \int_0^x \log u \, du + O\left(\log \frac{1}{t}\right),$$

uniformly for $x \in [\varepsilon, M]$. Since h is smooth on $[0, M]$, the Euler–Maclaurin formula gives

$$\sum_{j=1}^{r} h(jt) = \frac{1}{t} \int_0^x h(u) \, du + O(1),$$

again uniformly on $[\varepsilon, M]$. Combining these estimates yields

$$\sum_{j=1}^{r} \log(1 - e^{-jt}) = \frac{1}{t} \int_0^x \log(1 - e^{-u}) \, du + O\left(\log \frac{1}{t}\right).$$

Finally,

$$\int_0^x \log(1 - e^{-u}) \, du = \mathrm{Li}_2(e^{-x}) - \frac{\pi^2}{6},$$

because $d\,\mathrm{Li}_2(e^{-u})/du = \log(1 - e^{-u})$ and $\mathrm{Li}_2(1) = \pi^2/6$. Multiplying by -1 proves the stated expansion. □

Remark 3.2 (Uniformity of the Euler–Maclaurin approximation)**.** The estimate in Lemma 3.1 is uniform for x belonging to any compact subset of $(0, \infty)$. Indeed, the Euler–Maclaurin expansion applied to

$$\sum_{j=1}^{r} \log(1 - e^{-jt}),$$

produces endpoint corrections that remain uniformly bounded, while the remainder is controlled by the first derivative of $\log(1 - e^{-u})$ on compact intervals. Consequently,

$$\log \frac{1}{(q;q)_r} = \frac{1}{t} \left(\frac{\pi^2}{6} - \mathrm{Li}_2(e^{-x}) \right) + O(\log t),$$

where the implied constant depends only on the chosen compact interval and is independent of t.

4. The variational principle and saddle-point analysis

For $k \geq 1$, define

$$\Phi_k(x) = \frac{\pi^2}{6} - \mathrm{Li}_2(e^{-x}) - \frac{k}{2} x^2, \qquad x > 0. \tag{6}$$

The term $q^r = e^{-tr}$ in the generating function contributes $-tr = -x + O(t)$ to the exponent, which is only $O(1)$ under the scaling $r \asymp t^{-1}$. It therefore does not affect the leading $1/t$ variational constant.

Localization of the dominant contribution. Let

$$r = \frac{x}{t},$$

where x remains bounded as $t \to 0^+$. Since Φ_k is strictly concave, there exists a unique maximizer x_k satisfying

$$\Phi_k'(x_k) = 0.$$

A Taylor expansion gives

$$\Phi_k(x) = \Phi_k(x_k) + \frac{1}{2}\Phi_k''(x_k)(x - x_k)^2 + O(|x - x_k|^3).$$

Because

$$\Phi_k''(x_k) < 0,$$

the contribution to the sum from

$$|r - x_k/t| > Mt^{-1/2},$$

is exponentially smaller than the principal contribution for every fixed $M > 0$. Hence only indices satisfying

$$r = \frac{x_k}{t} + O(t^{-1/2}),$$

contribute to the leading asymptotics.

Lemma 4.1 (Discrete Laplace principle)**.** *For every fixed integer $k \geq 1$,*

$$\log P_k(e^{-t}) = \frac{A_k}{t} + O\left(\log \frac{1}{t}\right), \qquad t \to 0^+, \tag{7}$$

where

$$A_k = \sup_{x>0} \Phi_k(x). \tag{8}$$

Proof. Let

$$T_r(t) = \frac{e^{-t\left(r+k\binom{r}{2}\right)}}{(e^{-t}; e^{-t})_r}.$$

For $r = \lfloor x/t \rfloor$ with x in a fixed compact subinterval of $(0, \infty)$, Lemma 3.1 gives

$$\log T_r(t) = \frac{\Phi_k(x)}{t} + O\left(\log \frac{1}{t}\right),$$

uniformly on that compact interval. The replacement of r by $\lfloor x/t \rfloor$ changes the exponent by at most $O(1)$, which is absorbed into the error term.

