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Localizing spanning trees and arbors to vertex-induced partitions

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ABSTRACT

Let G be a graph, or digraph, and let $P = \{P_1, \ldots, P_k\}$ be a partition of its vertex set. We solve the following problem: find the generating function, in the edge variables of G, of the spanning trees of G each of whose restrictions to every part P_i is a spanning tree of the induced subgraph $G[P_i]$; and, in the directed case with a fixed root $v_i \in P_i$ for each i, of the spanning arbors of G rooted at v_1 each of whose restrictions to P_i is a spanning arbor of $G[P_i]$ rooted at v_i. We show that both generating functions factor as a product of the local generating functions on the parts $G[P_i]$ (or P_i, in the digraph case) with the generating function of spanning trees/arbors of a naturally associated contracted multigraph whose vertices are the parts of P. This localizes the classical theorems of Kirchhoff and Tutte and yields an exact identity, and a family of lower bounds, for the number of spanning trees of G in terms of spanning-tree counts of smaller induced subgraphs.

Keywords: spanning trees, spanning arborescences, generating functions

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1. Introduction

We refer to [5] and [7] for general terminology and for proofs of the matrix-tree theorems of Kirchhoff and Tutte. A digraph is a graph with directed edges; a graph is a digraph in which (v_i, v_j) is an edge if and only if (v_j, v_i) is an edge. A *tree* is a connected, cycle-free undirected graph; it is a tree *of* a graph Γ if its edges are edges of Γ, and a *spanning tree* of Γ if in addition it has every vertex of Γ. A *forest* is a disjoint union of trees, and a *spanning forest* of Γ is a forest of Γ containing every vertex of Γ. An *arbor* rooted at v is, after

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forgetting edge directions, a tree in which every edge points toward v; an *arborescence* rooted at $v_1, \dots, v_k$ is a disjoint union of arbors rooted, respectively, at $v_1, \dots, v_k$. The notions of arbor, arborescence, spanning arbor and spanning arborescence *of* a digraph Γ are defined exactly as for the undirected case, see [6].

Kirchhoff's theorem and Tutte's theorem express the generating function of all spanning trees (respectively, spanning arbors rooted at a fixed vertex) of a graph as a single determinant of a reduced Laplacian. This paper answers the following localized version of that question.

Problem 1.1. *Let G be a graph (or digraph) and let $P = \{P_1, \dots, P_k\}$ be a partition of its vertex set into the vertex sets of the induced subgraphs $G[P_1], \dots, G[P_k]$. Find the generating function of the set of spanning trees of G (respectively, spanning arbors of G rooted at a fixed vertex v_1, given fixed roots $v_i \in P_i$) whose restriction to each part P_i is itself a spanning tree of $G[P_i]$ (respectively, a spanning arbor of $G[P_i]$ rooted at v_i).*

We show (Theorem 3.6 for digraphs, Theorem 4.4 for graphs) that this generating function factors as a product of $k+1$ determinants: one reduced-Laplacian determinant for each part P_i, and one further reduced-Laplacian determinant for a contracted multigraph Γ_P whose k vertices are the parts of P and whose edges are the edges of G that run between distinct parts. Because Γ_P has fewer vertices and fewer edges than G, this gives an inductive route to bounding and computing the number of spanning trees of G; we develop this in Section 5, obtaining an exact partition identity (Theorem 5.2) and a family of lower bounds (Theorem 5.5) on $t(G)$, the number of spanning trees of G.

The results specialize, when P has singleton parts, to the theorems of Kirchhoff and Tutte, and when P has a single non-trivial part, to a fixed-forest/fixed-arborescence contraction result (Propositions 3.3 and 4.2) in the spirit of [3].

2. Notation and conventions

We work with labeled vertices and allow multiple edges throughout.

Convention 2.1. If G is a digraph and there is an edge from vertex i to vertex j, we associate to it an indeterminate x_{ij}; if there are several parallel edges from i to j we label them with *distinct* indeterminates $x_{ij}^{(1)}, x_{ij}^{(2)}, \dots$, and write $\sum_r x_{ij}^{(r)}$ for their total. If G is undirected, a single edge $\{i, j\}$ is associated with one indeterminate, written $x_{ij} = x_{ji}$ (the same variable regardless of which endpoint is listed first); parallel undirected edges again receive distinct indeterminates. To a rooted arbor (or, in the undirected case, a tree) A with edges $i_1 j_1, i_2 j_2, \dots$ we associate the monomial $m(A) = x_{i_1 j_1} x_{i_2 j_2} \cdots$, and to a set S of such objects the generating function $f(S) = \sum_{A \in S} m(A)$.

