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On matrices whose entries are Stirling numbers of the second kind (II)

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ABSTRACT

For each integer $m \geq -1$, we study the family of matrices $\mathsf{S}^{(m)}(n) = [S(i+j+m, j)]_{1 \leq i,j \leq n}$, whose entries are Stirling numbers of the second kind. We derive several explicit matrix decompositions of $\mathsf{S}^{(m)}(n)$ in terms of the classical Stirling matrix and certain explicitly constructed upper triangular matrices. As a consequence, we obtain the closed-form determinant formula $\det \mathsf{S}^{(m)}(n) = \prod_{i=1}^{n} i^{i+m}$, which extends several previously known determinant evaluations. We also establish several identities involving both the Stirling numbers of the first and second kinds.

Keywords: stirling numbers of the first and second kinds, matrix decomposition, determinant

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1. Introduction

For integers n and k, let $c(n, k)$ denote the number of permutations of n distinct objects that have exactly k cycles. The number $c(n, k)$ is known as a *signless Stirling number of the first kind*. The *signed Stirling number of the first kind* is defined by $s(n, k) = (-1)^{n-k}c(n, k)$. We also denote by $S(n, k)$ the *Stirling number of the second kind*, which is defined as the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ into exactly k nonempty subsets. It follows from the definitions that for every integer $n \geq 1$,

$$s(n, k) = S(n, k) = 0, \quad k > n,$$

$$c(n, 0) = s(n, 0) = S(n, 0) = 0, \qquad S(n, 1) = 1,$$

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and for every $n \geq 0$,

$$c(n, n) = s(n, n) = S(n, n) = 1.$$

It is well known (see [15, pages 26, 74] or [3, Theorems 8.2.4 and 8.2.8]) that for every $n, k \geq 1$ the numbers $c(n, k)$ and $S(n, k)$ satisfy the following recurrence relation:

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k), \tag{1}$$

and

$$S(n, k) = S(n-1, k-1) + kS(n-1, k). \tag{2}$$

Alternatively, the Stirling numbers of the first and second kinds may be defined as the coefficients in the following two expansions in the variable x (see [3, p. 282]):

$$[x]_n = \sum_{k=0}^{n} s(n, k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S(n, k)[x]_k, \tag{3}$$

where

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1),$$

is the falling factorial (with $[x]_0 = 1$).

The $n \times n$ *Stirling matrices of the first and second kinds* are defined by

$$s(n) = [s_{i,j}]_{1 \leq i,j \leq n}, \qquad s_{i,j} = \begin{cases} s(i, j), & \text{if } i \geq j; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$S(n) = [S_{i,j}]_{1 \leq i,j \leq n}, \qquad S_{i,j} = \begin{cases} S(i, j), & \text{if } i \geq j; \\ 0, & \text{otherwise,} \end{cases}$$

respectively. The Stirling matrices $s(5)$ and $S(5)$, for example, are

$$s(5) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & \cdot & \cdot \\ 2 & -3 & 1 & \cdot & \cdot \\ -6 & 11 & -6 & 1 & \cdot \\ 24 & -50 & 35 & -10 & 1 \end{bmatrix} \quad \text{and} \quad S(5) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 3 & 1 & \cdot & \cdot \\ 1 & 7 & 6 & 1 & \cdot \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}.$$

It follows immediately from (3) that the Stirling matrices of the first kind and the second kind are inverses of each other, that is

$$S(n)^{-1} = s(n) \quad \text{and} \quad s(n)^{-1} = S(n).$$

Stirling matrices have been studied extensively; see, for example, [2, 1, 4, 5, 6, 7, 8, 9, 12, 16, 17].

Let $m \geq 0$ be an integer. In [1], the authors introduced a family of matrices

$$\mathsf{S}^{[m]}(n) = \left[\mathsf{S}^{[m]}_{i,j}\right]_{1 \leq i,j \leq n},$$

whose entries are

$$\mathsf{S}^{[m]}_{i,j} = S(i+m,j) \quad \text{for} \quad 1 \le i,j \le n.$$

The entries $\mathsf{S}^{[m]}_{i,j}$ satisfy the following recurrence:

$$\mathsf{S}^{[m]}_{i,j} = \mathsf{S}^{[m]}_{i-1,j-1} + j\mathsf{S}^{[m]}_{i-1,j}, \quad \text{for} \quad 2 \le i,j \le n, \tag{4}$$

with the initial conditions given by:

$$\mathsf{S}^{[m]}_{i,1} = 1, \quad \mathsf{S}^{[m]}_{1,j} = S(m+1,j), \quad \text{for} \quad i,j \ge 1. \tag{5}$$

In [1, Theorem 2], explicit matrix decompositions of $\mathsf{S}^{[m]}(n)$ were established, leading to the determinant formula

$$\det \mathsf{S}^{[m]}(n) = \prod_{i=1}^{n} i^m = n!^m\,.$$

In this paper, we investigate a different family of matrices in which the Stirling numbers depend on both the row and column indices. This additional dependence leads to new matrix decompositions and determinant formulas for this family of matrices. Specifically, we study the matrices

$$\mathsf{S}^{(m)}(n) = [\mathsf{S}^{(m)}_{i,j}]_{1 \le i,j \le n},$$

whose entries are

$$\mathsf{S}^{(m)}_{i,j} = S(i+j+m,j) \quad \text{for} \quad 1 \le i,j \le n.$$

For example, the matrices $\mathsf{S}^{(-1)}(4)$ and $\mathsf{S}^{(0)}(4)$ are given by

$$\mathsf{S}^{(-1)}(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 \\ 1 & 7 & 25 & 65 \\ 1 & 15 & 90 & 350 \end{bmatrix} \quad \text{and} \quad \mathsf{S}^{(0)}(4) = \begin{bmatrix} 1 & 3 & 6 & 10 \\ 1 & 7 & 25 & 65 \\ 1 & 15 & 90 & 350 \\ 1 & 31 & 301 & 1701 \end{bmatrix}.$$

Note that the entries $\mathsf{S}^{(m)}_{i,j}$ satisfy the following recurrence:

$$\mathsf{S}^{(m)}_{i,j} = \mathsf{S}^{(m)}_{i,j-1} + j\mathsf{S}^{(m)}_{i-1,j}, \quad \text{for} \quad 2 \le i,j \le n, \tag{6}$$

with initial conditions given by:

$$\mathsf{S}^{(m)}_{i,1} = 1, \quad \mathsf{S}^{(m)}_{1,j} = S(1+j+m,j), \quad \text{for} \quad i,j \ge 1. \tag{7}$$

The recurrence relation (6), together with the initial conditions (7), yields explicit matrix decompositions of $\mathsf{S}^{(m)}(n)$ (see Theorem 3.3). As a consequence, we obtain the explicit determinant formula

$$\det \mathsf{S}^{(m)}(n) = \prod_{i=1}^{n} i^{i+m}.$$

We also derive several new identities involving the Stirling numbers of the first and second kinds.

The remainder of the paper is organized as follows. Additional notation, definitions, and auxiliary results are collected in Section 2. In Section 3, we establish the main decomposition theorem (Theorem 3.3) for the matrices $\mathsf{S}^{(m)}(n)$. In Section 4, we derive some identities involving the Stirling numbers of the first and second kinds.

2. Auxiliary results

In this section, we collect several auxiliary results that will be used throughout the sequel. We begin by introducing the $n \times n$ upper triangular matrix $V(n) = [V_{i,j}]_{1\leq i,j\leq n}$, whose entries are defined by

$$V_{i,j} = \begin{cases} j, & \text{if } j = i; \\ 1, & \text{if } j = i+1; \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma describes the entries of $V(n)^m$ for every nonnegative integer m.

