J. COMBIN. MATH. COMBIN. COMPUT. 131 (2026) 253–272

Journal of Combinatorial Mathematics
and Combinatorial Computing
www.combinatorialpress.com/jcmcc

Combinatorial Press

On local antimagic chromatic number of three disjoint cycles

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ABSTRACT

An edge labeling of a graph $G = (V, E)$ is said to be local antimagic if it is a bijection $f : E \to \{1, \ldots, |E|\}$ such that for any pair of adjacent vertices x and y, $f^+(x) \neq f^+(y)$, where the induced vertex label $f^+(x) = \sum f(e)$, with e ranging over all the edges incident to x. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of distinct induced vertex labels over all local antimagic labelings of G. In this paper, we study local antimagic labeling of three disjoint cycles with at least two odd cycles, one of which is C_3. We prove that (1) $\chi_{la}(C_3 + C_3 + C_3) = 5$ and $\chi_{la}(C_3+C_3+C_k) = 4$ for $k \geq 4$; (2) $\chi_{la}(C_3+C_4+C_{2k+1}) = 4$; (3) $\chi_{la}(C_3+C_6+C_{4k+1}) = 3$; (4) $\chi_{la}(C_3 + C_{10} + C_{8k-3}) = 3$; (5) $\chi_{la}(C_3 + C_5 + C_{14}) = 4$ and $\chi_{la}(C_3 + C_5 + C_{4k+2}) = 3$ for $k \neq 3$; (6) $\chi_{la}(C_3 + C_{4k+1} + C_{4k+5}) = 3$ and $\chi_{la}(C_3 + C_{4k+1} + C_{4k+6}) = 3$.

Keywords: local antimagic labeling, local antimagic chromatic number, cycles

2020 Mathematics Subject Classification: 05C78, 05C15.

1. Introduction

A graph $G = (V, E)$ is said to be *local antimagic* if it admits a *local antimagic edge labeling*, i.e., a bijection $f : E \to \{1, \ldots, |E|\}$ such that the induced vertex labeling $f^+ : V \to \mathbb{Z}$ given by $f^+(u) = \sum f(e)$, with e ranging over all the edges incident to u, has the property that any two adjacent vertices have distinct induced vertex labels (see [1, 2]). Thus, f^+ is a coloring of G. Clearly, the order of G must be at least 3. The vertex label $f^+(u)$ is called the *induced color* of u under f (the *color* of u, for short, if no ambiguity occurs). The number of distinct induced colors under f is denoted by $c(f)$, and is called the *color number* of

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Received 14 Jan 2026; Revised 04 Jun 2026; Accepted 14 Jun 2026; Published Online 20 Jul 2026.

DOI: 10.61091/jcmcc131-13

f. The labeling f is called a *local antimagic $c(f)$-coloring of G*. The *local antimagic chromatic number* of G, denoted by $\chi_{la}(G)$, is $\min\{c(f) \mid f$ is a local antimagic labeling of $G\}$. Haslegrave [4] proved that every connected graph other than K_2 is local antimagic. Hence $\chi_{la}(G)$ is well-defined for every connected or disconnected graph G not containing any isolated edge.

For graphs G and H, let $G+H$ be the disjoint union of G and H with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)$. For convenient, mG is the disjoint union of $m\ge 2$ copies of G. For any vertex $u\in V(G)$, denote the set of edges incident to u by $E(u)$. We shall use the notation $[a,b] := \{c\in\mathbb{Z} \mid a\le c\le b\}$. Unless stated otherwise, all graphs considered in this paper are simple, undirected and of order at least 3.

In [3], the authors gave lower and upper bounds for the local antimagic chromatic number of disjoint union of cycles.

Theorem 1.1. *For $m\ge 1$, let $G = C_{n_1}+C_{n_2}+\cdots+C_{n_m}$, where $n_i\ge 3$, $1\le i\le m$. Then $3\le \chi_{la}(G)\le m+2$.*

They also proved that if all but except maybe one cycle are even, then the local antimagic chromatic number is 3.

Theorem 1.2. *Let $G = C_{n_1}+C_{n_2}+\cdots+C_{n_m}$, where $n_i\ge 3$ for $1\le i\le m$, and n_i is even for $1\le i\le m-1$, then $\chi_{la}(G)=3$.*

The next natural step is to determine the local antimagic chromatic number of disjoint union of cycles with at least two odd cycles. In this paper, we study local antimagic edge labeling of three disjoint cycles with at least two odd cycles, one of which is C_3.

2. $C_3+C_3+C_a$

First, we consider disjoint union of cycles with at least two C_3's.

Theorem 2.1. *For $m\ge 1$, let $G = C_3+C_3+C_{n_1}+C_{n_2}+\cdots+C_{n_m}$, where $n_i\ge 3$, $1\le i\le m$. Then $4\le\chi_{la}(G)\le m+4$.*

Proof. Suppose $\chi_{la}(G)=3$. Let a,b,c be the induced vertex labels of G. Then both C_3's have edge labels $\frac{a+b-c}{2}$, $\frac{b+c-a}{2}$, $\frac{c+a-b}{2}$, a contradiction. The result follows from Theorem 1.1. □

Lemma 2.2. *Let $G=(V,E)$ be a k-regular graph. Suppose f is a local antimagic c-coloring of G. Define $g(e) := |E|+1-f(e)$ for all $e\in E$. Then g is a local antimagic c-coloring of G.*

Proof. Since f is a bijection from E to $[1,|E|]$, g is a bijection from E to $[1,|E|]$ as well. For all $u\in V$, $g^+(u) = \sum\limits_{e\in E(u)} g(e) = \sum\limits_{e\in E(u)} (|E|+1-f(e)) = k(|E|+1) - f^+(u)$. Thus, for all $uv \in E$, $g^+(u) \neq g^+(v)$. Hence, g is a local antimagic c-coloring of G. □

Theorem 2.3. $\chi_{la}(C_3 + C_3 + C_3) = 5$.

Proof. By Theorem 2.1, $4 \leq \chi_{la}(C_3 + C_3 + C_3) \leq 5$.

Suppose $\chi_{la}(C_3 + C_3 + C_3) = 4$. Let f be a local antimagic 4-coloring of $C_3 + C_3 + C_3$ with induced vertex labels a, b, c, d. Note that $a, b, c, d \in [3, 17]$ and are all distinct. Without loss of generality, assume the induced vertex labels of the three C_3 : H_1, H_2, H_3, are $\{a, b, c\}$, $\{a, b, d\}$ and $\{a, c, d\}$ respectively. Then the edge labels of H_1 are $\frac{a+b-c}{2}$, $\frac{b+c-a}{2}$, $\frac{c+a-b}{2}$. The edge labels of H_2 are $\frac{a+b-d}{2}$, $\frac{b+d-a}{2}$, $\frac{d+a-b}{2}$. The edge labels of H_3 are $\frac{a+c-d}{2}$, $\frac{c+d-a}{2}$, $\frac{d+a-c}{2}$. As $a + b \equiv c \equiv d \pmod 2$, $a + c \equiv b \equiv d \pmod 2$, $a + d \equiv b \equiv c \pmod 2$, we have $b \equiv c \equiv d \pmod 2$. Since $b + c \equiv a \pmod 2$, $a \equiv 0 \pmod 2$. Note that the sum of all the edge labels of H_1, H_2, H_3 are $\frac{a+b+c}{2}$, $\frac{a+b+d}{2}$, $\frac{a+c+d}{2}$ respectively. Hence,

$$\frac{a+b+c}{2} + \frac{a+b+d}{2} + \frac{a+c+d}{2} = \sum_{i=1}^{9} i$$
$$3a + 2(b + c + d) = 90.$$

Without loss of generality, assume $b < c < d$. Since a is even and appears 3 times as an induced vertex label, $a = 8, 10, 12$. Define $g(e) := 10 - f(e)$ for all $e \in E$. By Lemma 2.2, g is a local antimagic 4-coloring and $g^+(u) = 20 - f^+(u)$ for all $u \in V$. Thus, it suffices to consider $a = 8$ and $a = 10$. Also, since b, c, d all appear twice as an induced vertex label, $5 \leq b < c < d \leq 15$.