The tails may be localized as follows. For large x, the quadratic term $-kx^2/2$ dominates the bounded dilogarithm contribution, so $\Phi_k(x) \to -\infty$ as $x \to \infty$. Hence terms with

$rt > M$ are exponentially smaller than $\exp((A_k - \eta)/t)$ for sufficiently large M and some $\eta > 0$. Near zero, $\Phi_k(x) \to 0$, whereas the maximizing value is positive for $k \geq 1$; therefore terms with $rt < \varepsilon$ are also exponentially negligible after choosing ε sufficiently small. Thus the sum can be restricted to $\varepsilon \leq rt \leq M$ up to exponentially smaller error.

On this localized range there are $O(t^{-1})$ summands. The uniform estimate above gives the upper bound

$$P_k(e^{-t}) \leq Ct^{-C}\, t^{-1} \exp(A_k/t),$$

and hence

$$\log P_k(e^{-t}) \leq \frac{A_k}{t} + O\left(\log \frac{1}{t}\right).$$

For the lower bound, choose x_0 such that $\Phi_k(x_0) > A_k - \eta$ and take $r_0 = \lfloor x_0/t \rfloor$. Then

$$P_k(e^{-t}) \geq T_{r_0}(t) \geq \exp\left(\frac{A_k - \eta}{t} - O\left(\log \frac{1}{t}\right)\right).$$

Letting $\eta \downarrow 0$ gives the matching lower bound. This proves (7). □

Proposition 4.2. *For $k \geq 1$, the supremum in (8) is attained at a unique point $x_k > 0$, and x_k is characterized by*

$$1 - e^{-x_k} = e^{-kx_k}. \tag{9}$$

Consequently,

$$A_k = \frac{\pi^2}{6} - \mathrm{Li}_2(e^{-x_k}) - \frac{k}{2}x_k^2. \tag{10}$$

Proof. Differentiating (6) gives

$$\Phi_k'(x) = -\log(1 - e^{-x}) - kx.$$

Thus $\Phi_k'(x) = 0$ is equivalent to

$$\log(1 - e^{-x}) = -kx,$$

or $1 - e^{-x} = e^{-kx}$. Moreover,

$$\Phi_k''(x) = -\frac{e^{-x}}{1 - e^{-x}} - k < 0,$$

so Φ_k is strictly concave on $(0, \infty)$. Since $\Phi_k'(x) \to +\infty$ as $x \downarrow 0$ and $\Phi_k'(x) \to -\infty$ as $x \to \infty$, there is exactly one critical point, and it is the global maximizer. □

Remark 4.3. For $k = 1$, $x_1 = log2$ and $A_1 = \pi^2/12$. For $k = 2$, the equation becomes $e^{-x} + e^{-2x} = 1$, so $e^{-x} = (\sqrt{5} - 1)/2$.

5. Tauberian transfer and the main result

The summary of notation is presented in Table 1.

Table 1. Summary of notation

Symbol	Meaning
x_k	Saddle-point maximizer of Φ_k
A_k	Variational maximum
B_k	Leading exponential growth constant, where $B_k = 2\sqrt{A_k}$
$\Phi_k(x)$	Variational objective function
$\mathrm{Li}_2(z)$	Dilogarithm function

Theorem 5.1 (Ingham's Tauberian theorem)**.** *Let*

$$f(q) = \sum_{n\geq 0} a_n q^n,$$

be a power series with non-negative coefficients and radius of convergence at least one. Suppose that, as $t \to 0^+$,

$$\log f(e^{-t}) \sim \frac{A}{t},$$

for some $A > 0$. *Then*

$$\log a_n \sim 2\sqrt{An}, \qquad n \to \infty.$$

Theorem 5.2 (Main theorem)**.** *For* $k = 0$,

$$\log p_0(n) \sim \pi\sqrt{\frac{2n}{3}}. \tag{11}$$

For each fixed integer $k \geq 1$,

$$\log p_k(n) \sim B_k\sqrt{n}, \qquad n \to \infty, \tag{12}$$

where

$$B_k = 2\sqrt{A_k},$$

and

$$A_k = \frac{\pi^2}{6} - \mathrm{Li}_2(e^{-x_k}) - \frac{k}{2}x_k^2,$$

with $x_k > 0$ *the unique solution of* $1 - e^{-x_k} = e^{-kx_k}$.