Lemma 2.1. *Let m be a nonnegative integer and $V(n)^m = [V^m_{i,j}]_{1\leq i,j\leq n}$. Then $V(n)^m$ is upper triangular, and its entries satisfy*

$$V^m_{i,j} = \begin{cases} S(m+1, j), & \text{if } i = 1, j \geq 1; \\ V^m_{i-1,i-1} + V^m_{i-1,i}, & \text{if } j = i > 1; \\ V^m_{i-1,j-1} + (j-i+1)V^m_{i-1,j}, & \text{if } j > i > 1; \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Proof. Since every power of an upper triangular matrix is again upper triangular, it suffices to prove that the entries $V^m_{i,j}$ satisfy (8) whenever $j \geq i$. We proceed by induction on m. The statement is immediate for $m = 0$. Hence, assume that $m \geq 1$. Then, since $V(n)^m = V(n)V(n)^{m-1}$, it follows that

$$\begin{aligned} V^m_{1,j} &= \sum_{l=1}^{n} V_{1,l}V^{m-1}_{l,j} \\ &= V_{1,1}V^{m-1}_{1,j} + V_{1,2}V^{m-1}_{2,j} \quad \text{(by the structure of } V(n)) \\ &= 1 \cdot V^{m-1}_{1,j} + 1 \cdot (V^{m-1}_{1,j-1} + (j-1)V^{m-1}_{1,j}) \quad \text{(by the inductive hypothesis)} \\ &= V^{m-1}_{1,j-1} + jV^{m-1}_{1,j} \\ &= S(m, j-1) + jS(m, j) \\ &= S(m+1, j), \end{aligned}$$

for $j > 2$. The remaining cases $j = 1$ and $j = 2$ are handled separately.

$$V^m_{1,1} = \sum_{l=1}^{n} V_{1,l}V^{m-1}_{l,1} = V_{1,1}V^{m-1}_{1,1} = 1 \cdot S(m, 1) = 1 \cdot 1 = 1 = S(m+1, 1),$$

and

$$\begin{aligned} V^m_{1,2} &= \sum_{l=1}^{n} V_{1,l}V^{m-1}_{l,2} \\ &= V_{1,1}V^{m-1}_{1,2} + V_{1,2}V^{m-1}_{2,2} \quad \text{(by the structure of } V(n)) \\ &= 1 \cdot V^{m-1}_{1,2} + 1 \cdot (V^{m-1}_{1,1} + V^{m-1}_{1,2}) \quad \text{(by the inductive hypothesis)} \end{aligned}$$

$$
\begin{aligned}
&= V_{1,1}^{m-1} + 2V_{1,2}^{m-1} \\
&= S(m,1) + 2S(m,2) \quad \text{(by the inductive hypothesis)} \\
&= 1 + 2(2^{m-1} - 1) \quad (\text{because for } m \geq 1,\ S(m,2) = 2^{m-1} - 1) \\
&= 2^m - 1 \\
&= S(m+1,2).
\end{aligned}
$$

We next derive the recurrence for the diagonal entries. We have

$$
\begin{aligned}
V_{i,i}^m &= \sum_{l=1}^{n} V_{i,l} V_{l,i}^{m-1} = V_{i,i} V_{i,i}^{m-1} = i V_{i,i}^{m-1} = i(V_{i-1,i-1}^{m-1} + V_{i-1,i}^{m-1}) \\
&= (i-1)(V_{i-1,i-1}^{m-1} + V_{i-1,i}^{m-1}) + V_{i-1,i-1}^{m-1} + V_{i-1,i}^{m-1} \\
&= (i-1)V_{i-1,i-1}^{m-1} + (i-1)V_{i-1,i}^{m-1} + V_{i,i}^{m-1}, \ \text{(by the inductive hypothesis)}
\end{aligned}
$$

and

$$
\begin{aligned}
V_{i-1,i-1}^m + V_{i-1,i}^m &= \sum_{l=1}^{n} V_{i-1,l} V_{l,i-1}^{m-1} + \sum_{l=1}^{n} V_{i-1,l} V_{l,i}^{m-1} \\
&= V_{i-1,i-1} V_{i-1,i-1}^{m-1} + V_{i-1,i-1} V_{i-1,i}^{m-1} + V_{i-1,i} V_{i,i}^{m-1} \\
&= (i-1)V_{i-1,i-1}^{m-1} + (i-1)V_{i-1,i}^{m-1} + V_{i,i}^{m-1}.
\end{aligned}
$$

Comparing the two equalities yields

$$
V_{i,i}^m = V_{i-1,i-1}^m + V_{i-1,i}^m.
$$

It remains to consider the case $j > i > 1$. First, by the structure of $V(n)$, we have

$$
V_{i,j}^m = \sum_{l=1}^{n} V_{i,l} V_{l,j}^{m-1} = V_{i,i} V_{i,j}^{m-1} + V_{i,i+1} V_{i+1,j}^{m-1} = i V_{i,j}^{m-1} + V_{i+1,j}^{m-1}. \tag{9}
$$

Next, we obtain

$$
\begin{aligned}
V_{i-1,j-1}^m &= \sum_{l=1}^{n} V_{i-1,l} V_{l,j-1}^{m-1} \\
&= V_{i-1,i-1} V_{i-1,j-1}^{m-1} + V_{i-1,i} V_{i,j-1}^{m-1} \\
&= (i-1)V_{i-1,j-1}^{m-1} + V_{i,j-1}^{m-1},
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
V_{i-1,j}^m &= \sum_{l=1}^{n} V_{i-1,l} V_{l,j}^{m-1} \\
&= V_{i-1,i-1} V_{i-1,j}^{m-1} + V_{i-1,i} V_{i,j}^{m-1} \\
&= (i-1)V_{i-1,j}^{m-1} + V_{i,j}^{m-1}.
\end{aligned} \tag{11}
$$

Now, by Eqs. (9), (10), and (11) and the inductive hypothesis, we obtain

$$\begin{aligned}
&V^m_{i-1,j-1} + (j-i+1)V^m_{i-1,j}\\
&= (i-1)V^{m-1}_{i-1,j-1} + V^{m-1}_{i,j-1} + (j-i+1)[(i-1)V^{m-1}_{i-1,j} + V^{m-1}_{i,j}] \quad \text{(by (10) and (11))}\\
&= (i-1)[V^{m-1}_{i-1,j-1} + (j-i+1)V^{m-1}_{i-1,j}] + V^{m-1}_{i,j-1} + (j-i+1)V^{m-1}_{i,j}\\
&= (i-1)V^{m-1}_{i,j} + [V^{m-1}_{i,j-1} + (j-i)V^{m-1}_{i,j}] + V^{m-1}_{i,j} \quad \text{(by the inductive hypothesis)}\\
&= iV^{m-1}_{i,j} + V^{m-1}_{i+1,j} \quad \text{(by the inductive hypothesis)}\\
&= V^m_{i,j}. \quad \text{(by (9))}
\end{aligned}$$

This completes the proof. □

We now define an $n \times n$ matrix $U^{(-1)}(n) = \left[U^{(-1)}_{i,j}\right]_{1\le i,j\le n}$ whose entries satisfy the recurrence relation:

$$U^{(-1)}_{i,j} = U^{(-1)}_{i,j-1} + (i-1)U^{(-1)}_{i-1,j-1} + (j-i+1)U^{(-1)}_{i-1,j}, \qquad 2 \le i,j \le n, \tag{12}$$

with the initial conditions

$$U^{(-1)}_{1,1} = 1, \quad U^{(-1)}_{i,1} = 0, \quad U^{(-1)}_{1,j} = 1, \quad \text{for} \quad 2 \le i,j \le n. \tag{13}$$

For example, when $n = 5$,

$$U^{(-1)}(5) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ \cdot & 2 & 5 & 9 & 14\\ \cdot & \cdot & 9 & 37 & 97\\ \cdot & \cdot & \cdot & 64 & 369\\ \cdot & \cdot & \cdot & \cdot & 625 \end{bmatrix}.$$

The upper triangular matrix $U^{(-1)}(n)$ plays a crucial role in what follows. We now investigate some of its properties.