Convention 2.2. The *Laplacian* L of a digraph G is the vertex-by-vertex matrix with (i, j) entry, for $i \neq j$, equal to $-\sum_r x_{ij}^{(r)}$ if edges from i to j exist and 0 otherwise, and with mth diagonal entry equal to the negative of the sum of the off-diagonal entries in row m. For a vertex v of G, L_v denotes L with the row and column indexed by v deleted.

Convention 2.3 (Contraction and loops)**.** Whenever we contract a graph or digraph G along a partition $P = \{P_1, \dots, P_k\}$ (or along a spanning forest/arborescence, the special case in which each P_i is itself required to already carry a fixed local tree structure), the resulting multigraph has vertex set $\{P_1, \dots, P_k\}$ and, by definition, has as edges *only* those edges of G with endpoints in two different parts; edges of G with both endpoints in the same part are simply not part of the contraction. In particular no loops are ever created, and this does not need to be imposed separately. Edges retain their original labels $x_{ij}^{(r)}$ from G, so the contracted multigraph typically has parallel edges even when G does not.

Theorem 2.4 (Tutte)**.** *The generating function of the set of spanning arbors of a digraph G rooted at v is the formal expansion of* $\det L_v$.

Theorem 2.5 (Kirchhoff)**.** *The generating function of the set of spanning trees of a graph G is the formal expansion of* $\det L_v$*; the result does not depend on the choice of v.*

Given a digraph (or graph) G and $H \subseteq V(G)$, $G[H]$ denotes the induced subgraph on H: its vertex set is H and its edges are exactly the edges of G with both endpoints in H. A *partition of G* is a partition $P = \{P_1, \dots, P_k\}$ of $V(G)$, each part identified with the induced subgraph $G[P_i]$.

3. The embedded arborescence

Let G be a digraph and fix a spanning arborescence A of G with arbors $A_1, \dots, A_k$ rooted at $v_1, \dots, v_k$. We first isolate, among all spanning arbors of G rooted at v_1, those that contain A; this is the case of Theorem 3.6 below in which the local tree structure on each part is already fixed.

Definition 3.1. Let Γ_A be the digraph with vertex set $\{A_1, \dots, A_k\}$ and an edge $A_i \to A_j$ for every edge of G that originates at a root v_i, $i > 1$, and terminates at some vertex of A_j, $j \neq i$ (Convention 2.3); edges keep their original G-labels.

The restriction to edges originating at the roots v_i, $i > 1$, is not an arbitrary choice of convenience; it is forced by the requirement that the result be a spanning arbor.

Lemma 3.2. *Let T be a spanning arbor of G rooted at v_1 that contains A as a subdigraph. Then the edges of T not in A are exactly $k-1$ edges, one originating at each root v_i, $i > 1$, and each terminating at a vertex lying outside the arbor A_i that contains its tail.*

Proof. In a spanning arbor rooted at v_1, every vertex other than v_1 has out-degree exactly 1, and v_1 has out-degree 0. Inside A, every vertex of A_i other than the root v_i already has out-degree 1 (its edge toward v_i within A_i), so it can receive no further outgoing edge in T. The root v_1 needs no outgoing edge. Each remaining root v_i, $i > 1$, has out-degree 0 within A and so needs exactly one outgoing edge in $T \setminus A$. This accounts for exactly $k-1$ additional edges, one per root v_i, $i > 1$. Such an edge cannot terminate inside A_i itself: every vertex of A_i already has a directed path, within A_i, back to v_i, so an edge from v_i into A_i would close a cycle. Hence each additional edge terminates at a vertex of some A_j, $j \neq i$. □

Proposition 3.3. *Let G, A, $A_1, \ldots, A_k$, $v_1, \ldots, v_k$, and Γ_A be as above. The generating function of the spanning arbors of G rooted at v_1 that contain A is*

$$F_{G,A}^{v_1} = \Big(\prod_{ij \in A} x_{ij}\Big) \cdot \det L(\Gamma_A)_{A_1},$$

where $L(\Gamma_A)_{A_1}$ is the Laplacian of Γ_A with the row and column indexed by A_1 deleted.

