J. COMBIN. MATH. COMBIN. COMPUT. 131 (2026) 237–251

Journal of Combinatorial Mathematics
and Combinatorial Computing
www.combinatorialpress.com/jcmcc

Combinatorial Press

Invalid proofs in "edge k-product cordial labeling of graphs" and their revision

Wai Chee Shiu and Gee-Choon Lau*

ABSTRACT

This paper identifies and corrects invalid arguments in [Edge k-Product Cordial Labeling of Graphs, *Eur. J. Pure Appl. Math.*, **18(2)**, Article No. 5887 (2025)]. In particular, several published statements are shown to be false through explicit counterexamples. Revised versions of these results are established under appropriate conditions, often involving structural properties such as girth. Correct bounds or conditions are obtained for specific graph classes, including shadow graphs, splitting graphs of stars, and path unions of cycles. These corrections clarify the limitations of the earlier claims and provide a more accurate foundation for further study of edge k-product cordial labeling.

Keywords: edge product cordial

2020 Mathematics Subject Classification: 05C78, 05C15.

1. Introduction and general properties

A graph of order p and of size q is called a (p, q)-graph. For $k \geq 2$, we let $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ be the standard complete residue system modulo k. Suppose $G = (V, E)$ is a (p, q)-graph without isolated vertex. Let $f : E \to \mathbb{Z}_k$ be an edge labeling of G. Let the vertex labeling $f^* : V \to \mathbb{Z}_k$ be defined by $f^*(v) \equiv \prod_{uv \in E} f(uv) \pmod{k}$. An edge $e \in E$ and a vertex $v \in V$ are called an *i-edge* and *j-vertex* (under f), if $f(e) = i$ and $f^*(v) = j$, respectively, where $i, j \in \mathbb{Z}_k$. Let $e_f(i)$ and $v_f(j)$ denote the number of i-edges and j-vertices, respectively.

For $k \geq 2$, an edge labeling $f : E \to \mathbb{Z}_k$ of $G = (V, E)$ is said to be an *edge k-product cordial labeling of G* if $|e_f(i_1) - e_f(i_2)| \leq 1$ and $|v_f(j_1) - v_f(j_2)| \leq 1$ for $i_1, i_2, j_1, j_2 \in \mathbb{Z}_k$.

* Corresponding author.

Received 02 Apr 2026; Revised 19 May 2026; Accepted 10 Jun 2026; Published Online 20 Jul 2026.

DOI: 10.61091/jcmcc131-12

We say G is *edge k-product cordial* if G admits an edge k-product cordial labeling. In particular, if f is an edge k-product cordial labeling of G, then

$$\left\lfloor \frac{|E|}{k} \right\rfloor \le e_f(i) \le \left\lceil \frac{|E|}{k} \right\rceil \quad \text{and} \quad \left\lfloor \frac{|V|}{k} \right\rfloor \le v_f(j) \le \left\lceil \frac{|V|}{k} \right\rceil, \tag{1}$$

for all $i, j \in \mathbb{Z}_k$.

In [3], the authors repeatedly used the incorrect inequality "$v_f(0) \ge e_f(0) + 1$" without proper justification. As a result, several statements in [3] are false. In this paper, we idenfity these incorrect statements and provide explicit counterexamples. Corrected statements and proofs are also given.

Statement 1.1. [3, Theorem 5] A graph G with $k \le |V| \le |E|$ does not admit an edge k-product cordial labeling if $|V| \equiv 0 \pmod{k}$.

We now present counterexamples to Statement 1.1.

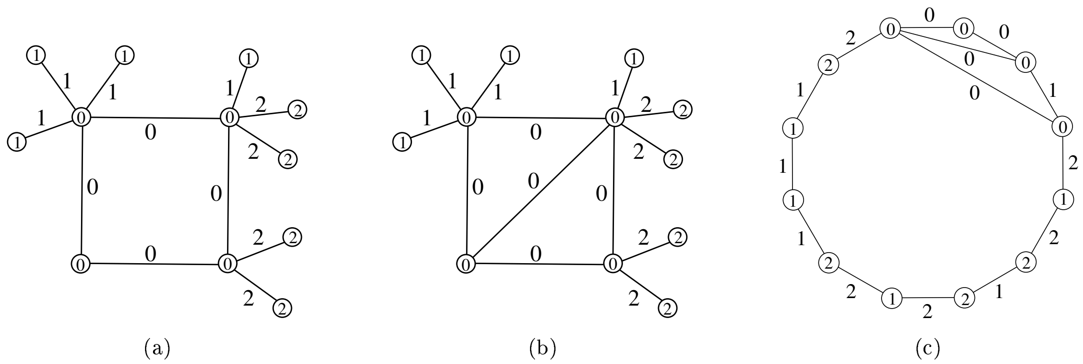


Fig. 1. Edge 3-product cordial labelings f for graphs of order 12 and sizes 12, 13, 14

For convenience in verifying the cordiality conditions, we introduce the notation

$$E_f = (e_f(0), e_f(1), \ldots, e_f(k-1)), \quad V_f = (v_f(0), v_f(1), \ldots, v_f(k-1)),$$

which record the distributions of edge labels and vertex labels, respectively. For example, in Figure 1(a), $E_f = (4,4,4)$ and $V_f = (4,4,4)$; in Figure 1(b), $E_f = (5,4,4)$ and $V_f = (4,4,4)$; and in Figure 1(c), $E_f = (4,5,5)$ and $V_f = (4,4,4)$.

Moreover, there exist infinitely many counterexamples to Statement 1.1. Let H_n be the graph obtained from $C_n = u_1u_2 \cdots u_nu_1$ by adding one chord. Let $v_1, v_2, \ldots, v_{(k-1)n}$ be additional vertices, where $k \ge 2$. For each v_j, add an edge connecting v_j to an arbitrary vertex of H_n. Denote the resulting graph by G_n. So G_n is a $(kn, kn+1)$-graph.