Lemma 2.2. *Let $j > i \ge 2$. Then, using the notation introduced above, the following identity holds.*

$$\sum_{k=i}^{j} kU^{(-1)}_{i,k} = iU^{(-1)}_{i,j} + U^{(-1)}_{i+1,j}.$$

Proof. We proceed by induction on j. If $j = i+1$, then

$$\begin{aligned}
\sum_{k=i}^{i+1} kU^{(-1)}_{i,k} &= iU^{(-1)}_{i,i} + (i+1)U^{(-1)}_{i,i+1}\\
&= iU^{(-1)}_{i,i} + (i+1)(U^{(-1)}_{i,i} + (i-1)U^{(-1)}_{i-1,i} + 2U^{(-1)}_{i-1,i+1})\\
&\quad \text{(by (12) with } j = i+1)
\end{aligned}$$

$$
\begin{aligned}
&= i(U_{i,i}^{(-1)} + (i-1)U_{i-1,i}^{(-1)} + 2U_{i-1,i+1}^{(-1)}) \\
&+ iU_{i,i}^{(-1)} + U_{i,i}^{(-1)} + (i-1)U_{i-1,i}^{(-1)} + 2U_{i-1,i+1}^{(-1)} \\
&= iU_{i,i+1}^{(-1)} + iU_{i,i}^{(-1)} + U_{i,i+1}^{(-1)} \quad \text{(by (12))} \\
&= iU_{i,i+1}^{(-1)} + U_{i+1,i+1}^{(-1)}, \quad \text{(by (12))},
\end{aligned}
$$

and we are done in this case. We can assume, therefore, that $j > i+1$. Then, we have

$$
\begin{aligned}
\sum_{k=i}^{j} kU_{i,k}^{(-1)} &= \sum_{k=i}^{j-1} kU_{i,k}^{(-1)} + jU_{i,j}^{(-1)} \\
&= iU_{i,j-1}^{(-1)} + U_{i+1,j-1}^{(-1)} + jU_{i,j}^{(-1)} \quad \text{(by the inductive hypothesis)} \\
&= iU_{i,j}^{(-1)} + (U_{i+1,j-1}^{(-1)} + iU_{i,j-1}^{(-1)} + (j-i)U_{i,j}^{(-1)}) \\
&= iU_{i,j}^{(-1)} + U_{i+1,j}^{(-1)}, \qquad \text{(by (12) with } i \text{ replaced by } i+1),
\end{aligned}
$$

completing the induction. $\square$

Finally, we define an upper triangular matrix $C(n) = [C_{i,j}]_{1\le i,j\le n}$ whose entries satisfy

$$
C_{i,j} = \begin{cases} i, & \text{if } i \le j, \\ 0, & \text{otherwise.} \end{cases} \tag{14}
$$

The next lemma establishes a close relationship among the matrices $V(n)$, $U^{(-1)}(n)$, and $C(n)$. In particular, it shows that $V(n)$ and $C(n)$ are similar. Since $U^{(-1)}(n)$ is upper triangular with diagonal entries

$$
U_{i,i}^{(-1)} = i^i \neq 0, \qquad 1 \le i \le n,
$$

it is invertible.

Lemma 2.3. *With the above notation, we have $U^{(-1)}(n) \cdot C(n) = V(n)U^{(-1)}(n)$.*

Proof. Let $C = C(n)$, $U = U^{(-1)}(n)$, and $V = V(n)$. Since the product of two upper triangular matrices is again an upper triangular matrix, it follows that each of the matrices $U \cdot C$ and $V \cdot U$ is an upper triangular matrix. Therefore, it suffices to prove that

$$
(U \cdot C)_{i,j} = (V \cdot U)_{i,j},
$$

for every $j \ge i$. We distinguish three cases. We first consider the case $j = i$. Then

$$
(U \cdot C)_{i,i} = \sum_{k=1}^{n} U_{i,k}C_{k,i} = U_{i,i}C_{i,i} = U_{i,i}i,
$$

and

$$
(V \cdot U)_{i,i} = \sum_{k=1}^{n} V_{i,k}U_{k,i} = V_{i,i}U_{i,i} = iU_{i,i},
$$

and so $(U \cdot C)_{i,i} = (V \cdot U)_{i,i}$, as required. Next suppose that $j > i = 1$. In this case, it follows that

$$(U \cdot C)_{1,j} = \sum_{k=1}^{n} U_{1,k} C_{k,j} = \sum_{k=1}^{n} C_{k,j} = \sum_{k=1}^{j} k = \binom{j+1}{2},$$

and

$$(V \cdot U)_{1,j} = \sum_{k=1}^{n} V_{1,k} U_{k,j} = V_{1,1} U_{1,j} + V_{1,2} U_{2,j} = U_{1,j} + U_{2,j}.$$

However, applying the special case of (12) with $i = 2$ and using the fact that $U_{1,j-1} = U_{1,j} = 1$, we get $U_{2,j} = U_{2,j-1} + j$, which implies that $U_{2,j} = \sum_{k=2}^{j} k$. Thus

$$U_{1,j} + U_{2,j} = \sum_{k=1}^{j} k = \binom{j+1}{2}.$$

We therefore conclude that $(U \cdot C)_{1,j} = (V \cdot U)_{1,j}$, as required. Finally, assume that $j > i \geq 2$. We have

$$(U \cdot C)_{i,j} = \sum_{k=1}^{n} U_{i,k} C_{k,j} = \sum_{k=i}^{j} U_{i,k} C_{k,j} = \sum_{k=i}^{j} U_{i,k} k.$$

Furthermore, using Lemma 2.2, we obtain

$$(V \cdot U)_{i,j} = \sum_{k=1}^{n} V_{i,k} U_{k,j} = V_{i,i} U_{i,j} + V_{i,i+1} U_{i+1,j} = i U_{i,j} + U_{i+1,j} = \sum_{k=i}^{j} k U_{i,k}.$$

Hence $(U \cdot C)_{i,j} = (V \cdot U)_{i,j}$, completing the proof. □

We conclude this section with the following simple identity involving the Stirling numbers of the second kind. Although straightforward, it plays an important role in the proof of Lemma 3.2.

Lemma 2.4. *Let $m \geq 0$ be an integer. Then, for every integer $j \geq 1$, we have*

$$\sum_{k=1}^{j} k S(k+m, k) = S(1+j+m, j).$$

Proof. We prove the identity by induction on j. If $j = 1$, then

$$\sum_{k=1}^{1} k S(k+m, k) = S(1+m, 1) = 1 = S(2+m, 1).$$

We can thus assume that $j > 1$. Now, using the inductive hypothesis and (2), we obtain

$$\sum_{k=1}^{j} k S(k+m, k) = \sum_{k=1}^{j-1} k S(k+m, k) + j S(j+m, j)$$

$$= S(j+m, j-1) + jS(j+m, j)$$
$$= S(j+1+m, j),$$

and the proof of the lemma is complete. □

3. The matrices $\mathsf{S}^{(m)}(n)$

In this section, we investigate the $n \times n$ matrices $\mathsf{S}^{(m)}(n) = [\mathsf{S}_{i,j}^{(m)}]_{1 \le i,j \le n}$, introduced in the Introduction. In particular, we derive several matrix decompositions, together with a closed formula for their determinants. We begin with the following observation.

