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Construction of GDD$(n_1, n_2, 4; \lambda, 1)$

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ABSTRACT

In this paper we study group divisible designs (GDDs) with block size 4 and two groups of different sizes when $\lambda_2 = 1$. We obtain necessary conditions for the existence of such GDDs and prove that these necessary conditions are sufficient in several cases. Further, we present general constructions using resolvable designs.

Keywords: balanced incmplete block designs, group divisible designs, nonuniform group divisible designs, resolvable designs

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1. Introduction

A set system or design is a pair $(X, \mathcal{B})$, where X is a non-empty finite set and $\mathcal{B}$ is a collection of subsets of X, called blocks. Among all combinatorial designs, probably the most widely studied design is a Balanced Incomplete Block Design (BIBD) (See [15]).

Definition 1.1. A Balanced Incomplete Block Design BIBD(v, k, λ) is an arrangement of elements of a v-set X into b blocks of size $(k < v)$ each, such that every element appears in exactly r blocks and every pair of distinct elements occurs together in exactly λ blocks.

The numbers v, b, r, k and λ are called parameters of a BIBD and satisfy the conditions or relationships

$$bk = rv \quad \text{and} \quad r(k-1) = \lambda(v-1).$$

These conditions are called necessary conditions for the existence of a BIBD(v, k, λ).

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Example 1.2. A famous BIBD is the Fano plane : a BIBD$(7,3,1)$. The blocks are obtained from the triple $\{1,3,4\}$ by adding the elements of $X = \{0,1,2,3,4,5,6\}$ modulo 7. So the blocks of the design are given in columns as follows:

1	2	3	4	5	6	0
3	4	5	6	0	1	2
4	5	6	0	1	2	3

In 1961, Hanani [9] proved that the necessary conditions are sufficient for the existence of BIBDs with block size three as well as four. Specifically he proved:

Theorem 1.3. *A BIBD$(v,4,\lambda)$ exists if and only if $\lambda \equiv 1,5 \pmod 6$ and $v \equiv 1,4 \pmod{12}$; $\lambda \equiv 2,4 \pmod 6$ and $v \equiv 1 \pmod 3$; $\lambda \equiv 3 \pmod 6$ and $v \equiv 0,1 \pmod 4$; $\lambda \equiv 0 \pmod 6$ and $v \geq 4$.*

Definition 1.4. A parallel class in a BIBD(v,k,λ) is a set of disjoint blocks of the BIBD whose union is X. A partition of the collection of all blocks into r parallel classes is called a resolution. A BIBD is said to be a resolvable BIBD, denoted RBIBD(v,k,λ), if it has a resolution.

A BIBD is called α-resolvable BIBD if its blocks can be partitioned into classes in which each element occurs α times.

Example 1.5. To construct a RBIBD$(9,3,1)$, the blocks of a BIBD$(9,3,1)$ on $\{1,2,3,4,5,6,7,8,9\}$ are partitioned into four parallel classes as demonstrated below:

1	4	7	1	2	3	1	2	3	1	2	3
2	5	8	4	5	6	5	6	4	6	4	5
3	6	9	7	8	9	9	7	8	8	9	7

Theorem 1.6. [3] *A resolvable BIBD$(v,3,1)$ exists if and only if $v \equiv 3 \pmod 6$.*

A resolvable BIBD$(v,3,1)$ is called Kirkman Triple System of order v, denoted by KTS(v). There are $\frac{v-1}{2}$ parallel classes in a KTS(v).

Let α be a positive integer. An α-parallel class or α-resolution class in a design is a set of blocks containing every point of the design exactly α times [1].

In [1] the following necessary conditions are given:

Theorem 1.7. *Necessary conditions for the existence of an α-resolvable BIBD(v,k,λ) are (1) $\lambda(v-1) \equiv 0 \pmod{(k-1)\alpha}$; (2) $\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$ and (3) $\alpha v \equiv 0 \pmod k$.*

For block size 3, following result is given in [1].

Theorem 1.8. *The necessary conditions for the existence of an α-resolvable BIBD$(v,3,\lambda)$ are sufficient except for $v = 6$, $\alpha = 1$, and $\lambda \equiv 2 \pmod 4$.*

From Theorem 1.8, we have the following corollary.

Corollary 1.9. *For $n \equiv 0 \pmod 3$, a resolvable BIBD$(n, 3, n-2)$ exists.*

Example 1.10. Let $X = \{1, 2, 3, 4, 5, 6\}$. By Corollary 1.9, BIBD(6, 3, 4) is a resolvable with 10 resolution classes; each class contains 2 mutually disjoint blocks which partition the point set X as follows. The blocks are written in columns.