Proof. By Lemma 3.2, a spanning arbor T of G rooted at v_1 containing A consists of the edges of A together with a choice, for each root v_i, $i > 1$, of exactly one outgoing edge to a vertex outside A_i, subject to the constraint that the resulting digraph is acyclic. Collapsing each arbor A_i to the single vertex it represents in Γ_A, such a choice is precisely a spanning arbor $\overline{T}$ of Γ_A rooted at A_1: each vertex A_i, $i > 1$, has exactly one outgoing edge, A_1 has none, and acyclicity of T on the roots corresponds to acyclicity of $\overline{T}$. Conversely, any spanning arbor $\overline{T}$ of Γ_A rooted at A_1 lifts uniquely to a spanning arbor $T \supseteq A$ of G rooted at v_1, since within each A_i all edges already point toward v_i. The correspondence $T \leftrightarrow \overline{T}$ is therefore a bijection between $\{T : T \text{ a spanning arbor of } G \text{ rooted at } v_1,\ T \supseteq A\}$ and the spanning arbors of Γ_A rooted at A_1, and $m(T) = (\prod_{ij \in A} x_{ij}) \cdot m(\overline{T})$. Summing over T and applying Tutte's theorem (Theorem 2.4) to Γ_A gives the stated formula. □

Setting all $x_{ij} = 1$ shows that the number of spanning arbors of G rooted at v_1 containing A is $\det L(\Gamma_A)_{A_1}$; and when every arbor A_i is a single vertex, $\Gamma_A = G$ and Proposition 3.3 recovers Tutte's theorem.

Example 3.4. Write i for vertex v_i and ij for the edge from v_i to v_j. Let G be the digraph on six vertices with edges $12, 21, 32, 34, 24, 25, 41, 54, 61, 65$. Fix the arborescence A with arbors $A_1 = \{54, 41\}$ rooted at 1, $A_2 = \{32\}$ rooted at 2, and $A_3 = \{6\}$ rooted at 6. Then Γ_A has vertex set $\{A_1, A_2, A_3\}$ and edges $21, 25, 24$ (from root 2) and $61, 65$ (from root 6). Proposition 3.3 gives

$$\begin{aligned} F_{G,A}^{1} =& (x_{54}x_{41}x_{32}) \cdot \det L(\Gamma_A)_{A_1} \\ =& (x_{54}x_{41}x_{32})\,(x_{21}x_{61} + x_{21}x_{65} + x_{25}x_{61} + x_{25}x_{65} + x_{24}x_{61} + x_{24}x_{65}), \end{aligned}$$

a sum of six monomials, confirming by direct inspection that there are exactly six spanning arbors of G rooted at 1 that contain A.

We now localize to an arbitrary partition, rather than a single fixed arborescence: instead of fixing the local arbors A_i in advance, we ask for the union, over *all* choices of a local arbor in each part, of the resulting spanning arbors of G.

Definition 3.5. Let G be a digraph and $P = \{P_1, \dots, P_k\}$ a partition of G with fixed vertices $v_i \in P_i$. Write $v = (v_1, \dots, v_k)$. Define

$$S(P, v) = \{\, T : T \text{ is a spanning arbor of } G \text{ rooted at } v_1, \text{and } T[P_i] \text{ is a spanning arbor of } P_i \text{ rooted at } v_i \text{ for every } i \,\}.$$

where $T[P_i]$ denotes the restriction of T to edges with both endpoints in P_i. Let Γ_P be the digraph with vertex set $\{P_1, \dots, P_k\}$ and an edge $P_i \to P_j$ for every edge of G that originates at v_i, $i > 1$, and terminates in P_j, $j \neq i$.