Lemma 3.1. *Let $i \ge 2$, $j \ge 1$ and $m \ge -1$ be integers. Then the following identity holds:*

$$\sum_{k=1}^{j} k\mathsf{S}_{i-1,k}^{(m)} = \mathsf{S}_{i,j}^{(m)}.$$

Proof. Since there is nothing to prove if $j = 1$, we can assume that $j > 1$, and we proceed by induction on j. It will then follow from (6) and the inductive hypothesis that

$$\sum_{k=1}^{j} k\mathsf{S}_{i-1,k}^{(m)} = \sum_{k=1}^{j-1} k\mathsf{S}_{i-1,k}^{(m)} + j\mathsf{S}_{i-1,j}^{(m)} = \mathsf{S}_{i,j-1}^{(m)} + j\mathsf{S}_{i-1,j}^{(m)} = \mathsf{S}_{i,j}^{(m)},$$

as required. □

The following lemma establishes a useful matrix decomposition of $\mathsf{S}^{(m)}(n)$, which plays a central role in the proof of Theorem 3.3.

Lemma 3.2. *Let $m \ge 0$ and $n \ge 1$ be integers. Then, the matrix $\mathsf{S}^{(m)}(n)$ has the following matrix decomposition:*

$$\mathsf{S}^{(m)}(n) = \mathsf{S}^{(m-1)}(n) \cdot C(n). \tag{15}$$

Proof. Let $\mathsf{S}^{(m-1)} = \mathsf{S}^{(m-1)}(n)$ and $C = C(n)$. To prove (15), we compute the (i, j)-entry of the product $\mathsf{S}^{(m-1)} \cdot C$:

$$(\mathsf{S}^{(m-1)} \cdot C)_{i,j} = \sum_{k=1}^{j} \mathsf{S}_{i,k}^{(m-1)} C_{k,j},$$

and, in view of Eqs. (7) and (6), it suffices to verify that

$$(\mathsf{S}^{(m-1)} \cdot C)_{i,1} = 1, \quad (\mathsf{S}^{(m-1)} \cdot C)_{1,j} = S(j+m+1, j), \quad \text{for} \quad i, j \ge 1, \tag{16}$$

and

$$(\mathsf{S}^{(m-1)} \cdot C)_{i,j} = (\mathsf{S}^{(m-1)} \cdot C)_{i,j-1} + j(\mathsf{S}^{(m-1)} \cdot C)_{i-1,j}, \quad \text{for} \quad 2 \le i, j \le n. \tag{17}$$

First consider the case $j = 1$. We obtain

$$(\mathsf{S}^{(m-1)} \cdot C)_{i,1} = \sum_{k=1}^{1} \mathsf{S}_{i,k}^{(m-1)} C_{k,1} = \mathsf{S}_{i,1}^{(m-1)} C_{1,1} = 1 \cdot 1 = 1.$$

Next consider the case $i = 1$. By Lemma 2.4, we get

$$(\mathsf{S}^{(m-1)} \cdot C)_{1,j} = \sum_{k=1}^{j} \mathsf{S}_{1,k}^{(m-1)} C_{k,j} = \sum_{k=1}^{j} S(k+m,k)k = S(1+j+m,j),$$

and the proof of (16) is complete.

Finally, let $2 \le i, j \le n$. In this case, we have

$$\begin{aligned}
(\mathsf{S}^{(m-1)} \cdot C)_{i,j} &= \sum_{k=1}^{j} \mathsf{S}_{i,k}^{(m-1)} C_{k,j} = \sum_{k=2}^{j} \mathsf{S}_{i,k}^{(m-1)} C_{k,j} + \mathsf{S}_{i,1}^{(m-1)} C_{1,j} \\
&= \sum_{k=2}^{j} (\mathsf{S}_{i,k-1}^{(m-1)} + k\mathsf{S}_{i-1,k}^{(m-1)}) C_{k,j} + 1 \cdot 1 \\
&= \sum_{k=2}^{j} \mathsf{S}_{i,k-1}^{(m-1)} C_{k,j} + \sum_{k=2}^{j} k\mathsf{S}_{i-1,k}^{(m-1)} C_{k,j} + 1 \\
&= \sum_{k=1}^{j-1} \mathsf{S}_{i,k}^{(m-1)} C_{k+1,j} + \sum_{k=2}^{j} k\mathsf{S}_{i-1,k}^{(m-1)} C_{k,j} + 1 \\
&= \sum_{k=1}^{j-1} \mathsf{S}_{i,k}^{(m-1)} (k+1) + \sum_{k=2}^{j} k\mathsf{S}_{i-1,k}^{(m-1)} C_{k,j} + 1 \\
&= \sum_{k=1}^{j-1} \mathsf{S}_{i,k}^{(m-1)} k + \sum_{k=1}^{j-1} \mathsf{S}_{i,k}^{(m-1)} + \sum_{k=2}^{j} k\mathsf{S}_{i-1,k}^{(m-1)} C_{k,j} + 1 \\
&= \sum_{k=1}^{j-1} \mathsf{S}_{i,k}^{(m-1)} C_{k,j-1} + \sum_{k=2}^{j} \mathsf{S}_{i,k-1}^{(m-1)} + \sum_{k=2}^{j} k\mathsf{S}_{i-1,k}^{(m-1)} C_{k,j} + 1 \\
&= (\mathsf{S}^{(m-1)} \cdot C)_{i,j-1} + \sum_{k=2}^{j} (\mathsf{S}_{i,k-1}^{(m-1)} + k^2\mathsf{S}_{i-1,k}^{(m-1)}) + 1.
\end{aligned}$$

Therefore, to establish (17), it remains to show that

$$\sum_{k=2}^{j} \left[\mathsf{S}_{i,k-1}^{(m-1)} + k^2 \mathsf{S}_{i-1,k}^{(m-1)} \right] = j \sum_{k=1}^{j} k\mathsf{S}_{i-1,k}^{(m-1)} - 1.$$

For $j = 2$, the identity follows by direct computation, and so we can assume that $j > 2$ and we proceed by induction on j. By the inductive hypothesis, we obtain

$$\sum_{k=2}^{j} (\mathsf{S}_{i,k-1}^{(m-1)} + k^2\mathsf{S}_{i-1,k}^{(m-1)}) = \sum_{k=2}^{j-1} (\mathsf{S}_{i,k-1}^{(m-1)} + k^2\mathsf{S}_{i-1,k}^{(m-1)}) + \mathsf{S}_{i,j-1}^{(m-1)} + j^2\mathsf{S}_{i-1,j}^{(m-1)}$$

$$
\begin{aligned}
&= (j-1)\sum_{k=1}^{j-1} k\mathsf{S}_{i-1,k}^{(m-1)} - 1 + \mathsf{S}_{i,j-1}^{(m-1)} + j^2\mathsf{S}_{i-1,j}^{(m-1)} \\
&= j\sum_{k=1}^{j} k\mathsf{S}_{i-1,k}^{(m-1)} - \sum_{k=1}^{j-1} k\mathsf{S}_{i-1,k}^{(m-1)} - 1 + \mathsf{S}_{i,j-1}^{(m-1)} \\
&= j\sum_{k=1}^{j} k\mathsf{S}_{i-1,k}^{(m-1)} - 1, \quad \text{(by Lemma 3.1)}
\end{aligned}
$$

as required. This completes the proof. □

We now turn to the evaluation of the determinants of the matrices $\mathsf{S}^{(m)}(n)$. The cases $m=-1$ and $m=0$ have been studied previously. We briefly recall them here because they illustrate the underlying matrix decompositions and motivate the general theorem proved in the next section.