(1) $a = 8$, $b + c + d = 33$. This implies b, c, d are all odd and $b \leq 9$. Thus, $b = 5$, $b = 7$ or $b = 9$.

(1.1) $b = 5$, $c + d = 28$. This implies $c = 13$ and $d = 15$. But H_1 has an edge label $\frac{a+b-c}{2} = 0$ which is impossible.

(1.2) $b = 7$, $c + d = 26$. This implies $c = 11$ and $d = 15$. But H_2 has an edge label $\frac{a+b-d}{2} = 0$ which is impossible.

(1.3) $b = 9$, $c + d = 24$. This implies $c = 11$ and $d = 13$. But H_1 has an edge label $\frac{a+b-c}{2} = 3$ and H_3 has an edge label $\frac{a+c-d}{2} = 3$, a contradiction.

(2) $a = 10$, $b + c + d = 30$. This implies b, c, d are all even and $\{b, c, d\} \subset \{6, 8, 12, 14\}$. But there is no solution.

Therefore, $\chi_{la}(C_3 + C_3 + C_3) = 5$. □

Theorem 2.4. *For $k \geq 2$, $\chi_{la}(C_3 + C_3 + C_{2k}) = 4$.*

Proof. By Theorem 2.1, $4 \leq \chi_{la}(C_3 + C_3 + C_{2k}) \leq 5$. We construct a local antimagic 4-coloring of $C_3 + C_3 + C_{2k}$ for $k \geq 2$. Note that there are $2k + 6$ edges.

For the first C_3, let e_i be the i-th edge in the clockwise order for $i \in [1, 3]$. Define f by $f(e_1) = 1, f(e_2) = 2k + 6, f(e_3) = 2k + 5$. The induced vertex labels are $2k + 6, 2k + 7, 4k + 11$.

For the second C_3, let e_i be the i-th edge in the clockwise order for $i \in [1,3]$. Define f by $f(e_1) = k+2, f(e_2) = k+3, f(e_3) = k+4$. The induced vertex labels are $2k+5, 2k+6, 2k+7$.

For C_{2k}, let e_i be the i-th edge in the clockwise order for $i \in [1,2k]$. Define f by

$$\begin{aligned}
f(e_{2i-1}) &= 2i, & i &\in \left[1, \left\lceil \frac{k}{2} \right\rceil\right] \\
f(e_{2i}) &= 2k+6-(2i+1), & i &\in \left[1, \left\lfloor \frac{k}{2} \right\rfloor\right] \\
f(e_{2k+1-2i}) &= 2i+1, & i &\in \left[1, \left\lfloor \frac{k}{2} \right\rfloor\right] \\
f(e_{2k+2-2i}) &= 2k+6-2i, & i &\in \left[1, \left\lceil \frac{k}{2} \right\rceil\right].
\end{aligned}$$

When k is odd, the edge labels are:

$$2,4,\dots,k+1; \quad 2k+3, 2k+1, \dots, k+6; \quad 3,5,\dots,k; \quad 2k+4, 2k+2, \dots, k+5.$$

The induced vertex labels are:

$$\begin{aligned}
f^+(e_{2i-1} \cap e_{2i}) &= 2k+5, & i &\in \left[1, \frac{k-1}{2}\right] \\
f^+(e_{2i} \cap e_{2i+1}) &= 2k+7, & i &\in \left[1, \frac{k-1}{2}\right] \\
f^+(e_{2k+2-2i} \cap e_{2k+1-2i}) &= 2k+7, & i &\in \left[1, \frac{k-1}{2}\right] \\
f^+(e_{2k+1-2i} \cap e_{2k-2i}) &= 2k+5, & i &\in \left[1, \frac{k-1}{2}\right] \\
f^+(e_1 \cap e_{2k}) &= 2+2k+4 = 2k+6 \\
f^+(e_k \cap e_{k+1}) &= k+1+k+5 = 2k+6.
\end{aligned}$$

When k is even, the edge labels are:

$$2,4,\dots,k; \quad 2k+3, 2k+1, \dots, k+5; \quad 3,5,\dots,k+1; \quad 2k+4, 2k+2, \dots, k+6.$$

The induced vertex labels are:

$$\begin{aligned}
f^+(e_{2i-1} \cap e_{2i}) &= 2k+5, & i &\in \left[1, \frac{k}{2}\right] \\
f^+(e_{2i} \cap e_{2i+1}) &= 2k+7, & i &\in \left[1, \frac{k}{2}-1\right] \\
f^+(e_{2k+2-2i} \cap e_{2k+1-2i}) &= 2k+7, & i &\in \left[1, \frac{k}{2}\right] \\
f^+(e_{2k+1-2i} \cap e_{2k-2i}) &= 2k+5, & i &\in \left[1, \frac{k}{2}-1\right] \\
f^+(e_1 \cap e_{2k}) &= 2+2k+4 = 2k+6
\end{aligned}$$

$$f^+(e_k \cap e_{k+1}) = k + 5 + k + 1 = 2k + 6.$$

Therefore, f is a local antimagic labeling of $C_3 + C_3 + C_{2k}$ with induced vertex labels $2k + 5$, $2k + 6$, $2k + 7$, $4k + 11$. □

Theorem 2.5. *For $k \geq 1$, $\chi_{la}(C_3 + C_3 + C_{4k+1}) = 4$.*

Proof. By Theorem 2.1, $4 \leq \chi_{la}(C_3 + C_3 + C_{4k+1}) \leq 5$. We construct a local antimagic 4-coloring of $C_3 + C_3 + C_{4k+1}$ for $k \geq 1$. Note that there are $4k + 7$ edges.

For the first C_3, let e_i be the i-th edge in the clockwise order for $i \in [1, 3]$. Define f by $f(e_1) = 1, f(e_2) = 4k + 6, f(e_3) = 7$. The induced vertex labels are $8, 4k + 7, 4k + 13$.

For the second C_3, let e_i be the i-th edge in the clockwise order for $i \in [1, 3]$. Define f by $f(e_1) = 2, f(e_2) = 4k + 7, f(e_3) = 6$. The induced vertex labels are $8, 4k + 9, 4k + 13$.

For C_{4k+1}, let e_i be the i-th edge in the clockwise order for $i \in [1, 4k + 1]$. Define f by

$$\begin{aligned}
f(e_1) &= 3, & \\
f(e_{2i}) &= 4k + 8 - 4i, & i \in [1, k], \\
f(e_{2i+1}) &= 4i + 5, & i \in [1, k], \\
f(e_{2k+2}) &= 4, & \\
f(e_{2(k+i)+1}) &= 4k + 7 - 4i, & i \in [1, k - 1], \\
f(e_{2(k+i)+2}) &= 4i + 6, & i \in [1, k - 1], \\
f(e_{4k+1}) &= 5. &
\end{aligned}$$

The edge labels are:

$$3; \quad 4k{+}4, 4k, \ldots, 8; \quad 9, 13, \ldots, 4k{+}5; \quad 4; \quad 4k{+}3, 4k{-}1, \ldots, 11; \quad 10, 14, \ldots, 4k{+}2; \quad 5.$$

The induced vertex labels are:

$$\begin{aligned}
f^+(e_1 \cap e_2) &= 3 + 4k + 4 = 4k + 7, & \\
f^+(e_{2i} \cap e_{2i+1}) &= 4k + 13, & i \in [1, k], \\
f^+(e_{2i+1} \cap e_{2i+2}) &= 4k + 9, & i \in [1, k], \\
f^+(e_{2k+2} \cap e_{2k+3}) &= 4 + 4k + 3 = 4k + 7, & \\
f^+(e_{2(k+i)+1} \cap e_{2(k+i)+2}) &= 4k + 13, & i \in [1, k - 1], \\
f^+(e_{2(k+i)+2} \cap e_{2(k+i)+3}) &= 4k + 9, & i \in [1, k - 2], \\
f^+(e_{4k} \cap e_{4k+1}) &= 4k + 2 + 5 = 4k + 7, & \\
f^+(e_1 \cap e_{4k+1}) &= 3 + 5 = 8. &
\end{aligned}$$

Therefore, f is a local antimagic labeling of $C_3 + C_3 + C_{4k+1}$ with induced vertex labels 8, $4k + 7$, $4k + 9$, $4k + 13$. □

Theorem 2.6. *For $k \geq 1$, $\chi_{la}(C_3 + C_3 + C_{4k+3}) = 4$.*

Proof. By Theorem 2.1, $4 \le \chi_{la}(C_3+C_3+C_{4k+3}) \le 5$. We construct a local antimagic 4-coloring of $C_3+C_3+C_{4k+3}$ for $k \ge 1$. Note that there are $4k+9$ edges.