Partition the $(k-1)n$ pendant edges into $k-1$ groups of size n, denoted $Y_1 \ldots, Y_{k-1}$. Label all edges of H_n by 0 and label the edges in Y_i by i, for $1 \le i \le k-1$. Let f be this labeling. Thus,

$$e_f(0) = n+1, e_f(i) = n \text{ for } 1 \le i \le k-1, \text{ and } v_f(j) = n \text{ for all } j.$$

Hence, G_n is edge k-product cordial. Figure 1(b) shows an example for $k = 3$ and $n = 4$.

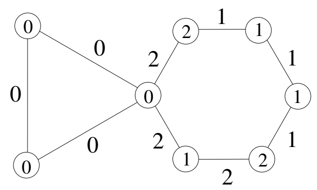
If the chord is removed from G_n, we obtain a class of (kn, kn)-graphs that are edge k-product cordial. Figure 1(a) shows an example for $k = 3$ and $n = 4$.

Therefore, all statements in [3] that rely on Statement 1.1 are invalid. In the subsequent sections, we provide corrected proofs for several affected results.

We now introduce some notation. Suppose $f : E \to \mathbb{Z}_k$ is an edge labeling of $G = (V, E)$. Let E_0 be the set of all 0-edges under f and let $G_0 = G[E_0]$ be the subgraph induced by E_0. Clearly $e_f(0) = |E_0|$ and $v_f(0) \geq |V(G_0)|$. The equality holds if k is prime. Otherwise, there may be 0-vertices not in G_0. For example, the induced label of a vertex incident to a 2-edge and a 3-edge is zero when $k = 6$. These notations will be used throughout the paper. Notation and concepts not defined in this paper are referred to the book [1].

Statement 1.2. [3, Theorem 19] Any graph $G = (V, E)$ with $\left\lfloor \frac{|V|}{k} \right\rfloor < \left\lfloor \frac{|E|}{k} \right\rfloor$ does not admit an edge k-product cordial labeling.

A counterexample to Statement 1.2 is shown below.



Here $|V|= 8$, $|E|= 9$ and $k = 3$. We have $2 = \left\lfloor \frac{|V|}{3} \right\rfloor < \left\lfloor \frac{|E|}{3} \right\rfloor = 3$ with $E_f = (3, 3, 3)$ and $V_f = (3, 3, 2)$ where f denotes the above labeling.

Lemma 1.3. *Suppose g is the girth of G and f is an edge labeling from $E(G) \to \mathbb{Z}_k$. If $e_f(0) < g$, then $v_f(0) \geq e_f(0) + 1$.*

Proof. Since $e_f(0) < g$, G_0 is acyclic. Thus, $|V(G_0)| \geq |E_0| + 1$. Hence $v_f(0) \geq e_f(0) + 1$. $\square$

We provide the following correction to Statement 1.2.

Theorem 1.4. *Let $G = (V, E)$ be a graph whose girth is greater than $\left\lceil \frac{|E|}{k} \right\rceil$. If $\left\lfloor \frac{|V|}{k} \right\rfloor < \left\lfloor \frac{|E|}{k} \right\rfloor$, then G is not edge k-product cordial.*

Proof. We prove the contrapositive of the theorem. Suppose G admits an edge k-product cordial labeling f. Since the girth of G is greater than $\left\lceil \frac{|E|}{k} \right\rceil$ and $e_f(0) = |E_0| \leq \left\lceil \frac{|E|}{k} \right\rceil$, G_0 is acyclic. Thus $v_f(0) \geq |V(G_0)| \geq |E_0| + 1 = e_f(0) + 1$.

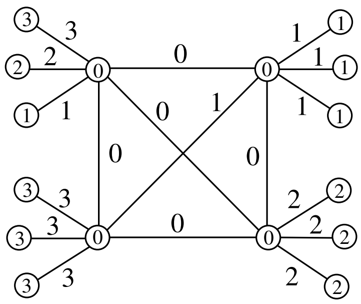
Since $e_f(0) \geq \left\lfloor \frac{|E|}{k} \right\rfloor$ and $v_f(0) \leq \left\lfloor \frac{|V|}{k} \right\rfloor + 1$, $\left\lfloor \frac{|V|}{k} \right\rfloor + 1 \geq \left\lceil \frac{|V|}{k} \right\rceil \geq v_f(0) \geq e_f(0) + 1 \geq \left\lfloor \frac{|E|}{k} \right\rfloor + 1$. Thus $\left\lfloor \frac{|V|}{k} \right\rfloor \geq \left\lfloor \frac{|E|}{k} \right\rfloor$. $\square$

Lemma 1.5. *For each $n \geq 3$ and $k \geq 2$, there is a graph admitting an edge k-product cordial labeling f such that $e_f(0) - v_f(0) = \left\lceil \frac{n^2-3n}{2k} \right\rceil$.*

Proof. Let G be the graph obtained from K_n by attaching $(k-1)n$ pendant edges arbitrary. So G is a $(kn, kn + \frac{1}{2}(n^2 - 3n))$-graph. We label an n-cycle in K_n by 0, and label all pendant edges by $1, 2, \ldots, k-1$ evenly. At this moment, the number of i-edges is n, $0 \leq i \leq k-1$. We still have $\frac{1}{2}(n^2 - n) - n = \frac{1}{2}(n^2 - 3n)$ unlabeled edges in K_n. Lastly, we label them by $0, 1, \ldots, k-1$ in order and evenly. Let the resulting labeling be f. This construction guarantees that the difference between the number of i-edges and j-edges is at most 1 when $i \neq j$ and $e_f(0) = n + \left\lceil \frac{n^2-3n}{2k} \right\rceil$. Clearly, $v_f(j) = n$ for $0 \leq j \leq k-1$. Thus f is an edge k-product cordial labeling of G. Hence, we have the lemma. $\square$

Thus, for any fixed k, there is a graph admitting an edge k-product cordial labeling f such that $e_f(0) - v_f(0)$ is arbitrarily large.

Example 1.6. Let $n = 4$ and $k = 4$. According to the proof of Lemma 1.5, we have the following labeling f.



Clearly, $E_f = (5, 5, 4, 4)$ and $V_f = (4, 4, 4, 4)$.