Case 1. $m=-1$. In this case, we have

$$
\det \mathsf{S}^{(-1)}(n) = \prod_{i=1}^{n} i^{\,i-1}. \tag{18}
$$

Theorem 44 of [11] contains this result as a special case. As mentioned in [11], this theorem is due to E. Neuwirth, which originates from an unpublished manuscript. To evaluate determinants of this type, Neuwirth uses *matrix decompositions*. Note that Main Theorem (or Theorem 1) in [13] also contains $\det \mathsf{S}^{(-1)}(n)$ as a special case, and its proof is based on the *LU-decomposition method* (see [11, Sec. 4]). For the reader's convenience, we give a sketch of proof in this case. Following the proof of the Main Theorem of [13] (see also [7, Theorem 2]), we obtain

$$
\mathsf{S}^{(-1)}(n) = S(n)\cdot U^{(-1)}(n). \tag{19}
$$

For instance, when $n=5$, we have

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1\\
1 & 3 & 6 & 10 & 15\\
1 & 7 & 25 & 65 & 140\\
1 & 15 & 90 & 350 & 1050\\
1 & 31 & 301 & 1701 & 6951
\end{bmatrix}
=
\begin{bmatrix}
1 & \cdot & \cdot & \cdot & \cdot\\
1 & 1 & \cdot & \cdot & \cdot\\
1 & 3 & 1 & \cdot & \cdot\\
1 & 7 & 6 & 1 & \cdot\\
1 & 15 & 25 & 10 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 1 & 1 & 1 & 1\\
\cdot & 2 & 5 & 9 & 14\\
\cdot & \cdot & 9 & 37 & 97\\
\cdot & \cdot & \cdot & 64 & 369\\
\cdot & \cdot & \cdot & \cdot & 625
\end{bmatrix}.
$$

The Stirling matrix $S(n)$ is a lower triangular matrix with 1's on the diagonal, whereas $U^{(-1)}(n)$ is an upper triangular matrix with diagonal entries $1^0, 2^1, \ldots, n^{n-1}$. Consequently, the decomposition immediately yields (18).

Case 2. $m=0$. In this case (see [11, Section 2, (2.2)]), we have[1]

$$
\det \mathsf{S}^{(0)}(n) = \prod_{i=1}^{n} i^{i}. \tag{20}
$$

[1] In particular, this provides a determinant representation of the integer sequence of hyperfactorials $h_n = 1^1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n$ (A002109 in [14]).

This follows immediately from the following matrix decomposition (see [11, p. 69], with a misprint corrected):

$$[(-1)^i S(i+j,i)]_{1\le i,j\le n} = [(-1)^k k^i/(k!\,(i-k)!\,)]_{1\le i,k\le n} \cdot [k^j]_{1\le k,j\le n},$$

and applying [10, Theorem 26, (3.14)] to the first determinant, and applying the Vandermonde determinant evaluation to the second one.

Once again, let us use the LU-decomposition method. Arguing as in the proof of the Main Theorem of [13], we obtain

$$\mathsf{S}^{(0)}(n) = S(n) \cdot U^{(0)}(n),$$

where $S(n)$ is the $n \times n$ Stirling matrix, and where $U^{(0)}(n) = [U^{(0)}_{i,j}]_{1\le i,j\le n}$ with

$$U^{(0)}_{1,1} = 1,\ U^{(0)}_{i,1} = 0,\ U^{(0)}_{1,j} = S(j+1,j) \quad (2 \le i,j \le n),$$

and

$$U^{(0)}_{i,j} = U^{(0)}_{i,j-1} + (i-1)U^{(0)}_{i-1,j-1} + (j-i+1)U^{(0)}_{i-1,j} \quad (2 \le i,j \le n).$$

For instance, when $n = 5$, we have

$$\begin{bmatrix} 1 & 3 & 6 & 10 & 15 \\ 1 & 7 & 25 & 65 & 140 \\ 1 & 15 & 90 & 350 & 1050 \\ 1 & 31 & 301 & 1701 & 6951 \\ 1 & 63 & 966 & 7770 & 42525 \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 3 & 1 & \cdot & \cdot \\ 1 & 7 & 6 & 1 & \cdot \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 6 & 10 & 15 \\ \cdot & 4 & 19 & 55 & 125 \\ \cdot & \cdot & 27 & 175 & 660 \\ \cdot & \cdot & \cdot & 256 & 2101 \\ \cdot & \cdot & \cdot & \cdot & 3125 \end{bmatrix}.$$

The Stirling matrix $S(n)$ is a lower triangular matrix with 1's on the diagonal, whereas $U^{(0)}(n)$ is an upper triangular matrix with diagonal entries $1^1, 2^2, \ldots, n^n$. Consequently, the decomposition immediately yields the (20).

These two examples illustrate the matrix decomposition underlying the general case. We now establish the corresponding matrix decompositions and determinant formula for arbitrary integers $m \ge -1$.

Theorem 3.3. *Let $m \ge -1$ be an integer. Then, for every positive integer n, the following matrix decompositions hold:*

$$\mathsf{S}^{(m)}(n) = S(n) \cdot V(n)^{m+1} \cdot U^{(-1)}(n), \tag{21}$$

$$\mathsf{S}^{(m)}(n) = S(n) \cdot U^{(-1)}(n) \cdot C(n)^{m+1}, \tag{22}$$

$$\mathsf{S}^{(m)}(n) = \mathsf{S}^{(-1)}(n) \cdot C(n)^{m+1}. \tag{23}$$

Also, we have

$$\det \mathsf{S}^{(m)}(n) = \det_{1\le i,j\le n} [S(i+j+m,j)] = \prod_{i=1}^{n} i^{i+m}.$$

Proof. We prove (21) by induction on m. The case $m = -1$ follows from (19). Assume now that $m \geq 0$. We obtain

$$\begin{aligned}\mathsf{S}^{(m)}(n) &= \mathsf{S}^{(m-1)}(n) \cdot C(n) \quad \text{(by Lemma 3.2)}\\ &= S(n) \cdot V(n)^m \cdot U^{(-1)}(n) \cdot C(n) \quad \text{(by the inductive hypothesis)}\\ &= S(n) \cdot V(n)^{m+1} \cdot U^{(-1)}(n) \quad \text{(by Lemma 2.3)},\end{aligned}$$

which completes the proof of (21). Also,

$$\begin{aligned}\mathsf{S}^{(m)}(n) &= S(n) \cdot V(n)^{m+1} \cdot U^{(-1)}(n) \quad \text{(by (21))}\\ &= S(n) \cdot V(n)^m \cdot V(n) \cdot U^{(-1)}(n)\\ &= S(n) \cdot V(n)^m \cdot U^{(-1)}(n) \cdot C(n) \quad \text{(by Lemma 2.3)}\\ &= S(n) \cdot V(n)^{m-1} \cdot V(n) \cdot U^{(-1)}(n) \cdot C(n)\\ &= S(n) \cdot V(n)^{m-1} \cdot U^{(-1)}(n) \cdot C(n)^2 \quad \text{(by Lemma 2.3)}.\end{aligned}$$

Iterating this argument yields (22). The decomposition (23) follows by repeated applications of Lemma 3.2:

$$\mathsf{S}^{(m)}(n) = \mathsf{S}^{(m-1)}(n) \cdot C(n) = \mathsf{S}^{(m-2)}(n) \cdot C(n)^2 = \cdots = \mathsf{S}^{(-1)}(n) \cdot C(n)^{m+1}.$$

Taking determinants in (21) gives

$$\det \mathsf{S}^{(m)}(n) = \det S(n) \cdot \det V(n)^{m+1} \cdot \det U^{(-1)}(n).$$

Since $\det S(n) = 1$, $\det V(n) = \prod_{i=1}^n i$, and $\det U^{(-1)}(n) = \prod_{i=1}^n i^{i-1}$, we obtain

$$\det \mathsf{S}^{(m)}(n) = \prod_{i=1}^{n} i^{i+m},$$

which completes the proof. □

The following corollaries are immediate consequences of Theorem 3.3.