Theorem 3.6. *With G, P, v, and Γ_P as above, the generating function of $S(P, v)$ is*

$$\Big(\prod_{i=1}^{k} \det L(P_i)_{v_i}\Big) \cdot \det L(\Gamma_P)_{P_1}.$$

Proof. Fix, for each i, a spanning arbor A_i of P_i rooted at v_i; let $A = A_1 \cup \dots \cup A_k$, a spanning arborescence of $G[P_1 \cup \dots \cup P_k] = G$ rooted at $v_1, \dots, v_k$. By the argument of Lemma 3.2 and Proposition 3.3, applied verbatim with A_i in the role of the arbors of A, the elements of $S(P, v)$ that contain this particular choice of local arbors are in bijection with the spanning arbors of the digraph Γ_A (vertex set $\{A_1, \dots, A_k\}$, edges from roots v_i, $i > 1$, into other A_j) rooted at A_1. Because these edges of Γ_A are determined only by which *root* v_i they originate at and which *part* P_j they terminate in — not by the internal structure of A_i or A_j — the digraph Γ_A is identical, as a labeled multigraph, to Γ_P for every choice of local arbors $A_1, \dots, A_k$. Hence

$$S(P, v) = \bigsqcup_{\substack{A_i \in \mathcal{A}(P_i, v_i) \\ i = 1, \dots, k}} \{\, T : T \supseteq A_1 \cup \dots \cup A_k,\ T \text{ a spanning arbor of } G \text{ rooted at } v_1 \,\},$$

where $\mathcal{A}(P_i, v_i)$ is the set of spanning arbors of P_i rooted at v_i, and this union is disjoint because two spanning arbors T of G containing different tuples $(A_1, \dots, A_k)$ restrict to different arbors on some P_i. By Proposition 3.3, the generating function of the A-indexed block is $(\prod_i m(A_i)) \cdot \det L(\Gamma_P)_{P_1}$; the determinant is the same for every choice of $(A_1, \dots, A_k)$, since $\Gamma_A = \Gamma_P$ regardless. Summing over all choices,

$$\begin{aligned} f(S(P, v)) &= \Big(\sum_{A_1 \in \mathcal{A}(P_1, v_1)} m(A_1)\Big) \cdots \Big(\sum_{A_k \in \mathcal{A}(P_k, v_k)} m(A_k)\Big) \cdot \det L(\Gamma_P)_{P_1} \\ &= \Big(\prod_{i=1}^{k} \det L(P_i)_{v_i}\Big) \cdot \det L(\Gamma_P)_{P_1}, \end{aligned}$$

the last step by Tutte's theorem applied to each P_i. If some P_j has no spanning arbor rooted at v_j, then $\mathcal{A}(P_j, v_j) = \varnothing$, $S(P, v) = \varnothing$, and $\det L(P_j)_{v_j} = 0$ by Theorem 2.4, so the formula still holds (both sides are 0). $\square$

Example 3.7. Let G have edges $21, 32, 31, 34, 45, 53, 51$. Take $P = \{P_1, P_2\}$ with $P_1 = \{1, 2, 3\}$ (induced edges $21, 32, 31$) and $P_2 = \{4, 5\}$ (induced edge 45); root P_1 at $v_1 = 1$ and P_2 at $v_2 = 5$. Direct inspection shows the elements of $S(P, v)$ have generating function $x_{21}x_{31}x_{45}x_{51} + x_{21}x_{31}x_{45}x_{53} + x_{32}x_{21}x_{45}x_{51} + x_{32}x_{21}x_{45}x_{53}$. Theorem 3.6 gives, with Γ_P having the two edges $53, 51$ from root $v_2 = 5$ into P_1 (the third P_1–P_2 edge, 34, does not originate at v_2 and so is not an edge of Γ_P),

$$(\det L(P_1)_1 \cdot \det L(P_2)_5) \cdot \det L(\Gamma_P)_{P_1} = (x_{21}x_{31} + x_{32}x_{21}) \cdot x_{45} \cdot (x_{51} + x_{53}),$$

which agrees with the direct computation upon expanding.

Now instead take $P_1 = \{1, 5\}$ with induced edge 51 only, and $P_2 = \{2, 3, 4\}$ with induced edges $32, 34$; root P_1 at 1 and P_2 at 2. Then P_2 has no spanning arbor rooted at 2 (vertex 4 has no outgoing edge within P_2), so $S(P, v) = \varnothing$, and indeed $\det L(P_2)_2 = 0$.