For the first C_3, let e_i be the i-th edge in the clockwise order for $i \in [1,3]$. Define f by $f(e_1)=1, f(e_2)=4k+8, f(e_3)=7$. The induced vertex labels are $8, 4k+9, 4k+15$.

For the second C_3, let e_i be the i-th edge in the clockwise order for $i \in [1,3]$. Define f by $f(e_1)=2, f(e_2)=4k+9, f(e_3)=6$. The induced vertex labels are $8, 4k+11, 4k+15$.

For C_{4k+3}, let e_i be the i-th edge in the clockwise order for $i \in [1,4k+3]$. Define f by, for $i \in [1,k]$,

$$\begin{aligned} f(e_1) &= 3, \\ f(e_{2i}) &= 4k+10-4i, \\ f(e_{2i+1}) &= 4i+5, \\ f(e_{2k+2}) &= 4, \\ f(e_{2(k+i)+1}) &= 4k+11-4i, \\ f(e_{2(k+i)+2}) &= 4i+4, \\ f(e_{4k+3}) &= 5. \end{aligned}$$

The edge labels are:

$3; \quad 4k+6, 4k+2, \dots, 10; \quad 9, 13, \dots, 4k+5; \quad 4; \quad 4k+7, 4k+3, \dots, 11; \quad 8, 12, \dots, 4k+4; \quad 5.$

The induced vertex labels are:

$$\begin{aligned} f^+(e_1 \cap e_2) &= 3+4k+6 = 4k+9, & & \\ f^+(e_{2i} \cap e_{2i+1}) &= 4k+15, & & i \in [1,k], \\ f^+(e_{2i+1} \cap e_{2i+2}) &= 4k+11, & & i \in [1,k-1], \\ f^+(e_{2k+1} \cap e_{2k+2}) &= 4k+5+4 = 4k+9, & & \\ f^+(e_{2(k+i)} \cap e_{2(k+i)+1}) &= 4k+11, & & i \in [1,k], \\ f^+(e_{2(k+i)+1} \cap e_{2(k+i)+2}) &= 4k+15, & & i \in [1,k], \\ f^+(e_{4k+2} \cap e_{4k+3}) &= 4k+4+5 = 4k+9, & & \\ f^+(e_1 \cap e_{4k+3}) &= 3+5 = 8. & & \end{aligned}$$

Therefore, f is a local antimagic labeling of $C_3+C_3+C_{4k+3}$ with induced vertex labels 8, $4k+9$, $4k+11$, $4k+15$. □

Combining all the above results, we have

Theorem 2.7. $\chi_{la}(C_3+C_3+C_3)=5$. *For* $k \ge 4$, $\chi_{la}(C_3+C_3+C_k)=4$.

3. $C_3+C_4+C_a$

By Theorem 1.2, $\chi_{la}(C_3+C_4+C_{2k})=3$ for $k \ge 2$. Here we prove that $\chi_{la}(C_3+C_4+C_{2k+1})=4$ for $k \ge 1$. First, we restate the following lemma in [5, Lemma 1] or [6, Lemma 2.1].

Lemma 3.1. *Let G be a graph and C be a component of G. Suppose there is a local antimagic labeling of G inducing a 2-vertex coloring of C with colors x and y, where $x < y$. Let X and Y be the sets of vertices in C colored x and y, respectively. Then C is a bipartite graph with bipartition (X, Y), $|X| > |Y|$, and $x|X| = y|Y|$.*

We also need the useful lemma proved in [3].

Lemma 3.2. *Let G be a disjoint union of cycles with n vertices. Suppose $\chi_{la}(G) = 3$. Then the edge labeled 1 is adjacent to the edge labeled n. Moreover, one vertex label is less than $n+1$, one is equal to $n+1$, and one is greater than $n+1$.*

Theorem 3.3. *For $k \geq 1$, $\chi_{la}(C_3 + C_4 + C_{2k+1}) = 4$.*

Proof. By Theorem 1.1, $3 \leq \chi_{la}(C_3+C_4+C_{2k+1}) \leq 5$. Suppose $\chi_{la}(C_3+C_4+C_{2k+1}) = 3$. Let f be a local antimagic 3-coloring of G, and a, b, c be the induced vertex labels. For C_4, let $e_i = v_i v_{i+1}$ be the i-th edge in the clockwise order for $i \in [1,4]$ where $v_5 = v_1$. By Lemma 3.1, all a, b, c must appear as induced vertex labels in C_4. Without loss of generality, assume $f^+(v_1) = a, f^+(v_2) = b, f^+(v_3) = c, f^+(v_4) = b$. Note that $a + c = 2b$. Thus, b lies between a and c. By Lemma 3.2, $b = 2k+9$. Since a, b, c are the induced vertex labels of C_3, the edge labels of C_3 are $\frac{a+b-c}{2}, \frac{b+c-a}{2}, \frac{c+a-b}{2}$. However, $\frac{c+a-b}{2} = \frac{b}{2} = \frac{2k+9}{2} \notin \mathbb{Z}$. Hence, $4 \leq \chi_{la}(C_3 + C_4 + C_{2k+1}) \leq 5$.

We construct a local antimagic 4-coloring of $C_3+C_4+C_{2k+1}$ for $k \geq 1$. Note that there are $2k + 8$ edges.

For C_3, let e_i be the i-th edge in the clockwise order for $i \in [1,3]$. Define f by $f(e_1) = k+3, f(e_2) = k+5, f(e_3) = k+4$. The induced vertex labels are $2k+7, 2k+8, 2k+9$.

For C_4, let e_i be the i-th edge in the clockwise order for $i \in [1,4]$. Define f by $f(e_1) = k+1, f(e_2) = k+7, f(e_3) = k+2, f(e_4) = k+6$. The induced vertex labels are $2k+7, 2k+8, 2k+9$.

For C_{2k+1}, let e_i be the i-th edge in the clockwise order for $i \in [1, 2k+1]$. Define f by $f(e_{2i-1}) = i$, $f(e_{2i}) = 2k+8-i$, for $i \in [1,k]$, and $f(e_{2k+1}) = 2k+8$.

The edge labels are:

$$1, 2, \ldots, k; \quad 2k+7, 2k+6, \ldots, k+8; \quad 2k+8.$$

The induced vertex labels are:

$$\begin{aligned}
f^+(e_{2i-1} \cap e_{2i}) &= 2k+8, & i &\in [1,k], \\
f^+(e_{2i} \cap e_{2i+1}) &= 2k+9, & i &\in [1,k-1], \\
f^+(e_{2k} \cap e_{2k+1}) &= k+8+2k+8 = 3k+16, \\
f^+(e_1 \cap e_{2k+1}) &= 1+2k+8 = 2k+9.
\end{aligned}$$

Therefore, f is a local antimagic labeling of $C_3+C_4+C_{2k+1}$ with induced vertex labels $2k+7$, $2k+8$, $2k+9$, $3k+16$. $\square$

4. $C_3 + C_6 + C_{4k+1}$

Theorem 4.1. *For $k \geq 1$, $\chi_{la}(C_3 + C_6 + C_{4k+1}) = 3$.*

Proof. By Theorem 1.1, $3 \leq \chi_{la}(C_3 + C_6 + C_{4k+1}) \leq 5$. We construct a local antimagic 3-coloring of $C_3 + C_6 + C_{4k+1}$ for $k \geq 1$. Note that there are $4k + 10$ edges.

For C_3, let e_i be the i-th edge in the clockwise order for $i \in [1, 3]$. Define f by $f(e_1) = k+2, f(e_2) = 3k+9, f(e_3) = 2k+5$. The induced vertex labels are $3k+7, 4k+11, 5k+14$.

For C_6, let e_i be the i-th edge in the clockwise order for $i \in [1, 6]$. Define f by $f(e_1) = k+1, f(e_2) = 3k+10, f(e_3) = 2k+4, f(e_4) = k+3, f(e_5) = 3k+8, f(e_6) = 2k+6$. The induced vertex labels are $3k + 7, 4k + 11, 5k + 14$.