1	4	1	3	1	3	1	3	1	2	1	2	1	2	1	2	1	2	1	2
2	5	2	5	2	4	2	4	3	5	3	4	3	4	4	3	4	3	5	3
3	6	4	6	5	6	6	5	4	6	5	6	6	5	5	6	6	5	6	4

Group divisible designs (GDDs) defined below play a role in the construction of BIBDs as well as other designs [10].

Definition 1.11. A group divisible design, GDD$(n_1, n_2, ..., n_m, k; \lambda_1, \lambda_2)$, is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a v-set, $\mathcal{G}$ is a partition of X into m subsets (called *groups*) of sizes $n_1, n_2, ..., n_m$ respectively and $\mathcal{B}$ is a collection of k-subsets of X (called *blocks*) such that each pair of points within the same group appear together in λ_1 blocks, whereas each pair of points from different groups appear together in λ_2 blocks. The points in the same group are called *first associate* of each other and points not in the same group are called *second associates* of each other.

Group divisible designs have been studied for their usefulness in statistics and for their universal applications in the constructions of new designs [16, 17]. Most research has concentrated on those cases where the group size or group type was critical. When the number of groups is smaller than the block size, however, certain difficulties are present and little has been done for block size four. Fu, Rodger and Sarvate [5, 6] obtained complete results on group divisible designs with block size 3 and m groups of size n. When all the groups are of the same size, the GDD is said to be uniform, otherwise it is said to be nonuniform. In 1992, Colbourn, Hoffman and Rees [4] proved the sufficiency of the necessary conditions for the existence of a GDD$(n, n, ..., n, u, 3; 0, 1)$. When GDDs are used to construct other combinatorial designs, the groups are preferred to have different sizes to fit in various situations [4]. Unfortunately comparing with uniform GDDs where group sizes are equal, less is known for the construction of nonuniform GDDs. The construction of nonuniform GDDs is a challenging problem as no appropriate algebraic or geometric structures have been found which are useful for these constructions [7].

This paper is specially motivated by [2] , [11] and [12]. In 2011, Pabhapote and Punnim [12] studied all triples of positive integers (n_1, n_2, λ) for which a GDD$(n_1, n_2, 3; \lambda, 1)$ exists. They proved that the necessary conditions are sufficient for the existence of a GDD$(n_1, n_2, 3; \lambda, 1)$. Later in 2012, Pabhapote [11] proved the existence of a GDD$(n_1, n_2, 3; \lambda_1, \lambda_2)$ for all $n_1 \neq 2$ and $n_2 \neq 2$ in which $\lambda_1 \geq \lambda_2$. In 2012, Chaiyasena, Hurd, Punim and Sarvate [2] investigated GDDs with two association classes with blocks of size 3 and groups of unequal sizes. They also obtained some general results for a GDD$(n_1, n_2, n_3, 3; \lambda_1, \lambda_2)$.

However, much less is known about the existence of nonuniform GDDs when the block size is four. A few results are available. For instance, in 2022, Sarvate and Woldemariam [13] studied the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ and proved that the necessary conditions are sufficient for its existence when $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ and $\lambda_1 \geq \lambda_2$. In 2023, Sarvate, Woldemariam and Zhang [14] investigated the existence of nonuniform GDDs with three groups and block size 4. Recently, Girma, Sarvate and Zhang [8] studied the existence of GDD$(1, n, 4; \lambda_1, \lambda_2)$ and showed the necessary conditions are sufficient in several cases when $\lambda_1 \geq \lambda_2$.

Despite these developments, the existence of GDDs with two unequal groups with block size 4 remains unknown when one of the indices λ_1 or λ_2 is fixed. In this paper, we study group divisible designs with block size 4 and two groups of sizes n_1 and n_2 when $\lambda_2 = 1$, namely a GDD$(n_1, n_2, 4; \lambda, 1)$.

Group divisible designs play a role in the construction of BIBDs, but the converse is also true, for example, an easy observation is the following result.