2. Shadow graph of star

Let the vertex and the edge sets of the star $K_{1,n}$ be $\{u, u_1, \ldots, u_n\}$ and $\{uu_i \mid 1 \leq i \leq n\}$, respectively. The shadow graph of $K_{1,n}$, denoted $D(K_{1,n})$, has vertex set $\{u, u'\} \cup \{u_i, u'_i \mid 1 \leq i \leq n\}$ and edge set $\{uu_i, uu'_i, u'u_i, u'u'_i \mid 1 \leq i \leq n\}$. Thus, $D(K_{1,n})$ is a $(2n+2, 4n)$-graph.

Theorem 2.1. *Let $n \geq 1$ and $k \geq 2$. Suppose $D(K_{1,n})$ admits an edge k-product cordial labeling f, then $k \geq 4n+1$ or $k = 2n+1$.*

Proof. Let $G = D(K_{1,n})$. Note that G is a $(2n+2, 4n)$-graph. Thus, $e_f(0) \geq \left\lfloor \dfrac{4n}{k} \right\rfloor$ and $v_f(0) \leq \left\lceil \dfrac{2n+2}{k} \right\rceil$.

(a) If G_0 is acyclic, then $v_f(0) \geq |V(G_0)| \geq |E(G_0)| + 1 = e_f(0) + 1$. We have $\left\lceil \dfrac{2n+2}{k} \right\rceil \geq \left\lfloor \dfrac{4n}{k} \right\rfloor + 1$ so that $\dfrac{2n+2}{k} + 1 \geq \left\lceil \dfrac{2n+2}{k} \right\rceil \geq \left\lfloor \dfrac{4n}{k} \right\rfloor + 1 > \dfrac{4n}{k}$. Thus, $k > 2n - 2$, i.e., $k \geq 2n-1$.

1. Consider $k \geq 2n+2$. If $k \geq 2n+3$, then $v_f(0) = 0, 1$. If $k = 2n+2$, then $v_f(i) = 1$ for $0 \leq i \leq 2n+1$. If $e_f(0) = 1$, then $v_f(0) \geq 2$ since the end-vertices of a 0-edge are 0-vertices. It is a contradiction. Thus, $e_f(0) = 0$. This implies that $e_f(i) = 0, 1$, where $1 \leq i \leq k-1$. So $D(K_{1,n})$ contains at most $k-1$ edges. Therefore, $k \geq 4n+1$.
2. Suppose $k = 2n+1$ so that $v_f(i) = 1, 2$ for $0 \leq i \leq k-1$. If $e_f(0) = 2$, then $v_f(0) \geq 3$. It is a contradiction. Thus, $e_f(0) = 1$ and $v_f(0) = 2$. Hence $v_f(i) = 1$ for $1 \leq i \leq k-1$.
3. Suppose $k = 2n$ so that $e_f(i) = 2$ for all i. If $n = 1$, then $D(K_{1,1}) \cong C_4$ which is not edge 2-product cordial. If $n \geq 2$, then $v_f(j) \leq 2$ for all j. However, $e_f(0) = 2$ implies that $v_f(0) \geq 3$, a contradiction.
4. Suppose $k = 2n-1$. Now, $k \geq 2$ implies that $n \geq 2$.

   If $n \geq 3$, then $v_f(0) = 1, 2$ and $e_f(0) = 2, 3$. Since $e_f(0) \geq 2$, $v_f(0) \geq 3$. It is a contradiction.

   If $n = 2$, then $k = 3$, $e_f(0) = 2, 3$ and $v_f(j) = 2$ for $j = 0, 1, 2$. It is also impossible.

(b) If G_0 contains a cycle, then both u and u' are vertices of G_0. Let $X = N_{G_0}(u) \cap N_{G_0}(u')$ and $Y = V(G_0) \setminus (\{u', u\} \cup X)$. Note that $|X| \geq 2$.

$$\begin{aligned} 2e_f(0) &= \deg_{G_0}(u) + \deg_{G_0}(u') + \sum_{x \in X} \deg_{G_0}(x) + \sum_{y \in Y} \deg_{G_0}(y) \\ &= (2|X| + |Y|) + 2|X| + |Y| = 4|X| + 2|Y|. \end{aligned}$$

So $e_f(0) = 2|X| + |Y|$. On the other hand,

$$\begin{aligned} 1 + \frac{2n+2}{k} > \left\lceil \frac{2n+2}{k} \right\rceil &\geq v_f(0) \\ &\geq |V(G_0)| = |X| + |Y| + 2 = \frac{e_f(0) - |Y|}{2} + |Y| + 2 \\ &= \frac{e_f(0)}{2} + 2 + \frac{|Y|}{2} \geq \frac{e_f(0)}{2} + 2 \\ &\geq \frac{1}{2} \left\lfloor \frac{4n}{k} \right\rfloor + 2 > \frac{1}{2}\left(\frac{4n}{k} - 1\right) + 2 = \frac{2n}{k} + \frac{3}{2}. \end{aligned}$$

We have $\frac{2}{k} > \frac{1}{2}$, i.e., $k = 2, 3$.

Suppose $k = 2$ so that $e_f(i) = 2n$ and $v_f(j) = n+1$ for all i, j. Now, $n + 1 = |X| + |Y| + 2 = \dfrac{e_f(0)}{2} + 2 + \dfrac{|Y|}{2} = n + 2 + \dfrac{|Y|}{2}$, which is impossible.

Suppose $k = 3$.

1. If $n = 3l + 2$ for some $l \geq 0$, then $\left\lfloor \dfrac{4n}{3} \right\rfloor = 4l + 2$ and $v_f(0) = \dfrac{6l+6}{3} = 2l + 2$.

   Thus, $2l + 2 \geq |V(G_0)| \geq \dfrac{e_f(0)}{2} + 2 \geq \dfrac{4l+2}{2} + 2$, which is impossible.