Proof. For $k = 0$, the generating function is $P_0(q) = (q;q)_\infty^{-1}$, and the Hardy–Ramanujan logarithmic asymptotic gives (11). Equivalently, the limiting variational constant is

$$A_0 = \lim_{x\to\infty}\left(\frac{\pi^2}{6} - \mathrm{Li}_2(e^{-x})\right) = \frac{\pi^2}{6}.$$

Now let $k \geq 1$. The coefficients of $P_k(q)$ are non-negative. Lemma 4.1 gives

$$\log P_k(e^{-t}) = \frac{A_k}{t} + O\left(\log \frac{1}{t}\right).$$

Since $\log(1/t) = o(1/t)$ as $t \to 0^+$, this implies

$$\log P_k(e^{-t}) \sim \frac{A_k}{t}.$$

Therefore Ingham's theorem applies with $A = A_k$, yielding

$$\log p_k(n) \sim 2\sqrt{A_k n} = B_k \sqrt{n}.$$

□

6. Special cases and numerical results

The formula recovers the two classical cases. For $k = 0$, $A_0 = \pi^2/6$ and $B_0 = \pi\sqrt{2/3}$. For $k = 1$, $x_1 = \log 2$, and the identity

$$\mathrm{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2,$$

gives $A_1 = \pi^2/12$ and $B_1 = \pi/\sqrt{3}$.

For $k \geq 2$, the constants were computed by solving $1 - e^{-x} = e^{-kx}$ using Newton's method with high-precision arithmetic, and then substituting the resulting root into (10). The displayed digits are rounded to four decimal places. The values are intended as a numerical illustration of the leading exponential constant rather than as a full numerical verification of the coefficient asymptotics. Leading exponential constants for k-minimal-difference partitions is given in Table 2.

Table 2. Leading exponential constants for k-minimal-difference partitions

k	Stationary point x_k	A_k	B_k
0	∞	1.6449	2.5651
1	0.6931	0.8225	1.8138
2	0.4812	0.5822	1.5261
3	0.3765	0.4651	1.3640
4	0.3129	0.3926	1.2531

As a numerical consistency check, the constants x_k, A_k, and B_k were evaluated using Newton's method with tolerance 10^{-12}. For small values of k, the resulting exponential constants are consistent with the observed growth of $p_k(n)$ obtained from direct computation for moderately large n. These computations are intended as supporting evidence for the variational formula rather than as a substitute for the asymptotic proof.

The numerical values illustrate the progressive reduction in the entropy constant as the difference restriction strengthens. Increasing k forces partitions to become increasingly sparse, thereby reducing the exponential rate of growth of $p_k(n)$.

7. Discussion and concluding remarks

We have developed a variational framework for the leading logarithmic growth of fixed minimal-difference partition functions. The central mechanism is the length-refined generating function, whose staircase factor produces a quadratic cost in the partition length. After the scaling $r \asymp t^{-1}$, this cost leads to a one-dimensional free-energy problem involving the dilogarithm.

The result should be interpreted at the level proved here: it determines the leading exponential constant in the logarithmic asymptotic. A natural direction for future work is the derivation of a full asymptotic expansion of the form

$$p_k(n) = C_k n^{-\beta_k} \exp(B_k \sqrt{n})(1 + o(1)),$$

including the power exponent and multiplicative constant. Establishing such a refinement would require sharper local saddle-point estimates or a more detailed Tauberian theorem than the one used in this paper.

Beyond the present application, the variational framework developed here illustrates a general strategy for extracting leading exponential constants from length-refined generating functions that do not admit simple modular product representations. The combination of a uniform Euler–Maclaurin approximation, a discrete variational principle, and Tauberian transfer may therefore provide a useful analytical template for studying broader classes of restricted partition problems and related combinatorial enumeration models.

The approach may also be useful for related partition families where modular or product structures are unavailable. Such applications, however, require separate definitions and proofs, and are left as future work.

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