Theorem 1.12. *If a BIBD$(n_1 + n_2 + ... + n_m, k, \lambda_2)$ and a BIBD(n_i, k, λ_1) exist for $i = 1, 2, ...m$, then a GDD$(n_1, n_2, ..., n_m, k; \lambda_1 + \lambda_2, \lambda_2)$ exists.*

Corollary 1.13. *If a BID$(n_1 + n_2, 4, \lambda_2)$ and a BIBD$(n_i, 4, \lambda_1)$ for $i = 1, 2$ exist, then a GDD$(n_1, n_2, 4; \lambda_1 + \lambda_2, \lambda_2)$ exists.*

The converse of Theorem 1.12 is not always true, for example, a GDD$(9, 24, 4; 6, 1)$ exists which can be constructed by using Corollary 3.5 but a BIBD$(9, 4, 5)$, a BIBD$(24, 4, 5)$ and a BIBD$(9 + 24, 4, 1)$ do not exist.

Notice that a GDD$(n_1, n_2, 4; 0, \lambda_2)$ does not exist as the number of groups is less than the block size.

2. Necessary conditions

Let r_i be the replication number of elements of the i-th group for $i = 1, 2$ and b be the required number of blocks if a GDD$(n_1, n_2, k; \lambda_1, \lambda_2)$ exists. By combinatorial argument,

$$r_1 = \frac{(n_1 - 1)\lambda_1 + n_2\lambda_2}{k - 1}, \tag{1}$$

$$r_2 = \frac{n_1\lambda_2 + (n_2 - 1)\lambda_1}{k - 1}, \tag{2}$$

$$b = \frac{[n_1^2 + n_2^2 - (n_1 + n_2)]\lambda_1 + \lambda_2(2n_1n_2)}{k(k - 1)}. \tag{3}$$

Theorem 2.1. *A necessary condition for the existence of a GDD$(n_1, n_2, 4; \lambda_1, \ \lambda_2)$ is $b \geq max(2r_i - \lambda_1)$.*

Proof. Let (x,y) be a first associate pair from G_i for some $i = 1, 2$. Then as both of them come together λ_1 times, there are r_i blocks containing x and $r_i - \lambda_1$ blocks which contain y but do not contain x. So the number of blocks must be at least $2r_i - \lambda_1$ to accommodate x and y r_i times in the design. □

Theorem 2.2. *A necessary condition for the existence of a GDD*$(n_1, n_2, 4; \lambda_1,\ \lambda_2)$ *is* $b \geq r_1 + r_2 - \lambda_2$.

Proof. Let (x,y) be a second associate pair, say $x \in G_1$ and $y \in G_2$. As both of them come together λ_2 times, there are $r_1 + r_2 - \lambda_2$ blocks containing x or y but do not contain both x and y. So the number of blocks must be at least $r_1 + r_2 - \lambda_2$. □

Suppose a GDD$(n_1, n_2, 4; \lambda_1, \lambda_2)$ exists. Then the number of first associate pairs equals $\binom{n_1}{2}\lambda_1 + \binom{n_2}{2}\lambda_1 = \frac{n_1^2+n_2^2-(n_1+n_2)}{2}\lambda_1$ and the number of second associate pairs equals $(n_1n_2)\lambda_2$. As the block size is 4, there are three types of block configurations: $(4, 0)$, $(3, 1)$ and $(2, 2)$. Let x, y and z be the number of blocks of type $(4, 0)$, $(3, 1)$ and $(2, 2)$ respectively. Note $x + y + z = b$ and they contribute $6x + 3y + 2z$ first associate pairs in the design. Hence,

$$6x + 3y + 2z = \frac{n_1^2 + n_2^2 - (n_1 + n_2)}{2}\lambda_1. \tag{4}$$

Similarly, counting second associate pairs,

$$3y + 4z = n_1n_2\lambda_2. \tag{5}$$

From Eqs. (4) and (5), we have $2z - 6x = n_1n_2\lambda_2 - \frac{n_1^2+n_2^2-(n_1+n_2)}{2}\lambda_1 \leq 2z \leq 2b$. Using Eq. (3), we obtain $\lambda_2 \leq \frac{n_1^2+n_2^2-(n_1+n_2)}{n_1n_2}\lambda_1$. Thus, we have the following theorem.

Theorem 2.3. *A necessary condition for the existence of a GDD*$(n_1, n_2, 4; \lambda_1,\ \lambda_2)$ *is* $\lambda_2 \leq \frac{n_1^2+n_2^2-(n_1+n_2)}{n_1n_2}\lambda_1$.

As every block of a GDD$(n_1, n_2, 4; \lambda_1, \lambda_2)$ contains at least two first associate pairs, hence $\left[\binom{n_1}{2} + \binom{n_2}{2}\right]\lambda_1 \geq 2b$. Thus, we have

Theorem 2.4. *A necessary condition for the existence of a GDD*$(n_1, n_2, 4; \lambda_1,\ \lambda_2)$ *is* $b \leq \frac{n_1^2+n_2^2-(n_1+n_2)}{4}\lambda_1$.