2. If $n = 3l+1$ for some $l \geq 0$, then $\left\lfloor \dfrac{4n}{3} \right\rfloor = 4l+1$ and $2l+2 = \left\lceil \dfrac{6l+4}{3} \right\rceil \geq v_f(0)$. Thus, $2l+2 \geq |V(G_0)| \geq \dfrac{e_f(0)}{2} + 2 \geq \dfrac{4l+1}{2} + 2$, which is impossible.
3. If $n = 3l$ for some $l \geq 1$, then $\left\lfloor \dfrac{4n}{3} \right\rfloor = 4l$ and $2l+1 = \left\lceil \dfrac{6l+2}{3} \right\rceil \geq v_f(0)$. Thus, $2l+1 \geq |V(G_0)| \geq \dfrac{e_f(0)}{2} + 2 \geq \dfrac{4l}{2} + 2$, which is impossible.

Consequently, either $k \geq 4n+1$ or $k = 2n+1$. □

Remark 2.2. 1) From the proof of Theorem 2.1, G_0 is acyclic if G is edge k-product cordial.

2) For $k \geq 4n+1$, Figure 2(a) shows an edge 9-product cordial labeling of $D(K_{1,2})$.
3) For $k = 2n+1$, label the edges of $D(K_{1,1}) \cong C_4$ by 0, 2, 1, 1. Thus, the vertex labels are 0, 0, 2, 1. Therefore, $D(K_{1,1})$ is edge 3-product cordial. Moreover, Figure 2(b) shows an edge 5-product cordial labeling of $D(K_{1,2})$.

(a) Edge 9-product cordial labeling f of $D(K_{1,2})$ (b) Edge 5-product cordial labeling g of $D(K_{1,2})$

Fig. 2.

Hence, Figure 2(a) has $E_f = (0,1,1,1,1,1,1,1,1)$ and $V_f = (1,1,1,1,0,0,1,1,0)$; Figure 2(b) has $E_g = (1,2,1,2,2)$ and $V_g = (2,1,1,1,1)$.

4) Note that Theorem 2.1 implies that Theorem 6 in [3] is true although the proof contains an error. By applying the sufficient part of the corresponding proofs in [3], Figure 2(b) and Theorem 2.1, we also obtain Theorems 7, 8, and 9 in [3].

Combining Theorem 2.1 and the results in [3], we have

Theorem 2.3. *Suppose $k = 3, 5$. $D(K_{1,n})$ is edge k-product cordial if and only if $n = \frac{k-1}{2}$.*

Corollary 2.4. *$D(K_{1,1}) \cong C_4$ is edge k-product cordial if and only if $k = 3$ or $k \geq 6$.*

Proof. From the proof of Theorem 9 in [3], the graph C_4 is not edge 5-product cordial. Combining this with Theorem 2.1, we obtain the necessity.

An edge 3-product cordial labeling of C_4 is given in Remark 2.2(3). For $k \geq 6$, label the edges of C_4 by $1, 3, 2, 4$ in cyclic order. The induced vertex labels are $3, 6, 8, 4$, which are pairwise distinct modulo k. Hence, this labeling is an edge k-product cordial, proving the sufficiency. $\square$

Question 2.5. *Is $D(K_{1,n})$ edge $(2n+1)$-product cordial for $n \geq 3$?*

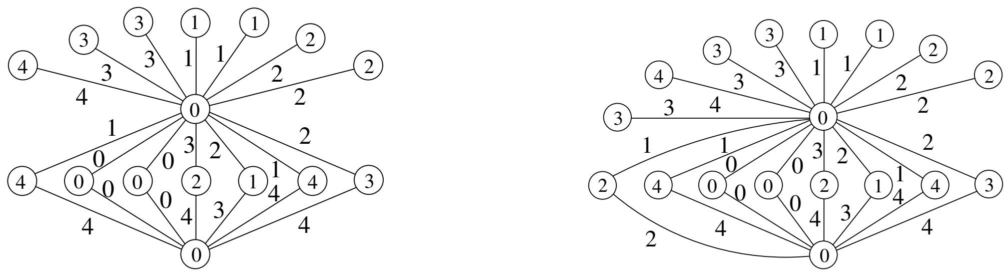
Question 2.6. *Is $D(K_{1,n})$ edge k-product cordial for $n \geq 2$ when $k \geq 4n+1$?*

3. Splitting graph of Star

In this section, $S(K_{1,n})$ denotes the splitting graph of the star $K_{1,n}$. The vertex set of $S(K_{1,n})$ is $\{u, v\} \cup \{u_i, v_i \mid 1 \leq i \leq n\}$ and its edge set is $\{uu_i, vu_i, uv_i \mid 1 \leq i \leq n\}$. Thus $S(K_{1,n})$ is a $(2n+2, 3n)$-graph.

Statement 3.1. [3, Theorem 10] The graph $S(K_{1,n})$ does not admit an edge k-product cordial labeling for $k \leq n$.

This statement is false, as demonstrated by explicit counterexamples below.



(a) Edge 5-product cordial labeling f of $S(K_{1,7})$ (b) Edge 5-product cordial labeling g of $S(K_{1,8})$

Fig. 3.

Here, $E_f = (4, 4, 4, 4, 5)$, $V_f = (4, 3, 3, 3, 3)$; and $E_g = (4, 5, 5, 5, 5)$, $V_g = (4, 3, 4, 4, 3)$.

Theorem 3.2. *Let $n \geq 1$ and $k \geq 2$. Suppose $G = S(K_{1,n})$ admits an edge k-product cordial labeling f.*

(1) *If G_0 is acyclic, then $k \geq 3n+1$ with $e_f(0) = 0$ or $n+2 \leq k \leq 2n+1$ with $e_f(0) = 1$.*

(2) *If G_0 contains a cycle, then $3k \leq 2n+1$.*

Proof. Since G is a $(2n+2, 3n)$-graph, we have $e_f(0) \geq \left\lfloor \dfrac{3n}{k} \right\rfloor$ and $\left\lfloor \dfrac{2n+2}{k} \right\rfloor \leq v_f(0) \leq \left\lceil \dfrac{2n+2}{k} \right\rceil$.

(a) Suppose G_0 is acyclic, then $v_f(0) \geq e_f(0)+1$. By a similar argument as in the proof of Theorem 2.1, we have $k > n-2$, i.e., $k \geq n-1$.