Corollary 3.4. *Let m be an integer with $m \geq -1$. Then, for any positive integer n, we have*

$$\det \mathsf{S}^{(m)}(n) = n!^{\,m+1} \det \mathsf{S}^{(-1)}(n).$$

Corollary 3.5. *Let m be an integer with $m \geq -1$. Then, for any positive integer n, we have*

$$\mathsf{S}^{(m)}(n) = \mathsf{S}^{[m+1]}(n) \cdot U^{(-1)}(n).$$

Proof. First, consider the case when $m = -1$. Using (19) and the fact that $\mathsf{S}^{[0]}(n) = S(n)$, we obtain

$$\mathsf{S}^{(-1)}(n) = S(n) \cdot U^{(-1)}(n) = \mathsf{S}^{[0]}(n) \cdot U^{(-1)}(n).$$

Now assume that $m \geq 0$. It follows from [1, Theorem 2] that

$$\mathsf{S}^{[m+1]}(n) = S(n) \cdot V(n)^{m+1}. \tag{24}$$

Combining (24) with (21) yields the result. □

Corollary 3.6. *Let $m_1, m_2 \geq -1$ be integers. Then, for any positive integer n, we have*

$$\mathsf{S}^{(m_1+m_2+1)}(n) = \mathsf{S}^{(m_1)}(n) \cdot \mathsf{S}^{(-1)}(n)^{-1} \cdot \mathsf{S}^{(m_2)}(n).$$

Proof. It follows from (23) that for any integer $m \geq -1$, we have

$$C(n)^{m+1} = \mathsf{S}^{(-1)}(n)^{-1} \cdot \mathsf{S}^{(m)}(n).$$

Since

$$C(n)^{m_1+m_2+2} = C(n)^{m_1+1} \cdot C(n)^{m_2+1},$$

we obtain

$$\mathsf{S}^{(-1)}(n)^{-1} \cdot \mathsf{S}^{(m_1+m_2+1)}(n) = \mathsf{S}^{(-1)}(n)^{-1} \cdot \mathsf{S}^{(m_1)}(n) \cdot \mathsf{S}^{(-1)}(n)^{-1} \cdot \mathsf{S}^{(m_2)}(n).$$

The result follows by left-multiplying both sides by $\mathsf{S}^{(-1)}(n)$. □

Although Corollaries 3.4–3.6 follow directly from Theorem 3.3, we state them separately because they emphasize different aspects of the main decomposition. Corollary 3.4 gives an explicit multiplicative relation between the determinants of $\mathsf{S}^{(m)}(n)$ and the fundamental case $m = -1$. Corollary 3.5 establishes a direct connection between the matrices $\mathsf{S}^{(m)}(n)$ and the family $\mathsf{S}^{[m]}(n)$ studied previously, whereas Corollary 3.6 shows that the family $\{\mathsf{S}^{(m)}(n)\}_{m\geq -1}$ satisfies a composition law, which may be of independent interest.

4. Several identities involving the Stirling numbers

In this section, we derive several identities involving the Stirling numbers from the matrix decompositions established in Theorem 3.3. Rather than being isolated summation formulas, these identities arise naturally from the matrix factorizations of the matrices $\mathsf{S}^{(m)}(n)$. In particular, multiplying one decomposition by the inverse Stirling matrix yields a shifted Stirling inversion formula, while another decomposition leads to a family of convolution identities for the Stirling numbers of the second kind.

Theorem 4.1. *Let $i, j \geq 1$ and $m \geq 0$ be integers. Then,*

$$\sum_{l=1}^{i} s(i,l) S(l+j+m, j) = \begin{cases} i^{i+m}, & \text{if } j = i; \\ 0, & \text{if } j < i. \end{cases}$$

Proof. Let $i, j \geq 1$ be fixed and choose the integer n so that $1 \leq i, j \leq n$. In view of the decomposition (21) in Theorem 3.3 and the fact that $S^{-1}(n) = s(n)$, we have

$$V(n)^{m+1} \cdot U^{(-1)}(n) = S^{-1}(n) \cdot \mathsf{S}^{(m)}(n) = s(n) \cdot \mathsf{S}^{(m)}(n).$$

This implies that the (i, j)-entry on both sides must be equal, that is,

$$\left(V(n)^{m+1} \cdot U^{(-1)}(n)\right)_{i,j} = \left(s(n) \cdot \mathsf{S}^{(m)}(n)\right)_{i,j}. \tag{25}$$

If $j < i$, since $V(n)^{m+1}$ and $U^{(-1)}(n)$ are both upper triangular matrices, then the left-hand side of this equation is equal to zero, and if $j = i$, it is equal to

$$(V(n)^{m+1} \cdot U^{(-1)}(n))_{i,i} = \sum_{l=1}^{n} V_{i,l}^{m+1} U_{l,i}^{(-1)} = V_{i,i}^{m+1} U_{i,i}^{(-1)} = (i^{m+1})(i^{i-1}) = i^{i+m},$$

where the second and third equalities are obtained by noting that $V(n)^{m+1}$ and $U^{(-1)}(n)$ are both upper triangular matrices with diagonal entries

$$1^{m+1}, 2^{m+1}, 3^{m+1}, \ldots, n^{m+1},$$

and

$$1^0, 2^1, 3^2, \ldots, n^{n-1},$$

respectively. On the other hand, the right-hand side of (25) is simplified as

$$(s(n) \cdot \mathsf{S}^{(m)}(n))_{i,j} = \sum_{l=1}^{n} s_{i,l} \mathsf{S}_{l,j}^{(m)} = \sum_{l=1}^{i} s_{i,l} \mathsf{S}_{l,j}^{(m)} = \sum_{l=1}^{i} s(i,l) S(l+j+m, j),$$

where the second equality is obtained by noting that $s(n)$ is a lower triangular matrix. This completes the proof. □

Theorem 4.1 is a direct consequence of the LU-decomposition in Theorem 3.3. It may be regarded as a shifted version of the classical Stirling inversion formula.

We now turn to Theorem 4.5. Its proof requires several auxiliary matrices and two technical lemmas. We first recall the Vandermonde-type matrix $E(n)$ introduced in [1], together with two upper triangular matrices that naturally arise in the proof.

We introduce the following matrices:

- the Vandermonde-type matrix $E(n) = [E_{i,j}]_{1 \le i,j \le n}$, where $E_{i,j} = (j+1)^{i-1}$.
- the upper triangular matrix $Q^{[N]}(n) = [Q_{i,j}^{[N]}]_{1 \le i,j \le n}$ with

$$Q_{i,j}^{[N]} = \begin{cases} (-1)^{j-i+1} \frac{i^{N-1}}{(i-1)!(j-i-1)!}, & \text{if } j \ge i+1; \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

- the upper triangular matrix $\widehat{Q}^{[N]}(n) = [\widehat{Q}_{i,j}^{[N]}]_{1 \le i,j \le n}$ with

$$\widehat{Q}_{i,j}^{[N]} = \begin{cases} (-1)^{j-i+1} \frac{i^{N-1}(i+1)^{j-1}}{(i-1)!(j-i-1)!}, & \text{if } j \ge i+1; \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

For instance, the matrices $Q^{[3]}(5)$ and $\widehat{Q}^{[3]}(5)$, respectively, are

$$Q^{[3]}(5) = \begin{bmatrix} \cdot & 1 & -1 & 1/2 & -1/6 \\ \cdot & \cdot & 4 & -4 & 2 \\ \cdot & \cdot & \cdot & 9/2 & -9/2 \\ \cdot & \cdot & \cdot & \cdot & 8/3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and} \quad \widehat{Q}^{[3]}(5) = \begin{bmatrix} \cdot & 2 & -4 & 4 & -8/3 \\ \cdot & \cdot & 36 & -108 & 162 \\ \cdot & \cdot & \cdot & 288 & -1152 \\ \cdot & \cdot & \cdot & \cdot & 5000/3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Our next goal is to derive a matrix decomposition of $\widehat{Q}^{[N]}(n)$. To do so, we first establish the following lemma.