4. The embedded spanning forest

We now treat the undirected analogue. Throughout this section all graphs are undirected; for an edge $\{i, j\}$ we always have $x_{ij} = x_{ji}$ (Convention 2.1), and we write ij for the unordered edge.

Definition 4.1. Let G be a graph and F a spanning forest of G with trees $T_1, \dots, T_k$. Let G/F be the multigraph with vertex set $\{T_1, \dots, T_k\}$ and, for $i \neq j$, an edge between T_i and T_j for every edge of G joining a vertex of T_i to a vertex of T_j (Convention 2.3); edges keep their original labels.

Proposition 4.2. *The generating function of the spanning trees of G that contain F is*

$$\sum_{\substack{T \text{ spanning tree of } G \\ F \subseteq T}} m(T) \;=\; \Big(\prod_{ij \in F} x_{ij}\Big) \cdot \det L(G/F)_v,$$

for any vertex v of G/F (the choice of v does not affect the value, by Theorem 2.5).

Proof. We exhibit mutually inverse maps between $\{T : T \text{ a spanning tree of } G,\ F \subseteq T\}$ and the spanning trees of G/F.

Forward map. Let $T \supseteq F$ be a spanning tree of G. The edge set $T \setminus F$, viewed as edges of G/F between the corresponding trees T_i, T_j, has $|T| - |F| = (n-1) - (n-k) = k-1$ edges, where $n = |V(G)|$. It contains no cycle: a cycle in G/F using edges of $T \setminus F$ would lift, via paths inside the relevant T_i's supplied by F, to a cycle in T, contradicting that T is a tree. A graph on k vertices with $k-1$ edges and no cycle is connected, hence $T \setminus F$ is a spanning tree $\overline{T}$ of G/F.

Backward map. Let $\overline{T}$ be a spanning tree of G/F, i.e. $k-1$ edges of G running between distinct T_i's, forming (as a graph on $\{T_1, \dots, T_k\}$) a tree. Set $T = F \cup \overline{T}$, viewed as a set of edges of G. Then T has $(n-k) + (k-1) = n-1$ edges and spans all n vertices; it is connected because F connects each T_i internally and $\overline{T}$ connects the T_i's to one another; and it is acyclic because a cycle would either lie inside some T_i (impossible, F

restricted to T_i is a tree) or would have to cross between at least two parts using at least two edges of $\overline{T}$ between the same pair of "super-vertices" in a way that closes a cycle in G/F, contradicting that $\overline{T}$ is a tree of G/F. So T is a spanning tree of G containing F.

These two maps are mutually inverse, giving a bijection $T \leftrightarrow \overline{T}$ with $m(T) = (\prod_{ij\in F} x_{ij}) \cdot m(\overline{T})$. Summing over T and applying Kirchhoff's theorem (Theorem 2.5) to G/F gives the stated identity. ☐

Setting $x_{ij} = 1$ for all ij, Proposition 4.2 says the number of spanning trees of G containing F is $\det L(G/F)_v$; when F consists solely of vertices of G this recovers Kirchhoff's theorem.

We next pass, exactly as in Section 3, from a single fixed forest F to a partition P, taking the union over all choices of local spanning tree in each part.

Definition 4.3. Let G be a graph and $P = \{P_1, \dots, P_k\}$ a partition of G. Set

$$S(P) = \{\, T : T \text{ a spanning tree of } G,\ T[P_i] \text{ a spanning tree of } G[P_i] \text{ for every } i \,\}.$$

Let G_P be the multigraph with vertex set $\{P_1, \dots, P_k\}$ and, for $i \neq j$, an edge between P_i and P_j for every edge of G joining a vertex of P_i to a vertex of P_j.

Theorem 4.4. *With G, P, G_P as above, and u_i any vertex of P_i, v any vertex of G_P, the generating function of $S(P)$ is*

$$\Big(\prod_{i=1}^{k} \det L(G[P_i])_{u_i}\Big) \cdot \det L(G_P)_v.$$

If some $G[P_i]$ is disconnected, $\det L(G[P_i])_{u_i} = 0$ *and* $S(P) = \varnothing$.