Example 2.5. A GDD$(7, 10, 4; 1, 3)$ does not exist as $\frac{n_1^2+n_2^2-(n_1+n_2)}{4}\lambda_1 = 33$ which is less than the required number of blocks $b = 46$.

Theorem 2.6. *A necessary condition for the existence of a GDD*$(n_1, n_2, 4; \lambda_1,\ \lambda_2)$ *is* $r_1 \geq \lambda_1$.

Proof. Let $x \in G_1$, as a first associate pair occurs λ_1 times, $r_1 \geq \lambda_1$ for $n_1 > 1$. □

Example 2.7. A GDD$(2, 5, 4; 4, 1)$ does not exists as $r_1 = 3 < 4 = \lambda_1$.

After a few observations on the necessary conditions for the existence of a GDD$(n_1, n_2, 4; \lambda_1, \lambda_2)$, next we will obtain the spectrum for the existence of a GDD$(n_1, n_2, 4; \lambda, 1)$.

Suppose $\lambda_1 = \lambda$ and $\lambda_2 = 1$. Then from Eqs. (1), (2) and (3), we have

$$r_1 = \frac{(n_1 - 1)\lambda + n_2}{3}, \tag{6}$$

$$r_2 = \frac{n_1 + (n_2 - 1)\lambda}{3}, \tag{7}$$

$$b = \frac{[n_1(n_1 - 1) + n_2(n_2 - 1)]\lambda + 2n_1 n_2}{12}. \tag{8}$$

Since r_1 and r_2 must be integers, from Eqs. (6) and (7), we have the following.

- If $\lambda \equiv 0 \pmod 3$, then $n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 0 \pmod 3$.
- If $\lambda \equiv 1 \pmod 3$, $n_1 + n_2 \equiv 1 \pmod 3$.
- If $\lambda \equiv 2 \pmod 3$, then there are no integer value for n_1 and n_2 that makes r_1 and r_2 integers.

Since b must be an integer, from Eq. (8), we have the following.

- If $\lambda \equiv 0 \pmod 6$, then $n_1 n_2 \equiv 0 \pmod 6$.
- If $\lambda \equiv 1 \pmod 6$, then $n_1 + n_2 \equiv 0, 1, 4, 9 \pmod{12}$.
- If $\lambda \equiv 2 \pmod 6$, then $(n_1 + n_1)(n_1 + n_2 - 1) - n_1 n_2 \equiv 0 \pmod 6$.
- If $\lambda \equiv 3 \pmod 6$, then $n_1 + n_2 \equiv 0, 1 \pmod 4$ and $n_1 n_2 \equiv 0 \pmod 3$.
- If $\lambda \equiv 4 \pmod 6$, then $n_1 + n_2 \equiv 0, 1 \pmod 3$ and $n_1 n_2 \equiv 0 \pmod 2$.
- If $\lambda \equiv 5 \pmod 6)$, then $5(n_1 + n_2)(n_1 + n_2 - 1) - 8n_1 n_2 \equiv 0 \pmod{12}$.

Combining the cases above, we obtain the following theorem:

Theorem 2.8. *Necessary conditions for the existence of a GDD$(n_1, n_2, 4; \lambda, 1)$ are*

a. $\lambda \equiv 0 \pmod 6$*:* $n_1 n_2 \equiv 0 \pmod 6$, $n_1 \equiv 0 \pmod 3$ *and* $n_2 \equiv 0 \pmod 3$.

b. $\lambda \equiv 1 \pmod 6$*:* $n_1 + n_2 \equiv 1, 4 \pmod{12}$.

c. $\lambda \equiv 3 \pmod 6$*:* $n_1 + n_2 \equiv 0, 9 \pmod{12}$ *and* $n_1 n_2 \equiv 0 \pmod 3$.

d. $\lambda \equiv 4 \pmod 6$*:* $n_1 + n_2 \equiv 1 \pmod 3$ *and* $n_1 n_2 \equiv 0 \pmod 2$.

Moreover, we have the following theorem from the divisiblity requirements of r_1 and r_2.

Theorem 2.9. *For* $\lambda \equiv 2, 5 \pmod 6$, a *GDD$(n_1, n_2, 4; \lambda, 1)$ is not admissible.*

Table 1 summarizes certain restrictions on λ where "None" means the design does not exist for any λ. Here n_1 and n_2 are given in modulo 12.