Lemma 4.2. *Let i, j be integers with $i \geq 1$ and $j \geq 2$. Then, we have*

$$(j-i-1)Q^{[N]}_{i,j} + Q^{[N]}_{i,j-1} = 0. \tag{28}$$

Proof. The proof is obtained by considering the three cases $j \leq i$, $j = i+1$, and $j \geq i+2$ separately:

- if $j \leq i$, then the result is trivial, because $Q^{[N]}_{i,j} = Q^{[N]}_{i,j-1} = 0$.
- if $j = i+1$, then both terms $(j-i-1)Q^{[N]}_{i,j}$ and $Q^{[N]}_{i,j-1} = Q^{[N]}_{i,1}$ equal zero, and again the result follows.
- if $j \geq i+2$, then by direct computation we have

$$(j-i-1)Q^{[N]}_{i,j} + Q^{[N]}_{i,j-1} = (-1)^{j-i}\frac{i^{N-1}}{(i-1)!\,(j-i-2)!}\,[-1+1] = 0.$$

The proof is complete. □

Lemma 4.3. *Let N, n be positive integers. Then, we have*

$$\widehat{Q}^{[N]}(n) = Q^{[N]}(n) \cdot U^{(-1)}(n). \tag{29}$$

Proof. Since $\widehat{Q}^{[N]}(n)$ and $Q^{[N]}(n)$ are strictly upper triangular matrices, it suffices to show that

$$\left(Q^{[N]}(n) \cdot U^{(-1)}(n)\right)_{i,j} = \widehat{Q}^{[N]}(n)_{i,j} \quad \text{for} \quad j \geq i+1 \geq 2.$$

If $j = i+1$, then a direct computation yields

$$\begin{aligned}\left(Q^{[N]}(n) \cdot U^{(-1)}(n)\right)_{i,i+1} &= \sum_{l=1}^{n} Q^{[N]}_{i,l} U^{(-1)}_{l,i+1} = Q^{[N]}_{i,i} U^{(-1)}_{i,i+1} + Q^{[N]}_{i,i+1} U^{(-1)}_{i+1,i+1} \\ &= 0 + \frac{i^{N-1}(i+1)^{i}}{(i-1)!} = \widehat{Q}^{[N]}_{i,i+1},\end{aligned}$$

as required.

Since the values of both matrices on the first superdiagonal have already been shown to coincide, it remains to compare the entries above the first superdiagonal. We do this by proving that both sets of entries satisfy the same recurrence relation together with the same initial values. Hence, from now on we assume $j \geq i+2$. It remains to show that the entries $\widehat{Q}^{[N]}_{i,j}$ and $\left(Q^{[N]}(n) \cdot U^{(-1)}(n)\right)_{i,j}$ satisfy the following recurrence relation:

$$(i+1)A_{i,j-1} + (j-i-1)A_{i,j} = 0, \quad \text{for} \quad j \geq i+2 \geq 3. \tag{30}$$

First, we verify that the entries of $\widehat{Q}^{[N]}$ satisfy the recurrence. Direct calculation gives

$$(i+1)\widehat{Q}^{[N]}_{i,j-1} + (j-i-1)\widehat{Q}^{[N]}_{i,j} = \frac{i^{N-1}(i+1)^{j-1}}{(i-1)!\,(j-i-2)!}\left[(-1)^{j-i} + (-1)^{j-i+1}\right] = 0,$$

as required.

Next, we verify that the entries of $Q^{[N]}(n)U^{(-1)}(n)$ satisfy the same recurrence. To verify (30), we compute the left-hand side of the recurrence for the entries of $Q^{[N]}(n) \cdot U^{(-1)}(n)$ and simplify it using Lemma 4.2 together with the recurrence satisfied by $U^{(-1)}(n)$. We have

$$(i+1)\sum_{l=1}^{n} Q_{i,l}^{[N]} U_{l,j-1}^{(-1)} + (j-i-1)\sum_{l=1}^{n} Q_{i,l}^{[N]} U_{l,j}^{(-1)}$$
$$= \sum_{l=2}^{n} (i+1) Q_{i,l}^{[N]} (U_{l,j-1}^{(-1)} - U_{l,j}^{(-1)}) + j\sum_{l=1}^{n} Q_{i,l}^{[N]} U_{l,j}^{(-1)} \quad (\text{note that } Q_{i,1}^{[N]} = 0).$$

Using Lemma 4.2, we replace $(i+1)Q_{i,l}^{[N]}$ by $lQ_{i,l}^{[N]} + Q_{i,l-1}^{[N]}$.

$$= \sum_{l=2}^{n} (lQ_{i,l}^{[N]} + Q_{i,l-1}^{[N]})(U_{l,j-1}^{(-1)} - U_{l,j}^{(-1)}) + j\sum_{l=1}^{n} Q_{i,l}^{[N]} U_{l,j}^{(-1)}$$
$$= \sum_{l=1}^{n} lQ_{i,l}^{[N]} (U_{l,j-1}^{(-1)} - U_{l,j}^{(-1)}) + \sum_{l=2}^{n} Q_{i,l-1}^{[N]} (U_{l,j-1}^{(-1)} - U_{l,j}^{(-1)}) + j\sum_{l=1}^{n} Q_{i,l}^{[N]} U_{l,j}^{(-1)}$$

(since $Q_{i,1}^{[N]} = 0$ for each i).

We may rewrite the second term as

$$\sum_{l=2}^{n} Q_{i,l-1}^{[N]} (U_{l,j-1}^{(-1)} - U_{l,j}^{(-1)})$$
$$= \sum_{l=2}^{n} (l-1) Q_{i,l-1}^{[N]} (U_{l-1,j}^{(-1)} - U_{l-1,j-1}^{(-1)}) - j\sum_{l=2}^{n} Q_{i,l-1}^{[N]} U_{l-1,j}^{(-1)} \quad (\text{by } (12)).$$

After the index substitution $l \mapsto l-1$, the summation index runs from 1 to $n-1$.

$$= -\sum_{l=1}^{n-1} lQ_{i,l}^{[N]} (U_{l,j-1}^{(-1)} - U_{l,j}^{(-1)}) - j\sum_{l=1}^{n-1} Q_{i,l}^{[N]} U_{l,j}^{(-1)} \quad (\text{index substitution}).$$

Substituting the previous identity into the expression above, we observe that all terms with indices $l = 1, \ldots, n-1$ cancel pairwise, leaving only the contribution corresponding to $l = n$.

$$(i+1)\sum_{l=1}^{n} Q_{i,l}^{[N]} U_{l,j-1}^{(-1)} + (j-i-1)\sum_{l=1}^{n} Q_{i,l}^{[N]} U_{l,j}^{(-1)}$$
$$= nQ_{i,n}^{[N]} (U_{n,j-1}^{(-1)} - U_{n,j}^{(-1)}) + jQ_{i,n}^{[N]} U_{n,j}^{(-1)}$$
$$= (j-n) Q_{i,n}^{[N]} U_{n,j}^{(-1)}, \quad (\text{note that } U_{n,j-1}^{(-1)} = 0 \text{ for each } j \le n)$$
$$= 0.$$

Indeed, if $j \le n-1$, then $U_{n,j}^{(-1)} = 0$ because $U^{(-1)}(n)$ is upper triangular, whereas if $j = n$, then the coefficient $j - n$ vanishes. Hence the remaining term is zero in either case. This completes the proof in situation (b). □

Finally, we recall the following matrix decomposition established in [1], which will be used in the proof of Theorem 4.5.