Proof. Exactly as in the proof of Theorem 3.6: fixing a spanning tree T_i of each $G[P_i]$ gives a spanning forest $F = T_1 \cup \cdots \cup T_k$ of G, and by Proposition 4.2 the spanning trees of G containing this F biject with the spanning trees of G/F; the multigraph G/F depends only on the partition $\{P_1, \dots, P_k\}$ induced by F's components, not on the particular local trees T_i chosen, so $G/F = G_P$ for every choice. Hence $S(P)$ is the disjoint union, over all tuples $(T_1, \dots, T_k)$ with T_i a spanning tree of $G[P_i]$, of the (bijective, by Proposition 4.2) fibers indexed by the spanning trees of G_P, and

$$f(S(P)) = \Big(\sum_{T_1} m(T_1)\Big) \cdots \Big(\sum_{T_k} m(T_k)\Big) \cdot \det L(G_P)_v = \Big(\prod_{i=1}^{k} \det L(G[P_i])_{u_i}\Big) \cdot \det L(G_P)_v,$$

using Kirchhoff's theorem on each $G[P_i]$; the choice of u_i, v is immaterial by Theorem 2.5. If $G[P_i]$ is disconnected for some i, it has no spanning tree, $\det L(G[P_i])_{u_i} = 0$ by Kirchhoff's theorem, and $S(P) = \varnothing$ since no $T \in S(P)$ can restrict to a spanning tree on a disconnected part. ☐

Setting $x_{ij} = 1$ in Proposition 4.2 (not Theorem 4.4, which counts a different, larger set) gives the number of spanning trees of G containing a *fixed* forest F as $\det L(G/F)_v$.

Example 4.5. We recover Cayley's formula, and a refinement of it, from Proposition 4.2 applied to $G = K_n$. Fix a spanning forest F of K_n with trees $T_1, \ldots, T_k$ on $n_1, \ldots, n_k$ vertices, $\sum_i n_i = n$. By Convention 2.3, K_n/F has vertices $1, \ldots, k$ (relabeling $T_1, \ldots, T_k$) with exactly $n_i n_j$ parallel edges between vertices i and j ($i \neq j$), all edges of K_n between T_i and T_j being present. Its Laplacian L has off-diagonal entries $L_{ij} = -n_i n_j$ ($i \neq j$) and diagonal entries

$$L_{ii} = \sum_{j \neq i} n_i n_j = n_i(n - n_i),$$

by Convention 2.2. Writing $\mathbf{n} = (n_1, \ldots, n_k)^\mathsf{T}$ and $D = \operatorname{diag}(n_1, \ldots, n_k)$, this is

$$L = nD - \mathbf{n}\mathbf{n}^\mathsf{T}.$$

Delete the row and column indexed by k; write $D' = \operatorname{diag}(n_1, \ldots, n_{k-1})$ and $\mathbf{n}' = (n_1, \ldots, n_{k-1})^\mathsf{T}$, so $L_k = nD' - \mathbf{n}'\mathbf{n}'^\mathsf{T}$. Since D' is invertible with $D'^{-1}\mathbf{n}' = (1, \ldots, 1)^\mathsf{T} =: \mathbf{1}$, the matrix determinant lemma $\det(A - uv^\mathsf{T}) = \det(A)(1 - v^\mathsf{T} A^{-1} u)$, applied with $A = nD'$, $u = v = \mathbf{n}'$, gives

$$\begin{aligned}
\det L_k &= \det(nD')\Big(1 - \frac{1}{n}\mathbf{n}'^\mathsf{T} D'^{-1}\mathbf{n}'\Big) \\
&= n^{k-1} \prod_{i=1}^{k-1} n_i \cdot \Big(1 - \frac{1}{n}\sum_{i=1}^{k-1} n_i\Big) \\
&= n^{k-1} \prod_{i=1}^{k-1} n_i \cdot \frac{n_k}{n} \\
&= n^{k-2} \prod_{i=1}^{k} n_i,
\end{aligned}$$

using $\sum_{i=1}^{k-1} n_i = n - n_k$. By Proposition 4.2 at $x_{ij} \equiv 1$, this is exactly the number of spanning trees of K_n containing the fixed forest F:

$$\#\{\text{spanning trees of } K_n \text{ containing } F\} = n^{k-2} \prod_{i=1}^{k} n_i,$$