Table 1. The necessary conditions for a $\mathrm{GDD}(n_1, n_2, 4; \lambda, 1)$

$n_1 \backslash n_2$	0	1	2	3	4	5	6	7	8	9	10	11
0	$\lambda \equiv 0,3 \pmod 6$	$\lambda \equiv 1,4 \pmod 6$	None	$\lambda \equiv 0 \pmod 6$	$\lambda \equiv 1,4 \pmod 6$	None	$\lambda \equiv 0 \pmod 6$	$\lambda \equiv 4 \pmod 6$	None	$\lambda \equiv 0,3 \pmod 6$	$\lambda \equiv 4 \pmod 6$	None
1	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 1 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$	None	None	None	None	None
2	None	None	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$	None	None	$\lambda \equiv 1,4 \pmod 6$
3	$\lambda \equiv 0 \pmod 6$	$\lambda \equiv 1 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$	None	$\lambda \equiv 0,3 \pmod 6$	None	None	$\lambda \equiv 3 \pmod 6$	$\lambda \equiv 1,4 \pmod 6$	None
4	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$	None	None	$\lambda \equiv 1,4 \pmod 6$	None	None
5	None	None	$\lambda \equiv 4 \pmod 6$	None	None	None	None	None	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 1 \pmod 6$
6	$\lambda \equiv 0 \pmod 6$	$\lambda \equiv 4 \pmod 6$	None	$\lambda \equiv 0,3 \pmod 6$	$\lambda \equiv 4 \pmod 6$	None	$\lambda \equiv 0,3 \pmod 6$	$\lambda \equiv 1,4 \pmod 6$	None	$\lambda \equiv 0 \pmod 6$	$\lambda \equiv 1,4 \pmod 6$	None
7	$\lambda \equiv 1,4 \pmod 6$	None	None	None	None	None	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 1 \pmod 6$	None	None
8	None	None	$\lambda \equiv 4 \pmod 6$	None	None	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$
9	$\lambda \equiv 0,3 \pmod 6$	None	None	$\lambda \equiv 3 \pmod 6$	$\lambda \equiv 1,4 \pmod 6$	None	$\lambda \equiv 0 \pmod 6$	$\lambda \equiv 1 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$	None
10	$\lambda \equiv 4 \pmod 6$	None	None	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$	None	None
11	None	None	$\lambda \equiv 1,4 \pmod 6$	None	None	$\lambda \equiv 1 \pmod 6$	None	None	$\lambda \equiv 4 \pmod 6$	None	None	None

3. Existence of families of $\mathrm{GDD}(n_1, n_2, 4; \lambda, 1)$

Unless otherwise stated t is a nonnegative integer throughout this section.

3.1. Sufficient conditions

Theorem 3.1. *Necessary conditions are sufficient for the existence of a GDD*$(n_1, n_2, 4; \lambda, 1)$ *when* $\lambda \equiv 1 \pmod 6$ *and* $n_i \geq 4$ *for* $i = 1, 2$.

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. When $\lambda \equiv 1 \pmod 6$, from the necessary conditions, $n_1 + n_2 \equiv 1, 4 \pmod{12}$. Then by Theorem 1.3, a $\mathrm{BIBD}(n_1 + n_2, 4, 1)$ exists. Also when $n_i \geq 4$ for $i = 1, 2$, a $\mathrm{BIBD}(n_i, 4, 6)$ exists. So the blocks of a $\mathrm{BIBD}(n_1 + n_2, 4, 1)$ on points of $G_1 \cup G_2$ together with t copies of the blocks of a $\mathrm{BIBD}(n_i, 4, 6)$ on points of G_i provide the blocks of a $\mathrm{GDD}(n_1, n_2, 4; 6t + 1, 1)$. □

Theorem 3.2. *Necessary conditions are sufficient for the existence of a GDD*$(n_1, n_2, 4; \lambda, 1)$ *for* $\lambda \equiv 4 \pmod 6$ *when*

(i) $n_1 \equiv 0 \pmod{12}$ *and* $n_2 \equiv 1, 4 \pmod{12}$,

(ii) $n_1 \equiv 4 \pmod{12}$ *and* $n_2 \equiv 9 \pmod{12}$,

(iii) $n_1 \equiv 5, 8 \pmod{12}$ *and* $n_2 \equiv 8 \pmod{12}$.