1. Suppose $k = n-1$. Since $k \geq 2$, $n \geq 3$. Now,

$$\left\lceil \frac{2n+2}{n-1} \right\rceil = \left\lceil 2 + \frac{4}{n-1} \right\rceil = \begin{cases} 4 & \text{if } n = 3, 4; \\ 3 & \text{if } n \geq 5, \end{cases}$$

$$e_f(0) \geq \left\lfloor \frac{3n}{n-1} \right\rfloor = \left\lfloor 3 + \frac{3}{n-1} \right\rfloor = \begin{cases} 4 & \text{if } n = 3, 4; \\ 3 & \text{if } n \geq 5. \end{cases}$$

   Since $v_f(0) \geq e_f(0) + 1$, it is impossible.

2. Suppose $k = n$. Since $n \geq 3$,

$$e_f(0) \geq \left\lfloor \frac{3n}{n} \right\rfloor = 3 \text{ and } v_f(0) \leq \left\lceil \frac{2n+2}{n} \right\rceil = 3.$$

   Since $v_f(0) \geq e_f(0) + 1$, it is impossible.

3. Suppose $k = n+1$. Thus,

$$v_f(j) = \frac{2n+2}{n+1} = 2 \text{ for all } j,$$
$$e_f(0) \geq \left\lfloor \frac{3n}{n+1} \right\rfloor = \left\lfloor 2 + \frac{n-2}{n+1} \right\rfloor = 2.$$

   It is impossible.

4. Suppose $k \geq n+2$.

$$v_f(0) \leq \left\lceil \frac{2n+2}{k} \right\rceil \leq \left\lceil \frac{2n+2}{n+2} \right\rceil = 2$$

$$\left\lfloor \frac{3n}{k} \right\rfloor \leq \left\lfloor \frac{3n}{n+2} \right\rfloor = \left\lfloor 2 + \frac{n-4}{n+2} \right\rfloor = \begin{cases} 1 & \text{if } n = 2, 3; \\ 2 & \text{if } n \geq 4. \end{cases}$$

   Since $v_f(0) \geq e_f(0) + 1$, $e_f(0) \leq 1$.

   Suppose $e_f(0) = 0$. Then $e_f(i) \leq 1$ for $1 \leq i \leq k-1$. Hence G has at most $k-1$ edges, i.e., $k \geq 3n+1$.

   Suppose $e_f(0) = 1$, then $v_f(0) \geq 2$ and hence $v_f(j) \geq 1$ for $1 \leq j \leq k-1$. So G has at least $2 + (k-1) = k+1$ vertices, i.e., $k \leq 2n+1$.

(b) If G_0 contains a cycle, then both u and v are vertices of G_0. Hence $e_f(0) \geq 4$ and $v_f(0) \geq 4$.

Therefore, $\frac{2n+2}{k} + 1 > \left\lceil \frac{2n+2}{k} \right\rceil \geq v_f(0) \geq 4$. Thus, $3k < 2n+2$, i.e., $3k \leq 2n+1$.

□

Example 3.3. The examples in Figure 4 to Figure 6 verify conclusion (1) of Theorem 3.2.

When $e_f(0) = 0$, we have an edge 8-product cordial labeling for $S(K_{1,2})$ and an edge 10-product cordial labeling for $S(K_{1,3})$. Clearly, there is an edge 5-product cordial labeling for $S(K_{1,1}) \cong P_4$.

(a) Edge 8-product cordial labeling f of $S(K_{1,2})$ (b) Edge 10-product cordial labeling g of $S(K_{1,3})$

Fig. 4.

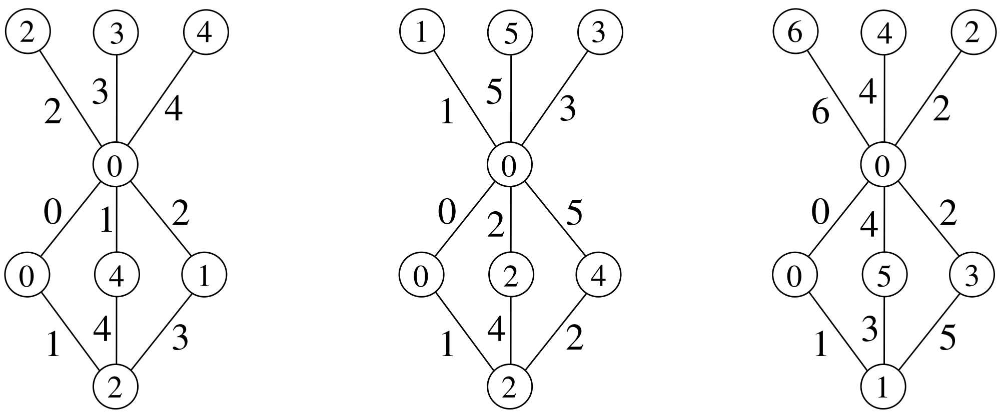
Here, Figure 4(a) has $E_f = (0,1,1,1,1,1,1,0)$, $V_f = (1,0,1,1,1,0,1,1)$; and Figure 4(b) has $E_g = (0,1,1,1,1,1,1,1,1,1)$, $V_g = (1,1,1,1,1,1,1,0,1,0)$.

When $e_f(0) = 1$, we have the following edge k-product cordial labelings for $S(K_{1,n})$ with $n = 2, 3$. Actually, when $n = 2$, $4 \leq k \leq 5$ and when $n = 3$, $5 \leq k \leq 7$.

(a) Labeling f (b) Labeling g

Fig. 5. Edge k-product cordial labeling of $S(K_{1,2})$, for $k = 4, 5$.

Here, Figure 5(a) has $E_f = (1,2,2,1)$, $V_f = (2,1,2,1)$; and Figure 5(b) has $E_g = (1,2,1,1,1)$, $V_g = (2,1,1,1,1)$.