depending only on the component sizes $n_1, \ldots, n_k$, not on the isomorphism type of the T_i or on which forest with these component sizes was chosen. Taking $k = n$ (all $n_i = 1$, $F = \varnothing$) recovers Cayley's theorem, $t(K_n) = n^{n-2}$ [4]. Combining with Theorem 4.4 and Cayley's formula applied inside each block ($t(K_{n_i}) = n_i^{n_i-2}$) also gives the number of spanning trees of K_n whose restriction to a *fixed partition* into blocks of sizes $n_1, \ldots, n_k$ is connected (any local tree allowed): $n^{k-2} \prod_i n_i \cdot \prod_i n_i^{n_i-2} = n^{k-2} \prod_i n_i^{n_i-1}$. See [3] for related enumeration.

Remark 4.6. It would be of interest to extend Theorems 3.6 and 4.4 to graphs and digraphs with colored edges, along the lines of [2, 1].

5. Inductive formulae and bounds for the number of spanning trees

We now use the results of Section 4 to relate $t(G)$, the number of spanning trees of G ($n = |V(G)|$ vertices), to spanning-tree counts of smaller induced subgraphs. For a spanning forest f of G, write $t(G/f)$ for the number of spanning trees of G containing f; by Proposition 4.2 at $x_{ij} \equiv 1$, $t(G/f) = \det L(G/f)_v$.

Theorem 5.1. *For any $1 \le k \le n$,*

$$t(G) = \binom{n-1}{n-k}^{-1} \sum_f t(G/f),$$

the sum over all spanning forests f of G with exactly k trees (isolated vertices count as one-vertex trees).

Proof. Count pairs

$$\mathcal{T} = \{(f,t) : f \text{ a spanning forest of } G \text{ with } k \text{ trees},\ f \subseteq t, t \text{ a spanning tree of } G\}$$

in two ways. Fixing a spanning tree t (which has $n-1$ edges), the forests $f \subseteq t$ with k trees are obtained by deleting exactly $n-k$ of its $n-1$ edges, in $\binom{n-1}{n-k}$ ways; so $|\mathcal{T}| = \binom{n-1}{n-k} t(G)$. Fixing f instead, the number of t with $f \subseteq t$ is $t(G/f)$ by definition; so $|\mathcal{T}| = \sum_f t(G/f)$. Equating the two counts and dividing gives the formula. □

Because G/f depends only on the *partition* of $V(G)$ into the vertex sets of f's components (not on the internal edges of f), and because the number of spanning forests f inducing a given partition $p = \{V_1, \ldots, V_k\}$ of $V(G)$ (each V_i connected) is $\prod_i t(G[V_i])$, Theorem 5.1 may be rewritten as a sum over set partitions rather than over forests.

Theorem 5.2 (Exact partition identity)**.** *For any $1 \le k \le n$,*

$$t(G) = \binom{n-1}{n-k}^{-1} \sum_p \Big(\prod_{i=1}^{k} t(G[V_{i,p}])\Big)\, t(G_p),$$

the sum over all partitions $p = \{V_{1,p}, \ldots, V_{k,p}\}$ of $V(G)$ into k (labeled or unlabeled – the summand is symmetric) blocks, where G_p is the contracted multigraph of Theorem 4.4 and $t(G[V_{i,p}]) = 0$ whenever $V_{i,p}$ does not induce a connected subgraph.

Proof. Group the sum in Theorem 5.1 by the partition p that a forest f induces: for fixed p with each block connected, the forests f inducing p are exactly the $\prod_i t(G[V_{i,p}])$ choices of a spanning tree in each block, and for every such f, $G/f = G_p$ (Convention 2.3: the contraction depends only on the blocks, not on which spanning tree was chosen within them), so $t(G/f) = t(G_p)$ for all of them. Summing $t(G/f) = t(G_p)$ over the $\prod_i t(G[V_{i,p}])$

forests inducing p, and then over all p (partitions with a disconnected block contribute 0, matching that no forest with k components induces them), reproduces $\sum_f t(G/f)$. □

Corollary 5.3 ($k = 2$)**.**

$$t(G) = \frac{1}{n-1} \sum_p t(G[V_{1,p}])\, t(G[V_{2,p}])\, e_p,$$

the sum over the (unordered) partitions $p = \{V_{1,p}, V_{2,p}\}$ *of* $V(G)$ *into two blocks, where* $t(G[V_{i,p}]) :=$ *the number of spanning trees of the induced subgraph* $G[V_{i,p}]$*, and* $e_p := |\{uv \in E(G) : u \in V_{1,p},\ v \in V_{2,p}\}|$.