For C_{4k+1}, let e_i be the i-th edge in the clockwise order for $i \in [1, 4k + 1]$. Define f by, for $i \in [1, k]$,

$$\begin{aligned} f(e_{4i-3}) &= k + 1 - i, \\ f(e_{4i-2}) &= 3k + 10 + i, \\ f(e_{4i-1}) &= 2k + 4 - i, \\ f(e_{4i}) &= 2k + 7 + i, \\ f(e_{4k+1}) &= 2k + 7. \end{aligned}$$

The edge labels are:

$$k, k - 1, \ldots, 1; 3k + 11, 3k + 12, \ldots, 4k + 10; 2k + 3, 2k + 2, \ldots, k + 4; 2k + 8, 2k + 9, \ldots, 3k + 7; 2k + 7.$$

The induced vertex labels are:

$$\begin{aligned} f^+(e_{4i-3} \cap e_{4i-2}) &= 4k + 11, && i \in [1, k], \\ f^+(e_{4i-2} \cap e_{4i-1}) &= 5k + 14, && i \in [1, k], \\ f^+(e_{4i-1} \cap e_{4i}) &= 4k + 11, && i \in [1, k], \\ f^+(e_{4i} \cap e_{4i+1}) &= 3k + 7, && i \in [1, k - 1], \\ f^+(e_{4k} \cap e_{4k+1}) &= 3k + 7 + 2k + 7 = 5k + 14. \end{aligned}$$

Therefore, f is a local antimagic labeling of $C_3 + C_6 + C_{4k+1}$ with induced vertex labels $3k + 7$, $4k + 11$, $5k + 14$. □

5. $C_3 + C_{10} + C_{8k-3}$

Theorem 5.1. *For $k \geq 1$, $\chi_{la}(C_3 + C_{10} + C_{8k-3}) = 3$.*

Proof. By Theorem 1.1, $3 \leq \chi_{la}(C_3 + C_{10} + C_{8k-3}) \leq 5$. We construct a local antimagic 3-coloring of $C_3 + C_{10} + C_{8k-3}$ for $k \geq 1$. Note that there are $8k + 10$ edges.

For C_3, let e_i be the i-th edge in the clockwise order for $i \in [1, 3]$. Define f by $f(e_1) = 4k+3, f(e_2) = 4k+8, f(e_3) = 4k+5$. The induced vertex labels are $8k+8, 8k+11, 8k+13$.

For C_{10}, let e_i be the i-th edge in the clockwise order for $i \in [1, 10]$. Define f by $f(e_1) = 1, f(e_2) = 8k+10, f(e_3) = 3, f(e_4) = 8k+8, f(e_5) = 5, f(e_6) = 8k+6, f(e_7) = 2, f(e_8) = 8k+9, f(e_9) = 4, f(e_{10}) = 8k+7$. The edge labels are $1, 2, 3, 4, 5; 8k+6, 8k+7, 8k+8, 8k+9, 8k+10$. The induced vertex labels are $8k+8, 8k+11, 8k+13$.

For C_{8k-3}, let e_i be the i-th edge in the clockwise order for $i \in [1, 8k-3]$. Define f by $f(e_1) = 6$.

For k odd, $f(e_{4k-2}) = 4k+9$, $f(e_{4k-1}) = 4k+4$, $f(e_{4k}) = 4k+7$, $f(e_{4k+1}) = 4k+6$.

For k even, $f(e_{4k-2}) = 4k+6$, $f(e_{4k-1}) = 4k+7$, $f(e_{4k}) = 4k+4$, $f(e_{4k+1}) = 4k+9$.

For $i \in [0, k-2]$,

$$f(e_{4i+2}) = \begin{cases} 8k+5-4i, & i \text{ even } (\equiv 5 \ mod\ 8), \\ 8k+2-4i, & i \text{ odd } (\equiv 6 \ mod\ 8), \end{cases}$$

$$f(e_{4i+3}) = \begin{cases} 8+4i, & i \text{ even } (\equiv 0 \ mod\ 8), \\ 9+4i, & i \text{ odd } (\equiv 5 \ mod\ 8), \end{cases}$$

$$f(e_{4i+4}) = \begin{cases} 8k+3-4i, & i \text{ even } (\equiv 3 \ mod\ 8), \\ 8k+4-4i, & i \text{ odd } (\equiv 0 \ mod\ 8), \end{cases}$$

$$f(e_{4i+5}) = \begin{cases} 10+4i, & i \text{ even } (\equiv 2 \ mod\ 8), \\ 7+4i, & i \text{ odd } (\equiv 3 \ mod\ 8), \end{cases}$$

$$f(e_{8k-3-4i}) = \begin{cases} 8k+2-4i, & i \text{ even } (\equiv 2 \ mod\ 8), \\ 8k+5-4i, & i \text{ odd } (\equiv 1 \ mod\ 8), \end{cases}$$

$$f(e_{8k-4-4i}) = \begin{cases} 9+4i, & i \text{ even } (\equiv 1 \ mod\ 8), \\ 8+4i, & i \text{ odd } (\equiv 4 \ mod\ 8), \end{cases}$$

$$f(e_{8k-5-4i}) = \begin{cases} 8k+4-4i, & i \text{ even } (\equiv 4 \ mod\ 8), \\ 8k+3-4i, & i \text{ odd } (\equiv 7 \ mod\ 8), \end{cases}$$

$$f(e_{8k-6-4i}) = \begin{cases} 7+4i, & i \text{ even } (\equiv 7 \ mod\ 8), \\ 10+4i, & i \text{ odd } (\equiv 6 \ mod\ 8). \end{cases}$$

Note that

(1) $7+4i \geq 7$ and $8k+5-4i \leq 8k+5$.

(2) $10+4i \leq 10+4(k-2) = 4k+2$ and $8k+2-4i \geq 8k+2-4(k-2) = 4k+10$.

(3) By considering mod 8, if two edge labels are equal, then either $8k+a-4i = a+4+4i' \implies i+i' = 2k-1$ or $8k+b-4i = b+8+4i' \implies i+i' = 2k-2$. Both are impossible. Thus all edge labels are distinct.

(4) $f^+(e_1 \cap e_2) = 6+8k+5 = 8k+11$ and $f^+(e_1 \cap e_{8k-3}) = 6+8k+2 = 8k+8$.

(5) For $i \in [0, k-2]$,

$$f^+(e_{4i+2} \cap e_{4i+3}) = \begin{cases} 8k+13, & i \text{ even}, \\ 8k+11, & i \text{ odd}, \end{cases}$$

$$f^+(e_{4i+3} \cap e_{4i+4}) = \begin{cases} 8k+11, & i \text{ even}, \\ 8k+13, & i \text{ odd}, \end{cases}$$

$$f^+(e_{4i+4} \cap e_{4i+5}) = \begin{cases} 8k+13, & i \text{ even}, \\ 8k+11, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k+2-4i} \cap e_{8k+1-4i}) = \begin{cases} 8k+11, & i \text{ even}, \\ 8k+13, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k+1-4i} \cap e_{8k-4i}) = \begin{cases} 8k+13, & i \text{ even}, \\ 8k+11, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k-4i} \cap e_{8k-1-4i}) = \begin{cases} 8k+11, & i \text{ even}, \\ 8k+13, & i \text{ odd}. \end{cases}$$

For $i \in [0, k-3]$,

$$f^+(e_{4i+5} \cap e_{4i+6}) = \begin{cases} 8k+8, & i \text{ even}, \\ 8k+8, & i \text{ odd}. \end{cases}$$

$$f^+(e_{8k-1-4i} \cap e_{8k-2-4i}) = \begin{cases} 8k+8, & i \text{ even}, \\ 8k+8, & i \text{ odd}. \end{cases}$$

(6) When k is even, $f^+(e_{4k-3} \cap e_{4k-2}) = 10 + 4(k-2) + 4k + 6 = 8k+8$, $f^+(e_{4k-2} \cap e_{4k-1}) = 8k+13$, $f^+(e_{4k-1} \cap e_{4k}) = 8k+11$, $f^+(e_{4k} \cap e_{4k+1}) = 8k+13$, and $f^+(e_{4k+1} \cap e_{4k+2}) = 4k+9+7+4(k-2) = 8k+8$.