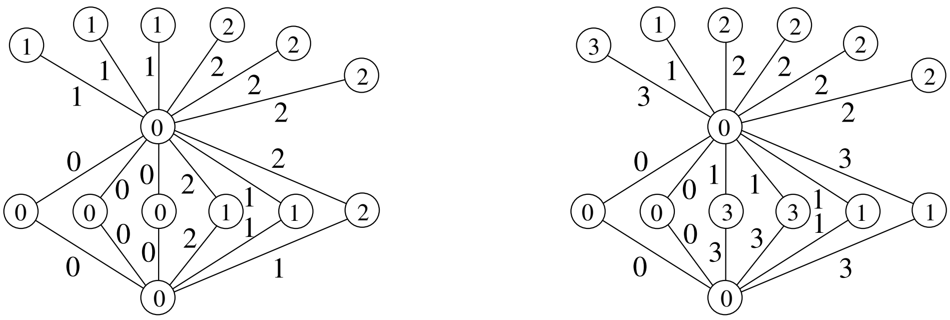


(a) Labeling f (b) Labeling g (c) Labeling h

Fig. 6. Edge k-product cordial labeling of $S(K_{1,3})$, for $k = 5, 6, 7$

Here, Figure 6(a) has $E_f = (1,2,2,2,2)$, $V_f = (2,1,2,1,2)$; Figure 6(b) has $E_g = (1,2,2,1,1,2)$, $V_g = (2,1,2,1,1,1)$; and Figure 6(c) has $E_h = (1,1,2,1,2,1,1)$, $V_h = (2,1,1,1,1,1,1)$.

Note that Figure 3 gives two examples that verify conclusion (2) of Theorem 3.2 when $k = 5$. Two more examples in Figure 7 verify conclusion (2) of Theorem 3.2 for $k = 3, 4$.



(a) $k = 3$ (b) $k = 4$

Fig. 7. Edge k-product cordial labelings f of $S(K_{1,6})$, for $k = 3, 4$

Here, Figure 7(a) has $E_f = (6,6,6)$, $V_f = (5,5,4)$; and Figure 7(b) has $E_f = (4,5,4,5)$, $V_f = (4,3,4,3)$.

3.1. Edge 3-product cordial

In this subsection, we point out some mistakes about edge 3-product cordial behavior of $S(K_{1,n})$ as shown in [3] and make some correction.

Statement 3.4. [3, Theorem 11] The graph $S(K_{1,n})$ admits an edge 3-product cordial labeling if and only if $n = 1$.

Statement 3.4 is incorrect, since there is an edge 3-product cordial labeling for the graph $S(K_{1,6})$ as shown in Figure 7(a).

Theorem 3.5. *If $G = S(K_{1,n})$ is edge 3-product cordial, then $n = 1$ or $n \geq 6$.*

Proof. By applying Theorem 3.2, we have the following results.

1. If G_0 is acyclic, then $n + 2 \leq 3 \leq 2n + 1$. Hence $n = 1$.

2. If G_0 contains a cycle, then $9 \leq 2n + 1$. Thus, $n \geq 4$.

   Consider $G = S(K_{1,4})$ which is a $(10, 12)$-graph. Suppose f is an edge 3-product cordial labeling. So $e_f(i) = 4$, for $i = 0, 1, 2$. Since G_0 contains a cycle, G_0 is a 4-cycle. Since $3 \leq v_f(0) \leq 4$, $v_f(0) = 4$. Thus, $v_f(1) = v_f(2) = 3$. Without loss of generality, we may assume that G_0 is the 4-cycle uu_1vu_2u.

   Suppose there is only one non-pendant 2-edge, then this edge is incident to a degree 2 vertex, say u_3. Thus, another edge incident to u_3 is a 1-edge. Therefore, $v_f(2) = 4$, a contradiction.

Suppose there are exactly two non-pendant 2-edges. If both are incident to u_3 (or else u_4), then $v_f(1) = 4$, a contradiction. Otherwise, $v_f(2) = 4$, also a contradiction.

Suppose there are at least three non-pendant 2-edges, then $v_f(1) \geq 4$, a contradiction.

Thus, f does not exist.

Consider $G = S(K_{1,5})$ which is a $(12, 15)$-graph. Suppose f is an edge 3-product cordial labeling. From (1), we have $e_f(0) = 5$ and $v_f(0) = 4$. Since every vertex incident to a 0-edge is also a 0-vertex, we must have $|V(G_0)|= 4$ so that $G_0 \cong K_{1,1,2}$. However, $S(K_{1,5})$ does not contain such a subgraph. Thus $n \geq 6$.

□

Remark 3.6. Label the edges of $S(K_{1,1}) \cong P_4$ by 0,2,1 in order we get an edge 3-product cordial labeling of it. Figure 7(a) shows an edge 3-product cordial labeling of $S(K_{1,6})$.

Finding a necessary and sufficient condition for $S(K_{1,n})$ being edge 3-product cordial is not a purpose of this paper. So we leave this problem to interested readers.

3.2. Edge 4-product cordial

Next, we study edge 4-product cordiality of $S(K_{1,n})$.

Statement 3.7. [3, Theorem 12] The graph $S(K_{1,n})$ admits an edge 4-product cordial labeling if and only if $n = 2$.

Statement 3.7 is incorrect, since there is an edge 4-product cordial labeling for the graph $S(K_{1,6})$ as shown in Figure 7(b).

Theorem 3.8. *If $G = S(K_{1,n})$ is edge 4-product cordial, then $n = 2$ or $n \geq 6$.*

Proof. Suppose $G = S(K_{1,n})$ admits an edge 4-product cordial labeling f. From Theorem 3.2, we have $n = 1$ with $e_f(0) = 0$, or $n = 2$, or $n \geq 6$. To complete the proof, we need only consider $n = 1$ with $e_f(0) = 0$ so that $G \cong P_4$ with $V_f = (1, 1, 1, 1)$. Since $E_f = (0, 1, 1, 1)$, it is routine to check that $v_f(0) = 0$, a contradiction. □

Remark 3.9. To verify the above theorem, there is an edge 4-product cordial labeling of $S(K_{1,2})$ which was shown in the proof of Theorem 12 of [3]; and Figure 7(b).

3.3. Edge 5-product cordial

Figure 3(a) and Figure 3(b) show that the following statement is incorrect.

Statement 3.10. [3, Theorem 13] The graph $S(K_{1,n})$ admits an edge 5-product cordial labeling if and only if $n \leq 3$.

From Theorem 3.2 we have the following result immediately.