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. When $\lambda \equiv 4 \pmod 6$, from the necessary conditions, $n_1+n_2 \equiv 1 \pmod 3$ and $n_1 n_2 \equiv 0 \pmod 2$. But $n_1+n_2 \equiv 1 \pmod 3$ implies $n_1 + n_2 \equiv 1, 4, 7, 10 \pmod{12}$. When $n_1 + n_2 \equiv 1, 4 \pmod{12}$, by Theorem 1.3, a $\mathrm{BIBD}(n_1 + n_2, 4, 1)$ exists. As $n_1 + n_2 \equiv 1, 4 \pmod{12}$ implies $n_1 \equiv 0 \pmod{12}$ and $n_2 \equiv 1, 4 \pmod{12}$; $n_1 \equiv 9 \pmod{12}$ and $n_2 \equiv 4 \pmod{12}$; and $n_1 \equiv 8 \pmod{12}$ and $n_2 \equiv 5, 8 \pmod{12}$. Also by Theorem 1.3 a $\mathrm{BIBD}(n_i, 4, 3)$ for $i = 1, 2$ exists. Since a $\mathrm{BIBD}(n_i, 4, 6)$ for $n_i \geq 4$ exists, the blocks of a $\mathrm{BIBD}(n_1 + n_2, 4, 1)$ on points of $G_1 \cup G_2$ together with the blocks of a $\mathrm{BIBD}(n_i, 4, 3)$ and t copies of the blocks of a $\mathrm{BIBD}(n_i, 4, 6)$ on points of G_i provide the blocks of a $\mathrm{GDD}(n_1, n_2, 4; 6t + 4, 1)$. □

Theorem 3.3. *The necessary conditions are sufficient for the existence a GDD*$(n_1, n_2, 4; \lambda, 1)$, *except possibly for the following cases:*

- $\lambda \equiv 0 \pmod 6$, $n_1 \equiv 0 \pmod{12}$ *and* $n_2 \equiv 0 \pmod 3$.
- $\lambda \equiv 0 \pmod 6$, $n_1 \equiv 0 \pmod 6$ *and* $n_2 \equiv 3, 9 \pmod{12}$.

- $\lambda \equiv 3 \pmod 6$, $n_1 \equiv 0 \pmod{12}$ *and* $n_2 \equiv 0, 9 \pmod{12}$.
- $\lambda \equiv 3 \pmod 6$, $n_1 \equiv 3 \pmod{12}$ *and* $n_2 \equiv 6, 9 \pmod{12}$.
- $\lambda \equiv 4 \pmod 6$, $n_1 \equiv 0, 6 \pmod{12}$ *and* $n_2 \equiv 7, 10 \pmod{12}$.
- $\lambda \equiv 4 \pmod 6$, $n_1 \equiv 1, 4 \pmod{12}$ *and* $n_2 \equiv 6 \pmod{12}$.
- $\lambda \equiv 4 \pmod 6$, $n_2 \equiv 2 \pmod{12}$ *and* $n_2 \equiv 2, 5, 8, 11 \pmod{12}$.
- $\lambda \equiv 4 \pmod 6$, $n_1 \equiv 3, 9 \pmod{12}$ *and* $n_2 \equiv 10 \pmod{12}$.
- $\lambda \equiv 4 \pmod 6$, $n_1 \equiv 8 \pmod{12}$ *and* $n_2 \equiv 11 \pmod{12}$.

Though the sufficiency of the above cases is not known, in the next subsection we establish the existence of certain infinite families of a GDD$(n_1, n_2, 4; \lambda, 1)$ when $\lambda \equiv 0, 3, 4 \pmod 6$ using resolvable designes.

3.2. Constructions using resolvable designs

Theorem 3.4. *If an α-resolvable BIBD(n_1, k, λ) with replication number r and a BIBD$(\frac{r}{\alpha}, k+1, \lambda)$ exist, then a GDD$(n_1, n_2 = \frac{r}{\alpha}, k+1; \lambda, \alpha)$ exists.*

Proof. Let G_1 and G_2 be groups of size n_1 and $n_2 = \frac{r}{\alpha}$ respectively. Suppose the blocks of the α-resolvable BIBD(n_1, k, λ) are partitioned into $\frac{r}{\alpha}$ classes namely: π_1, π_2, ..., $\pi_{\frac{r}{\alpha}}$ each of size $\frac{\alpha n_1}{k}$.