When k is odd, $f^+(e_{4k-3} \cap e_{4k-2}) = 7 + 4(k-2) + 4k + 9 = 8k+8$, $f^+(e_{4k-2} \cap e_{4k-1}) = 8k+13$, $f^+(e_{4k-1} \cap e_{4k}) = 8k+11$, $f^+(e_{4k} \cap e_{4k+1}) = 8k+13$, and $f^+(e_{4k+1} \cap e_{4k+2}) = 4k+6+10+4(k-2) = 8k+8$.

Therefore, f is a local antimagic labeling of $C_3 + C_{10} + C_{8k-3}$ with induced vertex labels $8k+8$, $8k+11$, $8k+13$. □

6. $C_3 + C_5 + C_{4k+2}$

Theorem 6.1. *For $a \geq 3$, $3 \leq \chi_{la}(C_3 + C_5 + C_a) \leq 4$.*

Proof. By Theorem 1.1, $3 \leq \chi_{la}(C_3 + C_5 + C_a) \leq 5$. First, we construct a local antimagic 4-coloring of $C_3 + C_5 + C_{2k+1}$ for $k \geq 1$. Note that there are $2k+9$ edges.

For C_3, let e_i be the i-th edge in the clockwise order for $i \in [1,3]$. Define f by $f(e_1) = 2, f(e_2) = k+6, f(e_3) = k+8$. The induced vertex labels are $k+8, k+10, 2k+14$.

For C_5, let e_i be the i-th edge in the clockwise order for $i \in [1,5]$. Define f by $f(e_1) = 1, f(e_2) = k+7, f(e_3) = 3, f(e_4) = k+5, f(e_5) = k+9$. The induced vertex labels are $k+8, k+10, 2k+14$.

For C_{2k+1}, let e_i be the i-th edge in the clockwise order for $i \in [1, 2k+1]$. Define f by $f(e_1) = 4$, $f(e_{2i}) = k+5-i$ and $f(e_{2i+1}) = k+9+i$, for $i \in [1, k]$.

The edge labels are:

$$4; \quad k+4, k+3, \ldots, 5; \quad k+10, k+11, \ldots, 2k+9.$$

The induced vertex labels are:

$$\begin{aligned} f^+(e_{2i} \cap e_{2i+1}) &= 2k+14, & i &\in [1, k], \\ f^+(e_{2i+1} \cap e_{2i+2}) &= 2k+13, & i &\in [1, k-1], \\ f^+(e_1 \cap e_2) &= 4+k+4 = k+8, \\ f^+(e_1 \cap e_{2k+1}) &= 4+2k+9 = 2k+13. \end{aligned}$$

Therefore, f is a local antimagic labeling of $C_3 + C_5 + C_{2k+1}$ with induced vertex labels $k+8$, $k+10$, $2k+13$, $2k+14$.

Next, we construct a local antimagic 4-coloring of $C_3 + C_5 + C_{2k}$ for $k \geq 2$. Note that there are $2k+8$ edges.

For C_3, let e_i be the i-th edge in the clockwise order for $i \in [1, 3]$. Define f by $f(e_1) = 2, f(e_2) = 2k+5, f(e_3) = 2k+7$. The induced vertex labels are $2k+7, 2k+9, 4k+12$.

For C_5, let e_i be the i-th edge in the clockwise order for $i \in [1, 5]$. Define f by $f(e_1) = 1, f(e_2) = 2k+6, f(e_3) = 3, f(e_4) = 2k+4, f(e_5) = 2k+8$. The induced vertex labels are $2k+7, 2k+9, 4k+12$.

For C_{2k}, let e_i be the i-th edge in the clockwise order for $i \in [1, 2k]$. Define f by $f(e_{2i-1}) = 2i+2$ and $f(e_{2i}) = 2k+5-2i$, for $i \in [1, k]$.

The edge labels are:

$$4, 6, \ldots, 2k+2; \quad 2k+3, 2k+1, \ldots, 5.$$

The induced vertex labels are:

$$\begin{aligned} f^+(e_{2i-1} \cap e_{2i}) &= 2k+7, & i &\in [1, k], \\ f^+(e_{2i} \cap e_{2i+1}) &= 2k+9, & i &\in [1, k-1], \\ f^+(e_1 \cap e_{2k}) &= 4+5 = 9. \end{aligned}$$

Therefore, f is a local antimagic labeling of $C_3 + C_5 + C_{2k}$ with induced vertex labels 9, $2k+7$, $2k+9$, $4k+12$.

Combining the above results, we have $3 \leq \chi_{la}(C_3 + C_5 + C_a) \leq 4$ for $a \geq 3$. □

Below we provide some families of graphs belonging to $C_3+C_5+C_a$ with local antimagic chromatic number 3.

Theorem 6.2. *For $k \geq 1$, $\chi_{la}(C_3 + C_5 + C_{8k+2}) = 3$.*

Proof. By Theorem 6.1, $3 \leq \chi_{la}(C_3 + C_5 + C_{8k+2}) \leq 4$. We construct a local antimagic 3-coloring of $C_3 + C_5 + C_{8k+2}$ for $k \geq 1$. Note that there are $8k+10$ edges.

For C_3, let e_i be the i-th edge in the clockwise order for $i \in [1,3]$. Define f by $f(e_1) = 4k+3, f(e_2) = 4k+8, f(e_3) = 4k+5$. The induced vertex labels are $8k+8, 8k+11, 8k+13$.

For C_5, let e_i be the i-th edge in the clockwise order for $i \in [1,5]$. Define f by $f(e_1) = 4k+2, f(e_2) = 4k+9, f(e_3) = 4k+4, f(e_4) = 4k+7, f(e_5) = 4k+6$. The induced vertex labels are $8k+8, 8k+11, 8k+13$.

For C_{8k+2}, let e_i be the i-th edge in the clockwise order for $i \in [1, 8k+2]$. Define f by $f(e_1) = 1$, $f(e_{4k+2}) = 4k+10$, and for $i \in [0, k-1]$,

$$f(e_{4i+2}) = \begin{cases} 8k+10-4i, & i \text{ even } (\equiv 2 \ mod\ 8), \\ 8k+7-4i, & i \text{ odd } (\equiv 3 \ mod\ 8), \end{cases}$$

$$f(e_{4i+3}) = \begin{cases} 3+4i, & i \text{ even } (\equiv 3 \ mod\ 8), \\ 4+4i, & i \text{ odd } (\equiv 0 \ mod\ 8), \end{cases}$$

$$f(e_{4i+4}) = \begin{cases} 8k+8-4i, & i \text{ even } (\equiv 0 \ mod\ 8), \\ 8k+9-4i, & i \text{ odd } (\equiv 5 \ mod\ 8), \end{cases}$$

$$f(e_{4i+5}) = \begin{cases} 5+4i, & i \text{ even } (\equiv 5 \ mod\ 8), \\ 2+4i, & i \text{ odd } (\equiv 6 \ mod\ 8), \end{cases}$$

$$f(e_{8k+2-4i}) = \begin{cases} 8k+7-4i, & i \text{ even } (\equiv 7 \ mod\ 8), \\ 8k+10-4i, & i \text{ odd } (\equiv 6 \ mod\ 8), \end{cases}$$

$$f(e_{8k+1-4i}) = \begin{cases} 4+4i, & i \text{ even } (\equiv 4 \ mod\ 8), \\ 3+4i, & i \text{ odd } (\equiv 7 \ mod\ 8), \end{cases}$$

$$f(e_{8k-4i}) = \begin{cases} 8k+9-4i, & i \text{ even } (\equiv 1 \ mod\ 8), \\ 8k+8-4i, & i \text{ odd } (\equiv 4 \ mod\ 8), \end{cases}$$

$$f(e_{8k-1-4i}) = \begin{cases} 2+4i, & i \text{ even } (\equiv 2 \ mod\ 8), \\ 5+4i, & i \text{ odd } (\equiv 1 \ mod\ 8). \end{cases}$$

Note that

(1) $4i+2 \geq 2$ and $8k+10-4i \leq 8k+10$.

(2) $5+4i \leq 5+4(k-1) = 4k+1$ and $8k+7-4i \geq 8k+7-4(k-1) = 4k+11$.