Theorem 3.11. *If $G = S(K_{1,n})$ is edge 5-product cordial, then $1 \le n \le 3$ or $n \ge 7$.*

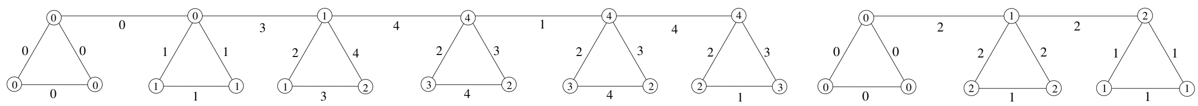
4. Path union of cycle

Let $G_1, \dots G_n$, $n \ge 2$, be n copies of a graph G. Let v be a vertex of G and $v_i \in V(G_i)$, $i = 1, \dots, n$, be the vertex corresponding to v. The graph $P(n.G^v)$ is the graph obtained from $G_1 + \cdots + G_n$ by adding the edge $v_i v_{i+1}$, $1 \le i \le n-1$. $P(n.G^v)$ is called the *path union of n copies of the graph G* [2].

Suppose $G = C_m$, $m \ge 3$. By symmetry, all $P(n.C_m^v)$ are isomorphic. We shall use $P(n.C_m)$ instead of $P(n.C_m^v)$. Note that $P(n.C_m)$ is an $(mn, mn+n-1)$-graph. We let the i-th copy of C_m be $v_{i,1}v_{i,2}\cdots v_{i,m}v_{i,1}$. We also let $v_{1,1}v_{2,1}\cdots v_{n,1}$ be a subpath of $P(n.C_m)$.

Statement 4.1. [3, Theorem 20] The graph $P(n.C_m)$ does not admit an edge k-product cordial labeling if $n \ge k$.

In Figure 8, we give two counterexamples to Statement 4.1.



(a) $n = 6$, $m = 3$ and $k = 5$

(b) $n = m = k = 3$

Fig. 8. Edge k-product cordial labeling f of $P(n.C_3)$

Here, Figure 8(a) has $E_f = (4, 5, 4, 5, 5)$, $V_f = (4, 4, 4, 3, 3)$; and Figure 8(b) has $E_f = (3, 4, 4)$, $V_f = (3, 3, 3)$.

Since Statement 4.1 is false, the proofs of Theorems 21, 22, and 23 in [3] are also invalid. We note that Theorems 21, 22, and 23 are in fact incorrect. Counterexamples to Theorems 21 and 22 are given in Figure 8(b), while a counterexample to Theorem 23 is given in Figure 9.

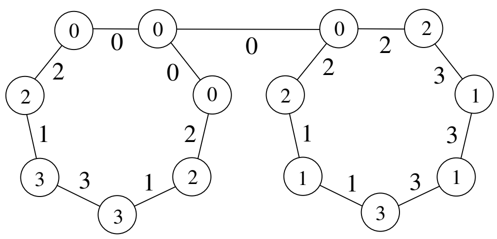


Fig. 9. Edge 4-product cordial labeling f of $P(2.C_7)$

Here, $E_f = (3, 4, 4, 4)$, $V_f = (4, 3, 4, 3)$.

Theorem 4.2. *Suppose graph $P(n.C_m)$ admits an edge k-product cordial labeling. If G_0*

is acyclic, then $k \geq n+1$. If G_0 contains a cycle, then $k \leq \lfloor \frac{mn-1}{m-1} \rfloor$. Moreover, the bounds are attainable.

Proof. Suppose G_0 is acyclic, then $\frac{mn}{k}+1 \geq \lceil \frac{mn}{k} \rceil \geq v_f(0) \geq e_f(0)+1 \geq \lfloor \frac{mn+n-1}{k} \rfloor + 1 > \frac{mn+n-1}{k}$. Thus, we have $k > n-1$, equivalently, $k \geq n$.

If $k = n$, then $v_f(j) = m$ for all j. However, $m = \lfloor \frac{mn+n-1}{n} \rfloor \leq e_f(0)$ implies $v_f(0) \geq m+1$. It is a contradiction. Hence $k \geq n+1$. Figure 10 shows that $P(4.C_4)$ admits an edge 5-product cordial labeling g.

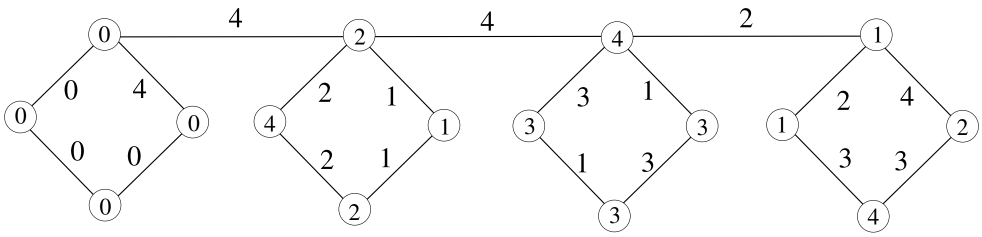


Fig. 10. Edge 5-product cordial labeling g of $P(4.C_4)$

Here, $E_g = (3,4,4,4,4)$, $V_g = (4,3,3,3,3)$. Since $n = 4, k = 5$, the bound is attainable.

Suppose G_0 contains a cycle, then $v_f(0) \geq m$. Now, $\frac{mn}{k}+1 > \lceil \frac{mn}{k} \rceil \geq v_f(0) \geq m$. So $mn > (m-1)k$, i.e., $mn-1 \geq (m-1)k$. Hence $k \leq \frac{mn-1}{m-1}$, i.e., $k \leq \lfloor \frac{mn-1}{m-1} \rfloor$. The graph in Figure 11 shows the bound is attainable.