First, we construct blocks of the required GDD by taking union of the i^{th} element of G_2 with each block of the i^{th} class for $i = 1, 2, ..., \frac{r}{\alpha}$. These blocks provide blocks of size $k+1$, where pairs from G_1 occur λ times and pairs from different groups occur α times. Then, these blocks together with the blocks of a BIBD$(n_2 = \frac{r}{\alpha}, k+1, \lambda)$ on the points of G_2 provide the blocks for GDD$(n_1, n_2 = \frac{r}{\alpha}, k+1; \lambda, \alpha)$. □

Corollary 3.5. *Suppose a KTS(n_1) with replication number r and a BIBD$(n_2, 4, \frac{n_2}{r})$ exist. Then there exists a GDD$(n_1, n_2, 4; \frac{n_2}{r}, 1)$.*

Proof. Let G_1 and G_2 be groups of size n_1 and n_2 respectively such that a KTS(n_1) with replication number r and a BIBD$(n_2, 4, \frac{n_2}{r})$ exist. Then a KTS(n_1) has r parallel classes namely: $\pi_1, \pi_2, ..., \pi_r$ each of size $\frac{n_1}{3}$ and $r \mid n_2$. Let $T_1, T_2, ..., T_{\frac{n_2}{r}}$ be partitions of G_2 into parts of size r each.

First we construct the blocks of the required GDD by taking union of the i^{th} element of T_j with each block of π_i for $i = 1, 2, ..., r$ and $j = 1, 2, ..., \frac{n_2}{r}$. These blocks provide blocks of size 4, where pairs from G_1 occur $\frac{n_2}{r}$ times and pairs from different groups occur only once. Then these blocks together with the blocks of a BIBD$(n_2, 4, \frac{n_2}{r})$ on points G_2 provides the blocks for GDD$(n_1, n_2, 4; \frac{n_2}{r}, 1)$. □

Theorem 3.6. *A GDD$(n_1, n_2, 4; \lambda, 1)$ exists for $\lambda \equiv 3 \pmod 6$ when*

(i) $n_1 = 24t + 9$ *and* $n_2 = 36t + 12$, *and*

(ii) $n_1 = 24t + 15$ *and* $n_2 = 36t + 21$.

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. Let $n_1 = 24t + 9$ for a nonnegative integer t. Then a $\mathrm{KTS}(n_1, 3, 1)$ with replication number $r = 12t + 4$ exists. As a $\mathrm{BIBD}(36t + 12, 4, \frac{36t+12}{r} = 3)$ exists, by Corollary 3.5, a $\mathrm{GDD}(n_1, n_2, 4; 3, 1)$ exists. Since a $\mathrm{BIBD}(n_i, 4, 6)$ for $i = 1, 2$ exists, the blocks of a $\mathrm{GDD}(n_1, n_2, 4; 3, 1)$ on points of $G_1 \cup G_2$ together with s copies of the blocks of a $\mathrm{BIBD}(n_i, 4, 6)$ on points of G_i provide the blocks of a $\mathrm{GDD}(n_1, n_2, 4; 3 + 6s, 1)$ for a nonnegative integer s. Similarly, by Corollary 3.5 and Theorem 1.3 we can construct a $\mathrm{GDD}(n_1, n_2, 4; 6s + 3, 1)$ for $n_1 = 24t + 15$ and $n_2 = 36t + 21$. □

Theorem 3.7. *A* $GDD(n_1, n_2, 4; \lambda, 1)$ *exists for* $\lambda \equiv 4 \pmod 6$ *when* $n_1 = 6t + 3$ *and* $n_2 = 12t + 4$.

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. Let $n_1 = 6t+3$. Then a $\mathrm{KTS}(n_1, 3, 1)$ with replication number $r = 3t + 1$ exists. As a $\mathrm{BIBD}(12t + 4, 4, \frac{12t+4}{r} = 4)$ exists, by Corollary 3.5, a $\mathrm{GDD}(n_1, n_2, 4; 4, 1)$ exists. Since a $\mathrm{BIBD}(n_i, 4, 6)$ for $i = 1, 2$ and $n_1 > 3$ exists, the blocks of a $\mathrm{GDD}(n_1, n_2, 4; 4, 1)$ on points of $G_1 \cup G_2$ together with s copies of the blocks of a $\mathrm{BIBD}(n_i, 4, 6)$ on points of G_i provide the blocks of a $\mathrm{GDD}(n_1, n_2, 4; 4 + 6s, 1)$ for a nonnegative integer s. □

Theorem 3.8. *A* $GDD(n_1, n_2, 4;\ \lambda, 1)$ *exists for* $\lambda \equiv 0 \pmod 6$ *when*

(i) $n_1 = 6t + 3$ *and* $n_2 = 18t + 6$, *and*

(ii) $n_1 = 12t + 9$ *and* $n_2 = 36t + 24$.