(3) By considering mod 8, if two edge labels are equal, then either $8k+a-4i = a-4+4i' \implies i+i' = 2k+1$ or $8k+b-4i = b-8+4i' \implies i+i' = 2k+2$. Both are impossible. Thus all edge labels are distinct.

(4) $f^+(e_1 \cap e_2) = 1+8k+10 = 8k+11$ and $f^+(e_1 \cap e_{8k+2}) = 1+8k+7 = 8k+8$.

(5) For $i \in [0, k-1]$,

$$f^+(e_{4i+2} \cap e_{4i+3}) = \begin{cases} 8k+13, & i \text{ even}, \\ 8k+11, & i \text{ odd}, \end{cases}$$

$$f^+(e_{4i+3} \cap e_{4i+4}) = \begin{cases} 8k+11, & i \text{ even}, \\ 8k+13, & i \text{ odd}, \end{cases}$$

$$f^+(e_{4i+4} \cap e_{4i+5}) = \begin{cases} 8k+13, & i \text{ even}, \\ 8k+11, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k+2-4i} \cap e_{8k+1-4i}) = \begin{cases} 8k+11, & i \text{ even}, \\ 8k+13, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k+1-4i} \cap e_{8k-4i}) = \begin{cases} 8k+13, & i \text{ even}, \\ 8k+11, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k-4i} \cap e_{8k-1-4i}) = \begin{cases} 8k+11, & i \text{ even}, \\ 8k+13, & i \text{ odd}. \end{cases}$$

For $i \in [0, k-2]$,

$$f^+(e_{4i+5} \cap e_{4i+6}) = \begin{cases} 8k+8, & i \text{ even}, \\ 8k+8, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k-1-4i} \cap e_{8k-2-4i}) = \begin{cases} 8k+8, & i \text{ even}, \\ 8k+8, & i \text{ odd}. \end{cases}$$

(6) When k is odd, $f^+(e_{4k+1} \cap e_{4k+2}) = 5 + 4(k-1) + 4k + 10 = 8k + 11$ and $f^+(e_{4k+2} \cap e_{4k+3}) = 4k + 10 + 2 + 4(k-1) = 8k + 8$.

When k is even, $f^+(e_{4k+1} \cap e_{4k+2}) = 2 + 4(k-1) + 4k + 10 = 8k + 8$ and $f^+(e_{4k+2} \cap e_{4k+3}) = 4k + 10 + 5 + 4(k-1) = 8k + 11$.

Therefore, f is a local antimagic labeling of $C_3 + C_5 + C_{8k+2}$ with induced vertex labels $8k+8$, $8k+11$, $8k+13$. □

Theorem 6.3. *For $k \geq 1$, $\chi_{la}(C_3 + C_5 + C_{8k+14}) = 3$.*

Proof. By Theorem 6.1, $3 \leq \chi_{la}(C_3 + C_5 + C_{8k+14}) \leq 4$. We construct a local antimagic 3-coloring of $C_3 + C_5 + C_{8k+14}$ for $k \geq 1$. Note that there are $8k + 22$ edges.

For C_3, let e_i be the i-th edge in the clockwise order for $i \in [1, 3]$. Define f by $f(e_1) = 4k + 9, f(e_2) = 4k + 14, f(e_3) = 4k + 11$. The induced vertex labels are $8k + 20, 8k + 23, 8k + 25$.

For C_5, let e_i be the i-th edge in the clockwise order for $i \in [1, 5]$. Define f by $f(e_1) = 4k + 8, f(e_2) = 4k + 15, f(e_3) = 4k + 10, f(e_4) = 4k + 13, f(e_5) = 4k + 12$. The induced vertex labels are $8k + 20, 8k + 23, 8k + 25$.

For C_{8k+14}, let e_i be the i-th edge in the clockwise order for $i \in [1, 8k + 14]$. Define f by $f(e_1) = 1$, $f(e_2) = 8k + 22$, $f(e_3) = 3$, $f(e_4) = 8k + 20$, $f(e_5) = 5$, $f(e_6) = 8k + 15$, $f(e_7) = 10$, $f(e_8) = 8k + 13$, $f(e_9) = 7$, $f(e_{10}) = 8k + 18$, $f(e_{11}) = 2$, $f(e_{12}) = 8k + 21$, $f(e_{13}) = 4$, $f(e_{14}) = 8k+16$, $f(e_{15}) = 9$, $f(e_{16}) = 8k+14$, $f(e_{17}) = 11$, $f(e_{4k+14}) = 4k+16$, $f(e_{8k+11}) = 8$, $f(e_{8k+12}) = 8k + 17$, $f(e_{8k+13}) = 6$, $f(e_{8k+14}) = 8k + 19$. The edge labels are $1, 2, \ldots, 11; 4k + 16; 8k + 13, 8k + 14, \ldots, 8k + 22$. The induced vertex labels are $8k + 20, 8k + 23, 8k + 25$.

For $i \in [0, k-2]$,

$$f(e_{4i+18}) = \begin{cases} 8k+9-4i, & i \text{ even } (\equiv 1 \; mod \; 8), \\ 8k+12-4i, & i \text{ odd } (\equiv 0 \; mod \; 8), \end{cases}$$

$$f(e_{4i+19}) = \begin{cases} 14+4i, & i \text{ even } (\equiv 6 \; mod \; 8), \\ 13+4i, & i \text{ odd } (\equiv 1 \; mod \; 8), \end{cases}$$

$$f(e_{4i+20}) = \begin{cases} 8k+11-4i, & i \text{ even } (\equiv 3 \; mod \; 8), \\ 8k+10-4i, & i \text{ odd } (\equiv 6 \; mod \; 8), \end{cases}$$

$$f(e_{4i+21}) = \begin{cases} 12+4i, & i \text{ even } (\equiv 4 \; mod \; 8), \\ 15+4i, & i \text{ odd } (\equiv 3 \; mod \; 8), \end{cases}$$

$$f(e_{8k+10-4i}) = \begin{cases} 8k+12-4i, & i \text{ even } (\equiv 4 \; mod \; 8), \\ 8k+9-4i, & i \text{ odd } (\equiv 5 \; mod \; 8), \end{cases}$$

$$f(e_{8k+9-4i}) = \begin{cases} 13+4i, & i \text{ even } (\equiv 5 \; mod \; 8), \\ 14+4i, & i \text{ odd } (\equiv 2 \; mod \; 8), \end{cases}$$

$$f(e_{8k+8-4i}) = \begin{cases} 8k+10-4i, & i \text{ even } (\equiv 2 \; mod \; 8), \\ 8k+11-4i, & i \text{ odd } (\equiv 7 \; mod \; 8), \end{cases}$$

$$f(e_{8k+7-4i}) = \begin{cases} 15+4i, & i \text{ even } (\equiv 7 \; mod \; 8), \\ 12+4i, & i \text{ odd } (\equiv 0 \; mod \; 8). \end{cases}$$

Note that

(1) $12 + 4i \geq 12$ and $8k + 12 - 4i \leq 8k + 12$.

(2) $15 + 4i \leq 15 + 4(k-2) = 4k + 7$ and $8k + 9 - 4i \geq 8k + 9 - 4(k-2) = 4k + 17$.

(3) By considering mod 8, if two edge labels are equal, then either $8k + a - 4i = a + 4 + 4i' \implies i + i' = 2k - 1$ or $8k + 12 - 4i = 12 + 4i' \implies i + i' = 2k$. Both are impossible. Thus all edge labels are distinct.

(4) $f^+(e_{17} \cap e_{18}) = 11+8k+9 = 8k+20$ and $f^+(e_{8k+10} \cap e_{8k+11}) = 8k+12+8 = 8k+20$.