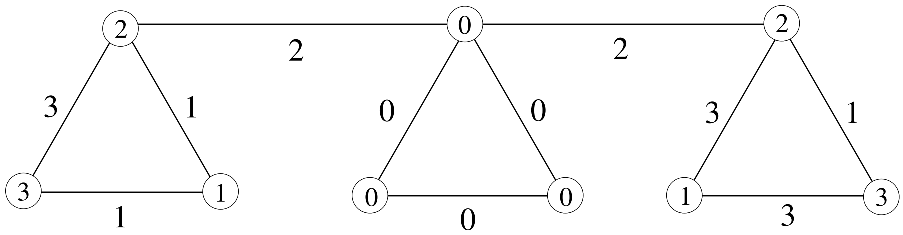


Fig. 11. Edge 4-product cordial labeling f of $P(3.C_3)$

Here, $E_f = (3,3,2,3)$, $V_f = (3,2,2,2)$. $\square$

Remark 4.3. Suppose $P(n.C_m)$ admits an edge n-product cordial labeling f, then $v_f(j) = m$. On the other hand, from the proof above G_0 contains a cycle C_m. Thus $G_0 \cong C_m$.

Let us consider some edge 3-product cordial labelings of the graph $P(n.C_3)$. Note that $P(n.C_3)$ is a $(3n, 4n-1)$-graph. Suppose $P(n.C_3)$ admits an edge 3-product cordial labeling f, then $v_f(j) = n$, for all $0 \leq j \leq 2$.

When $n = 2$, G_0 is acyclic. Now, $e_f(0) = 2, 3$ for any edge 3-product cordial labeling f. This implies that $v_f(0) \geq 3$. Thus $P(2.C_3)$ is not edge 3-product cordial.

When $n = 3$, Figure 8(b) shows an edge 3-product cordial labeling for $P(3.C_3)$.

Suppose $n \geq 4$ and $G = P(n.C_3)$ admits an edge 3-product cordial labeling f. By Theorem 4.2, G_0 contains a cycle which is C_3.

Suppose $n = 4$, then $v_f(j) = 4$ and $e_f(i) = 5$ for all i, j. If G_0 is connected, then it is unicyclic. Hence $|V(G_0)| = 5$ which is impossible. If G_0 is disconnected, then $|V(G_0)| \geq 6$ which is also impossible. Thus there is no edge 3-product cordial labeling for $P(4.C_3)$.

Suppose $n = 5$, then $v_f(j) = 5$ and $e_f(i) = 6$ or 7, for all i, j. By a similar argument as above, there is no edge 3-product cordial labeling for $P(5.C_3)$.

Suppose $n = 6$, then $v_f(j) = 6$ and $e_f(i) = 7$ or 8, for all i, j. We have an edge 3-product cordial labeling for $P(6.C_3)$ as in Figure 12.

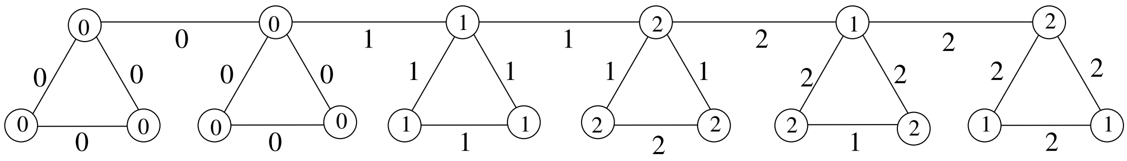


Fig. 12. Edge 3-product cordial labeling f of $P(6.C_3)$

Here, $E_f = (7, 8, 8)$, $V_f = (6, 6, 6)$.

The purpose of this paper is not to study the behavior of edge k-product cordiality of $P(n.C_m)$. We do not pursue in this direction further.

5. A remark on complete bipartite graph

The proof of the following statement is invalid because the authors applied the inequality $v_f(0) \geq e_f(0)$ without justification. Lemma 1.5 shows that this inequality may not hold in general.

Statement 5.1. [3, Theorem 4] A complete bipartite graph $K_{m,n}$ with $m \equiv r_1 \pmod k$ and $n \equiv r_2 \pmod k$ does not admit an edge k-product cordial labeling if $r_1 + r_2 < k \leq r_1 r_2$.

We may restate the above statement as follows: For $m \equiv r_1 \pmod k$ and $n \equiv r_2 \pmod k$, where $0 \leq r_1, r_2 < k$, if $K_{m,n}$ admits an edge k-product cordial labeling, then $k \leq r_1 + r_2$ or $k > r_1 r_2$.

Since $D(K_{1,s}) \cong K_{2,2s}$, Theorem 2.1 implies that if $K_{2,2s}$ admits an edge k-product cordial labeling, then $k = 2s+1$ or $k \geq 4s+1$. Thus, taking $m = r_1 = 2$ and $n = r_2 = 2s$, Statement 5.1 holds for $K_{2,2s}$. Since we believe that Statement 5.1 is true, we propose the following open problem.

Problem 5.2. *Show that the above reformulation of Statement 5.1 holds.*

6. Summary

Let us summarize the current status.

1. Theorem 5 in [3] is false. We gave infinitely many counterexamples.
2. Theorem 19 in [3] is false. A corrected version is provided under additional hypothe-

ses.

3. Theorem 6 in [3] contains an error in its proof. Theorem 2.1 gives a more general necessary condition that includes Theorem 6 as a special case.
4. Theorems 10, 11, 12 and 13 in [3] are false. Counterexamples and revised versions are given.
5. Theorem 20 in [3] is false. We gave counterexamples and a revised version with attainable bounds on k.
6. Theorem 4 in [3] contains an error in its proof. Its validity remains open.

References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications.* Macmillan, New York, 1976.

[2] P. Jeyanthi, K. J. Daisy, and A. Semaničová-Feňovčíková. $\mathbb{Z}_k$-magic labeling of path union of graphs. *CUBO, A Mathematical Journal*, 21(2):15–35, 2019. https://doi.org/10.4067/S0719-06462019000200015.

[3] N. M. NourEldeen, J. Jenisha, K. J. Daisy, P. Jeyanthi, and M. E. Abdel-Aal. Edge k-product cordial labeling of graphs. *European Journal of Pure and Applied Mathematics*, 18(2):5887, 2025. https://doi.org/10.29020/nybg.ejpam.v18i2.5887.

Wai Chee Shiu
Department of Mathematics, The Chinese University of Hong Kong
Shatin, Hong Kong, P.R. China
E-mail wcshiu@associate.hkbu.edu.hk

Gee-Choon Lau
77D, Jalan Suboh, 85000 Johor, Malaysia
E-mail geeclau@yahoo.com