Proof. Immediate from Theorem 5.2 at $k = 2$: the two-vertex multigraph G_p has $t(G_p) = e_p$, since any single one of its e_p parallel edges is by itself a spanning tree. □

Example 5.4. Let G have vertices $1, 2, 3, 4$ and all edges present except 24. There are 10 partitions of $\{1, 2, 3, 4\}$ into two blocks. Of these, four have 3 edges of G running between the blocks and six have 2 edges between the blocks, and in every one of these ten partitions both blocks induce connected subgraphs with a single spanning tree each; direct enumeration gives $\sum_p t(G[V_{1,p}])t(G[V_{2,p}])e_p = 4 \cdot 3 + 6 \cdot 2 = 24$. By Corollary 5.3, $t(G) = \frac{1}{3} \cdot 24 = 8$, which agrees with direct enumeration of the spanning trees of G.

For general k, Theorem 5.2 sums nonnegative terms over *all* partitions with k blocks; restricting the sum to a subset of these partitions can only decrease it, and every term already vanishes unless every block is connected. This gives a lower bound, correcting the direction claimed in an earlier draft of this result.

Theorem 5.5 (Lower bound)**.** *Let* $\mathcal{Q}$ *be any set of partitions of* $V(G)$ *into* k *blocks. Then*

$$\binom{n-1}{n-k}^{-1} \sum_{p \in \mathcal{Q}} \Big(\prod_{i=1}^{k} t(G[V_{i,p}]) \Big) t(G_p) \ \le\ t(G),$$

with equality if and only if $\mathcal{Q}$ *contains every partition of* $V(G)$ *into* k *blocks all of which induce connected subgraphs of* G *(partitions with a disconnected block may freely be included or omitted, as they contribute* 0 *either way).*

Proof. By Theorem 5.2, $\binom{n-1}{n-k}\, t(G) = \sum_p (\prod_i t(G[V_{i,p}]))t(G_p)$, the sum over *all* partitions p into k blocks, with every summand nonnegative. Restricting the sum to $p \in \mathcal{Q} \subseteq \{\text{all partitions into } k \text{ blocks}\}$ removes only nonnegative terms, so $\sum_{p \in \mathcal{Q}} (\cdots) \le \binom{n-1}{n-k} t(G)$; dividing gives the stated inequality. Equality holds exactly when the omitted terms are all 0, i.e. when every partition *not* in $\mathcal{Q}$ has some disconnected block. □

Remark 5.6. Theorem 5.5 is most useful when $\mathcal{Q}$ is a small, structured family of partitions – for instance, all partitions obtained by removing a fixed small edge cut, or all partitions into k blocks of prescribed sizes – for which the local factors $t(G[V_{i,p}])$ and the contracted count $t(G_p)$ can be computed or bounded on graphs substantially smaller than G.

6. Conclusion

We have shown that the generating function of the spanning trees of a graph G (or spanning arbors of a digraph, rooted appropriately) that respect a given vertex partition P – restricting to a spanning tree, or arbor, on each part – factors as the product of the local generating functions on the parts and the generating function of a naturally associated contracted multigraph on the parts. This localizes Kirchhoff's and Tutte's theorems (Theorems 3.6 and 4.4), specializes to a fixed-forest contraction result (Propositions 3.3 and 4.2) that recovers and refines Cayley's formula on K_n (Example 4.5), and yields both an exact identity (Theorem 5.2) and a family of lower bounds (Theorem 5.5) for $t(G)$ in terms of spanning-tree counts of induced subgraphs on fewer vertices.

Natural directions for further work include: an analogous treatment of edge-colored graphs and digraphs, in which the generating function tracks color multiplicities ([2, 1]); explicit families of partitions $\mathcal{Q}$ in Theorem 5.5 for which the lower bound can be shown to be tight or near-tight; and a corresponding upper-bound theory, which does not follow from the present argument and appears to require different techniques.

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