(5) For $i \in [0, k-2]$,

$$f^+(e_{4i+18} \cap e_{4i+19}) = \begin{cases} 8k+23, & i \text{ even}, \\ 8k+25, & i \text{ odd}, \end{cases}$$

$$f^+(e_{4i+19} \cap e_{4i+20}) = \begin{cases} 8k+25, & i \text{ even}, \\ 8k+23, & i \text{ odd}, \end{cases}$$

$$f^+(e_{4i+20} \cap e_{4i+21}) = \begin{cases} 8k+23, & i \text{ even}, \\ 8k+25, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k+10-4i} \cap e_{8k+9-4i}) = \begin{cases} 8k+25, & i \text{ even}, \\ 8k+23, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k+9-4i} \cap e_{8k+8-4i}) = \begin{cases} 8k+23, & i \text{ even}, \\ 8k+25, & i \text{ odd}, \end{cases}$$

$$f^+(e_{8k+8-4i} \cap e_{8k+7-4i}) = \begin{cases} 8k+25, & i \text{ even}, \\ 8k+23, & i \text{ odd}. \end{cases}$$

For $i \in [0, k-3]$,

$$f^+(e_{4i+21} \cap e_{4i+22}) = \begin{cases} 8k+20, & i \text{ even}, \\ 8k+20, & i \text{ odd}. \end{cases}$$

$$f^+(e_{8k+7-4i} \cap e_{8k+6-4i}) = \begin{cases} 8k+20, & i \text{ even}, \\ 8k+20, & i \text{ odd}. \end{cases}$$

(6) When k is even, $f^+(e_{4k+13} \cap e_{4k+14}) = 12 + 4(k-2) + 4k + 16 = 8k + 20$ and $f^+(e_{4k+14} \cap e_{4k+15}) = 4k + 16 + 15 + 4(k-2) = 8k + 23$.

When k is odd, $f^+(e_{4k+13} \cap e_{4k+14}) = 15 + 4(k-2) + 4k + 16 = 8k + 23$ and $f^+(e_{4k+14} \cap e_{4k+15}) = 4k + 16 + 12 + 4(k-2) = 8k + 20$.

Therefore, f is a local antimagic labeling of $C_3 + C_5 + C_{8k+14}$ with induced vertex labels $8k+20$, $8k+23$, $8k+25$. □

Combining Theorem 6.2, Theorem 6.3 and Theorem 4.1 (with $k = 1$), we have

Theorem 6.4. *For $k \geq 1$ and $k \neq 3$, $\chi_{la}(C_3 + C_5 + C_{4k+2}) = 3$.*

By using computer, we have checked that $\chi_{la}(C_3 + C_5 + C_{14}) = 4$. (Refer to Appendix)

7. $C_3 + C_{4k+1} + C_{4k+5}$ and $C_3 + C_{4k+1} + C_{4k+6}$

Theorem 7.1. *For $k \geq 1$, $\chi_{la}(C_3 + C_{4k+1} + C_{4k+5}) = 3$.*

Proof. By Theorem 1.1, $3 \leq \chi_{la}(C_3 + C_{4k+1} + C_{4k+5}) \leq 5$. We construct a local antimagic 3-coloring of $C_3 + C_{4k+1} + C_{4k+5}$ for $k \geq 1$. Note that there are $8k+9$ edges.

For C_3, let e_i be the i-th edge in the clockwise order for $i \in [1, 3]$. Define f by $f(e_1) = 2k+2, f(e_2) = 6k+8, f(e_3) = 4k+4$. The induced vertex labels are $6k+6, 8k+10, 10k+12$.

For C_{4k+1}, let e_i be the i-th edge in the clockwise order for $i \in [1, 4k+1]$. Define f by $f(e_{4k+1}) = 4k+6$, and for $i \in [1, k]$,

$$\begin{aligned} f(e_{4i-3}) &= 2k + 2 - 2i, \\ f(e_{4i-2}) &= 6k + 8 + 2i, \\ f(e_{4i-1}) &= 4k + 4 - 2i, \\ f(e_{4i}) &= 4k + 6 + 2i. \end{aligned}$$

The edge labels are $2k, 2k-2, \ldots, 2; 6k+10, 6k+12, \ldots, 8k+8; 4k+2, 4k, \ldots, 2k+4; 4k+8, 4k+10, \ldots, 6k+6; 4k+6$, which are all the even integers in $[1, 8k+9]$ except $2k+2, 4k+4, 6k+8$.

For $i \in [1, k]$,

$$f^+(e_{4i-3} \cap e_{4i-2}) = 8k + 10,$$
$$f^+(e_{4i-2} \cap e_{4i-1}) = 10k + 12,$$
$$f^+(e_{4i-1} \cap e_{4i}) = 8k + 10.$$

For $i \in [1, k-1]$,

$$f^+(e_{4i} \cap e_{4i+1}) = 6k + 6,$$

$f^+(e_{4k} \cap e_{4k+1}) = 6k + 6 + 4k + 6 = 10k + 12$, $f^+(e_{4k+1} \cap e_1) = 4k + 6 + 2k = 6k + 6$.

For C_{4k+5}, let e_i be the i-th edge in the clockwise order for $i \in [1, 4k+5]$. Define f by $f(e_{4k+5}) = 4k + 5$, and for $i \in [1, k+1]$,

$$f(e_{4i-3}) = 2k + 3 - 2i,$$
$$f(e_{4i-2}) = 6k + 7 + 2i,$$
$$f(e_{4i-1}) = 4k + 5 - 2i,$$
$$f(e_{4i}) = 4k + 5 + 2i.$$

The edge labels are $2k+1, 2k-1, \ldots, 1; 6k+9, 6k+11, \ldots, 8k+9; 4k+3, 4k+1, \ldots, 2k+3; 4k+7, 4k+9, \ldots, 6k+7; 4k+5$, which are all the odd integers in $[1, 8k+9]$.

For $i \in [1, k+1]$,

$$f^+(e_{4i-3} \cap e_{4i-2}) = 8k + 10,$$
$$f^+(e_{4i-2} \cap e_{4i-1}) = 10k + 12,$$
$$f^+(e_{4i-1} \cap e_{4i}) = 8k + 10.$$

For $i \in [1, k]$,

$$f^+(e_{4i} \cap e_{4i+1}) = 6k + 6,$$

$f^+(e_{4k+4} \cap e_{4k+5}) = 6k + 7 + 4k + 5 = 10k + 12$, $f^+(e_{4k+5} \cap e_1) = 4k + 5 + 2k + 1 = 6k + 6$.

Therefore, f is a local antimagic labeling of $C_3 + C_{4k+1} + C_{4k+5}$ with induced vertex labels $6k + 6$, $8k + 10$, $10k + 12$. □

Theorem 7.2. *For $k \geq 1$, $\chi_{la}(C_3 + C_{4k+1} + C_{4k+6}) = 3$.*

Proof. By Theorem 1.1, $3 \leq \chi_{la}(C_3 + C_{4k+1} + C_{4k+6}) \leq 5$. We construct a local antimagic 3-coloring of $C_3 + C_{4k+1} + C_{4k+6}$ for $k \geq 1$. Note that there are $8k + 10$ edges.

For C_3, let e_i be the i-th edge in the clockwise order for $i \in [1, 3]$. Define f by $f(e_1) = 3k+4, f(e_2) = 5k+7, f(e_3) = 4k+5$. The induced vertex labels are $7k+9, 8k+11, 9k+12$.

For C_{4k+1}, let e_i be the i-th edge in the clockwise order for $i \in [1, 4k+1]$. Define f by $f(e_{4k+1}) = 4k + 6$, and for $i \in [1, k]$,

$$f(e_{4i-3}) = 3k + 4 - i,$$
$$f(e_{4i-2}) = 5k + 7 + i,$$

$$f(e_{4i-1}) = 4k + 5 - i,$$
$$f(e_{4i}) = 4k + 6 + i.$$

The edge labels are $3k+3, 3k+2, \ldots, 2k+4; 5k+8, 5k+9, \ldots, 6k+7; 4k+4, 4k+3, \ldots, 3k+5; 4k+7, 4k+8, \ldots, 5k+6; 4k+6$, which are all the integers in $[2k+4, 6k+7]$ except $3k+4, 4k+5, 5k+7$.

For $i \in [1, k]$,

$$f^+(e_{4i-3} \cap e_{4i-2}) = 8k + 11,$$
$$f^+(e_{4i-2} \cap e_{4i-1}) = 9k + 12,$$
$$f^+(e_{4i-1} \cap e_{4i}) = 8k + 11.$$

For $i \in [1, k-1]$,

$$f^+(e_{4i} \cap e_{4i+1}) = 7k + 9,$$

$f^+(e_{4k} \cap e_{4k+1}) = 5k+6+4k+6 = 9k+12$ and $f^+(e_{4k+1} \cap e_1) = 4k+6+3k+3 = 7k+9$.