Lemma 4.4 (Theorem 1, [1]). *Let N, n be positive integers. Then, we have the following matrix identity:*

$$S(n)\cdot(V(n)-I_n)^N = E(n)\cdot Q^{[N]}(n).$$

Theorem 4.5. *Let $i, j, N \geq 1$ be integers. Then, we have*

$$\sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}S(i+j+l-1,j) = \sum_{k=1}^{j-1}(k+1)^{i-1}(-1)^{j-k+1}\frac{k^{N-1}(k+1)^{j-1}}{(k-1)!\,(j-k-1)!}.$$

Proof. Let i, j be fixed, and choose the integer $n \geq 1$ so that $1 \leq i, j \leq n$. We have

$$\begin{aligned}
E(n)\cdot\widehat{Q}^{[N]}(n) &= E(n)\cdot Q^{[N]}(n)\cdot U^{(-1)}(n) \quad \text{(by Lemma 4.3)}\\
&= S(n)\cdot(V(n)-I_n)^N\cdot U^{(-1)}(n) \quad \text{(by Lemma 4.4)}\\
&= S(n)\cdot\Big(\sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}V^l(n)\Big)\cdot U^{(-1)}(n) \quad \text{(by binomial theorem)}\\
&= \sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}\big(S(n)\cdot V^l(n)\cdot U^{(-1)}(n)\big)\\
&= \sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}\mathsf{S}^{(l-1)}(n). \quad \text{(by Theorem 3.3)}
\end{aligned}$$

Comparing the (i, j)-entries of both sides completes the proof once the two entries are evaluated explicitly. On the left-hand side

$$\big(E(n)\cdot\widehat{Q}^{[N]}(n)\big)_{i,j} = \sum_{k=1}^{n}E_{i,k}\widehat{Q}^{[N]}_{k,j} = \sum_{k=1}^{j-1}(k+1)^{i-1}(-1)^{j-k+1}\frac{k^{N-1}(k+1)^{j-1}}{(k-1)!\,(j-k-1)!}.$$

On the right-hand side

$$\left(\sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}\mathsf{S}^{(l-1)}(n)\right)_{i,j} = \sum_{l=0}^{N}(-1)^{N-l}\binom{N}{l}S(i+j+l-1,j),$$

because $\mathsf{S}^{(l-1)}(n)_{i,j} = S(i+j+l-1,j)$. □

Corollary 4.6. *Let $i, j \geq 1$ be integers. Then, we have*

$$S(i+j,j) - S(i+j-1,j) = \sum_{k=1}^{j-1}(-1)^{j-k+1}\frac{(k+1)^{i+j-2}}{(k-1)!\,(j-k-1)!}.$$

Proof. This follows immediately from Theorem 4.5 by taking $N = 1$. □

5. Conclusion

In this paper, we established several determinant identities for generalized Stirling matrices by deriving explicit matrix decompositions and factorizations. These results provide a unified framework for evaluating determinants of a broad class of matrices associated with generalized Stirling numbers and yield a number of explicit determinant formulas as direct consequences. We also illustrated the effectiveness of the proposed approach through corollaries and illustrative examples.

The decomposition techniques developed in this work may also prove useful in the study of determinant identities for other combinatorial matrices arising from recursive sequences or polynomial families. Possible directions for future research include extending these methods to broader classes of Stirling-type matrices and investigating further applications in combinatorial matrix theory and enumerative combinatorics.

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References

[1] M. Bahrami-Taghanaki, A. R. Moghaddamfar, N. Salehy, and N. Salehy. On matrices whose entries are stirling numbers of the second kind (I). *Integers*, 25:Paper No. A27, 15 pp. 2025. https://doi.org/10.5281/zenodo.15091069.

[2] M. Bahrami-Taghanaki, A. R. Moghaddamfar, N. Salehy, and N. Salehy. Some identities involving stirling numbers arising from matrix decompositions. *Journal of Integer Sequences*, 27:Article 24.5.3, 14 pp. 2024.

[3] R. A. Brualdi. *Introductory Combinatorics*. Pearson Prentice Hall, Upper Saddle River, NJ, 5th edition, 2010.

[4] M. Can and M. C. Dağlı. Extended bernoulli and stirling matrices and related combinatorial identities. *Linear Algebra and its Applications*, 444:114–131, 2014. https://doi.org/10.1016/j.laa.2013.11.031.

[5] G. S. Cheon and J. S. Kim. Stirling matrix via pascal matrix. *Linear Algebra and its Applications*, 329:49–59, 2001. https://doi.org/10.1016/S0024-3795(01)00234-8.

[6] G. S. Cheon, J. S. Kim, and H. W. Yoon. A note on pascal's matrix. *Journal of the Korea Society of Mathematical Education Series B: Pure and Applied Mathematics*, 6(2):121–127, 1999.

[7] E. Choi and J. Jo. On the determinants of the square-type stirling matrix and bell matrix. *International Journal of Mathematics and Mathematical Sciences*, 2021:7959370, 2021. https://doi.org/10.1155/2021/7959370.

[8] J. Engbers, D. Galvin, and C. Smyth. Restricted stirling and lah number matrices and their inverses. *Journal of Combinatorial Theory, Series A*, 161:271–298, 2019. https://doi.org/10.1016/j.jcta.2018.08.001.

[9] J. H. Jung. LU-factorization of the square-type matrix of the stirling matrix. *East Asian Mathematical Journal*, 39(5):523–528, 2023.

[10] C. Krattenthaler. Advanced determinant calculus. *Séminaire Lotharingien de Combinatoire*, 42:Article B42q, 67 pp. 1999. The Andrews Festschrift.

[11] C. Krattenthaler. Advanced determinant calculus: a complement. *Linear Algebra and its Applications*, 411:68–166, 2005. https://doi.org/10.1016/j.laa.2005.06.042.

[12] P. Maltais and T. A. Gulliver. Pascal matrices and stirling numbers. *Applied Mathematics Letters*, 11(2):7–11, 1998. https://doi.org/10.1016/S0893-9659(98)00002-0.

[13] A. R. Moghaddamfar, S. N. Salehy, and S. N. Salehy. The determinants of matrices with recursive entries. *Linear Algebra and its Applications*, 428:2468–2481, 2008. https://doi.org/10.1016/j.laa.2007.11.021.

[14] OEIS Foundation Inc. The on-line encyclopedia of integer sequences. https://oeis.org, 2026.

[15] R. P. Stanley. *Enumerative Combinatorics*, volume 1 of number 49 in *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2nd edition, 2012. https://doi.org/10.1017/CBO9781139058520.

[16] A. Xu. Determinants involving the numbers of the stirling-type. *Filomat*, 33:1659–1666, 2019. https://doi.org/10.2298/FIL1906659X.

[17] S.-L. Yang and H. You. On a connection between the pascal, stirling and vandermonde matrices. *Discrete Applied Mathematics*, 155:2025–2030, 2007. https://doi.org/10.1016/j.dam.2007.05.017.

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