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. Let $n_1 = 6t+3$. Then a $\mathrm{KTS}(n_1, 3, 1)$ with replication number $r = 3t + 1$ exists. As a $\mathrm{BIBD}(18t + 6, 4, \frac{18t+6}{r} = 6)$ exists, by Corollary 3.5, a $\mathrm{GDD}(n_1, n_2, 4; 6, 1)$ exists. Since a $\mathrm{BIBD}(n_i, 4, 6)$ for $i = 1, 2$ and $n_1 > 3$ exists, the blocks of a $\mathrm{GDD}(n_1, n_2, 4; 6, 1)$ on points of $G_1 \cup G_2$ together with s copies of the blocks of a $\mathrm{BIBD}(n_i, 4, 6)$ on points of G_i provide the blocks of a $\mathrm{GDD}(n_1, n_2, 4; 6 + 6s, 1)$ for a nonnegative integer s. Similarily, by Corollary 3.5 and Theorem 1.3 we can construct a $\mathrm{GDD}(n_1, n_2, 4; 6s + 6, 1)$ when $n_1 = 12t + 9$ and $n_2 = 36t + 24$. □

Example 3.9. As a KTS(9) with replication number 4 and a $\mathrm{BIBD}(12,\ 4, 12/4 = 3)$ exist, we have a $\mathrm{GDD}(9, 12, 4; 3, 1)$.

Example 3.10. As a KTS(15) with repliation number 7 and a $\mathrm{BIBD}(21, 4,\ 21/7 = 3)$ exist, we have a $\mathrm{GDD}(15, 21, 4; 3, 1)$

Theorem 3.11. *If a* $BIBD(\frac{(n-1)(n-2)}{2}, 4, n-2)$ *exists, then a* $GDD(n, \frac{(n-1)(n-2)}{2}, 4; n-2, 1)$ *exists for* $n \equiv 0 \pmod 3$.

Proof. Let G_1 be a set with n elements. For $n \equiv 0 \pmod 3$, a BIBD$(n, 3, n-2)$ is resolvable. By Corollary 1.9, a BIBD$(n, 3, n-2)$ has $N = \frac{(n-1)(n-2)}{2}$ parallel classes; each class consists of $\frac{n}{3}$ mutually disjoint blocks which partitions G_1. Let $\pi_1, \pi_2, \ldots, \pi_N$ be the resolution classes. Let G_2 be a set with N elements. Then the union of each block of π_i with the i^{th} element of G_2 provides $\frac{n(n-1)(n-2)}{6}$ blocks of size 4 in which pairs from G_1 occur $n-2$ times while pairs from G_1 and G_2 occur only once. If a BIBD$(\frac{(n-1)(n-2)}{2}, 4, n-2)$ exists, then the blocks of a BIBD$(\frac{(n-1)(n-2)}{2}, 4, n-2)$ on G_2 along with the above blocks just constructed provide the required GDD. □

Example 3.12. A GDD$(6, 10, 4; 4, 1)$ exists.

Proof. Let $G_1 = \{1, 2, 3, 4, 5, 6\}$ and $G_2 = \{a, b, c, d, e, f, g, h, i, j\}$ be two groups of sizes 6 and 10 respectively. By Example 1.10 a BIBD(6,3,4) on G_1 has 10 resolution classes, each with 2 pairwise disjoint blocks which partition G_1. Let $\pi_1, \pi_2, \ldots, \pi_{10}$ be the parallel classes. Then by taking the union of each block of π_l with the l^{th} element of G_2 for $l = 1, 2, \ldots, 10$, we have the following blocks of size 4 in which pairs of elements from G_1 occur 4 times while pairs from both groups occur only once.

1	4	1	3	1	3	1	3	1	2	1	2	1	2	1	2	1	2	1	2
2	5	2	5	2	4	2	4	3	5	3	4	3	4	4	3	4	3	5	3
3	6	4	6	5	6	6	5	4	6	5	6	6	5	5	6	6	5	6	4
a	a	b	b	c	c	d	d	e	e	f	f	g	g	h	h	i	i	j	j

Since a BIBD$(10, 4, 4)$ exists by Theorem 1.3, hence the blocks of BIBD$(10, 4, 4)$ on G_2 together with the above blocks just constructed provide the blocks of a GDD$(6, 10, 4; 4, 1)$. □

4. Conclusion

We obtained necessary conditions for the existence of a GDD$(n_1, n_2, 4; \lambda, 1)$ and proved that the necessary conditions were sufficient in several cases. Further, we presented several new general constructions for GDDs with two unequal groups of sizes n_1 and n_2 with block size k as well as block size 4 and $\lambda_2 = 1$.

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