For C_{4k+6}, let e_i be the i-th edge in the clockwise order for $i \in [1, 4k+6]$. Define f by $f(e_{4k+5}) = 2k+3$, $f(e_{4k+6}) = 6k+8$, and for $i \in [1, k+1]$,

$$f(e_{4i-3}) = k + 2 - i,$$
$$f(e_{4i-2}) = 7k + 9 + i,$$
$$f(e_{4i-1}) = 2k + 3 - i,$$
$$f(e_{4i}) = 6k + 8 + i.$$

The edge labels are $k+1, k-1, \ldots, 1; 7k+10, 7k+11, \ldots, 8k+10; 2k+2, 2k+1, \ldots, k+2; 6k+9, 6k+10, \ldots, 7k+9; 2k+3; 6k+8$, which are all the integers in $[1, 2k+3] \cup [6k+8, 8k+10]$.

For $i \in [1, k+1]$,

$$f^+(e_{4i-3} \cap e_{4i-2}) = 8k + 11,$$
$$f^+(e_{4i-2} \cap e_{4i-1}) = 9k + 12,$$
$$f^+(e_{4i-1} \cap e_{4i}) = 8k + 11.$$

For $i \in [1, k]$,

$$f^+(e_{4i} \cap e_{4i+1}) = 7k + 9,$$

$f^+(e_{4k+4} \cap e_{4k+5}) = 7k+9+2k+3 = 9k+12$, $f^+(e_{4k+5} \cap e_{4k+6}) = 2k+3+6k+8 = 8k+11$, $f^+(e_{4k+6} \cap e_1) = 6k+8+k+1 = 7k+9$.

Therefore, f is a local antimagic labeling of $C_3 + C_{4k+1} + C_{4k+6}$ with induced vertex labels $7k+9$, $8k+11$, $9k+12$. □

8. Conclusion

Summing up, the main results of this paper are:

(1) $\chi_{la}(C_3 + C_3 + C_3) = 5$ and $\chi_{la}(C_3 + C_3 + C_k) = 4$ for $k \geq 4$.

(2) $\chi_{la}(C_3 + C_4 + C_{2k+1}) = 4$.

(3) $\chi_{la}(C_3 + C_5 + C_{14}) = 4$ and $\chi_{la}(C_3 + C_5 + C_{4k+2}) = 3$ for $k \neq 3$.

(4) $\chi_{la}(C_3 + C_6 + C_{4k+1}) = 3$; $\chi_{la}(C_3 + C_{10} + C_{8k-3}) = 3$; $\chi_{la}(C_3 + C_{4k+1} + C_{4k+5}) = 3$ and $\chi_{la}(C_3 + C_{4k+1} + C_{4k+6}) = 3$.

The next reachable goal is to determine $\chi_{la}(C_3 + C_5 + C_{4k})$, $\chi_{la}(C_3 + C_5 + C_{2k+1})$ and $\chi_{la}(C_3 + C_6 + C_{4k+3})$. Future directions include studying local antimagic labeling of $C_3 + C_a + C_{2k}$ and $C_3 + C_a + C_{2k+1}$ where $a \geq 7$. This may not be easy as there are no general methods of constructing such labelings. Even more challenging questions concern local antimagic labeling of $C_a + C_b + C_{2k}$ and $C_a + C_b + C_{2k+1}$ where both a, b are odd integers greater than 3.

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Appendix

Here we give an algorithm to verify that $\chi_{la}(C_3 + C_5 + C_{14}) \neq 3$. First, we derive some useful results on the edge labels and induced vertex labels of $C_3 + C_5 + C_{2k}$ if $\chi_{la}(C_3 + C_5 + C_{2k}) = 3$.

Let f be a local antimagic 3-coloring of $C_3 + C_5 + C_{2k}$. Denote the induced vertex labels of $C_3 + C_5 + C_{2k}$ by a, b, c. By Lemma 3.2, let $a = 2k + 9$ and $b < a < c$. The edge labels on C_3 are $\frac{a+b-c}{2}$, $\frac{b+c-a}{2}$, $\frac{c+a-b}{2}$. We have $a + b \equiv c \pmod 2$, $b + c \equiv a \pmod 2$, $c + a \equiv b \pmod 2$. So one of b, c is odd and the other is even. By Lemma 2.2, we can assume that b is odd and c is even. Also, (1) $a + b > c$.

Algorithm 1 Check whether $\chi_{la}(C_3 + C_5 + C_{2k}) = 3$

```
a = 2k + 9
for odd integer b in [3, 2k + 7] do
    for even integer c in [2k + 10, 4k + 14] do
        Set all integers in [1, 2k + 8] as available
        Set f(e_i) = 0, f^+(u_i) = 0 for all i ∈ [1, 2k]
        Set p(i) = 0 for all i ∈ [1, 2k]        ▷ p(i) allows the vertex label of u_i to go through a, b, c
        if a, b, c satisfy conditions (1), (2) and (3) then
            Set v(0) = a, v(1) = b, v(2) = c
            Set (a+b−c)/2, (b+c−a)/2, (c+a−b)/2 as unavailable
            Set c/2, a − c/2, b − a + c/2, a − b + c/2, b − c/2 as unavailable
            Find the smallest available edge label d for C_{2k}
            for (f^+(u_1), f^+(u_2)) in {(a, b), (a, c), (b, c)} do
                if d < f^+(u_1) and d < f^+(u_2) then
                    f(e_1) = d and set f(e_1) as unavailable
                    if f^+(u_2) − d is available then
                        f(e_2) = f^+(u_2) − d and set f(e_2) as unavailable
                        i = 3        ▷ u_i is the current vertex under consideration
                        while i > 2 do
                            if i ≤ 2k then
                                if p(i) ≤ 2 then
                                    if p(i) ≠ p(i − 1) and v(p(i)) > f(e_{i−1}) and
                                       v(p(i)) − f(e_{i−1}) is available then
                                        f^+(u_i) = v(p(i))
                                        f(e_i) = f^+(u_i) − f(e_{i−1}) and set f(e_i) as unavailable
                                        i++
                                    else
                                        p(i)++
                                    end if
                                else
                                    p(i) = 0
                                    i− −
                                    Set f(e_i) as available
                                    p(i)++
                                end if
                            else
                                if f(e_{2k}) + f(e_1) = f^+(u_1) then
                                    Print out the local antimagic 3-coloring f of C_3 + C_5 + C_{2k}
                                end if
                                i− −
                                Set f(e_i) as available
                                p(i)++
                            end if
                        end while
                    end if
                end if
            end for
        end if
    end for
end for
```

By considering all possible arrangements of the induced vertex labels of C_5, each one has the pattern x, y, x, y, z in cyclic order where $\{x, y, z\} = \{a, b, c\}$. Solving the following system of linear equations

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & x \\ 0 & 1 & 1 & 0 & 0 & y \\ 0 & 0 & 1 & 1 & 0 & z \\ 0 & 0 & 0 & 1 & 1 & x \\ 1 & 0 & 0 & 0 & 1 & y \end{array}\right],$$

the edge labels of C_5 are $\frac{z}{2}$, $x - \frac{z}{2}$, $y - x + \frac{z}{2}$, $x - y + \frac{z}{2}$, $y - \frac{z}{2}$ in cyclic order. Hence, $z = c$. Therefore, the edge labels of C_5 are $\frac{c}{2}$, $a - \frac{c}{2}$, $b - a + \frac{c}{2}$, $a - b + \frac{c}{2}$, $b - \frac{c}{2}$ in cyclic order which satisfy (2) $b > \frac{c}{2}$ and $b - a + \frac{c}{2} > 0$.

Since all the edge labels on C_3 and C_5 are distinct, we have (3) $\frac{b+c-a}{2} \neq a - \frac{c}{2} \iff b + 2c \neq 3a$.

Algorithm 1 is the pseudocode for checking whether $\chi_{la}(C_3 + C_5 + C_{2k}) = 3$. Denote $C_{2k} = u_1 u_2 \cdots u_{2k} u_1$ where $e_i = u_i u_{i+1}$ for $i \in [1, 2k]$ and $u_{2k+1} = u_1$.

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