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Target Ramsey numbers in graphs

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ABSTRACT

The Ramsey number $R(G)$ of a graph G without isolated vertices is the minimum positive integer n such that for every red-blue coloring of the complete graph K_n of order n, there is a subgraph isomorphic to G all of whose edges are colored the same (a monochromatic G). A Ramsey chain in a graph G with a red-blue coloring is a sequence $G_1, G_2, \ldots, G_k$ of pairwise edge-disjoint monochromatic subgraphs of G such that G_i has i edges for $1 \le i \le k$ and G_i is isomorphic to a subgraph of G_{i+1} for $1 \le i \le k-1$. The subgraphs in a Ramsey chain are the links of the chain and the terminal subgraph G_k is the target link of the chain. A graph H without isolated vertices is called a target graph if there exists a positive integer n such that every red-blue coloring of K_n results in a Ramsey chain with target link H. The target Ramsey number $TR(H)$ of H is the minimum positive integer n such that every red-blue coloring of K_n results in a Ramsey chain with target link H. The target Ramsey number $TR(s)$ of a Ramsey chain s is the minimum positive integer n such that s is a Ramsey chain in every red-blue coloring of K_n. We investigate graphs H with the property that $TR(s) = TR(H) = R(H)$ for every Ramsey chain s with target link H. It is shown that every graph H with relatively small size has this property. Other results and open questions are also presented.

Keywords: red-blue coloring, Ramsey number, Ramsey chain, target graph, target Ramsey number

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1. Introduction

In a *red-blue coloring* of a graph G, every edge of G is colored red or blue. For two graphs F and H (without isolated vertices), the *Ramsey number* $R(F, H)$ of F and H is the minimum positive integer n such that for every red-blue coloring of the complete graph K_n of order n, there is either a subgraph of K_n isomorphic to F all of whose edges are colored red (a red F) or a subgraph of K_n isomorphic to H all of whose edges are colored blue (a blue H). It is a consequence of a theorem of Ramsey [18] that the number $R(F, H)$ exists for every two graphs F and H. If $F \cong H$, then the (*diagonal*) *Ramsey number* $R(F, H) = R(H, H) = R(H)$ is the minimum positive integer n such that every red-blue coloring of K_n results in a subgraph of K_n isomorphic to F all of whose edges are colored the same (a monochromatic F).

A *Ramsey chain* in a graph G with a red-blue coloring is a sequence G_1, G_2, ..., G_k of pairwise edge-disjoint subgraphs of G such that each subgraph G_i $(1 \leq i \leq k)$ is monochromatic of size i and G_i is isomorphic to a subgraph of G_{i+1} for $1 \leq i \leq k-1$. This concept was first introduced and studied in [2, 7]. The subgraphs in a Ramsey chain are the *links* of the chain and the terminal subgraph G_k of size k is the *target link* of the chain. A graph H without isolated vertices is called a *target graph* if there exists a positive integer n such that every red-blue coloring of K_n results in a Ramsey chain with target link H. It was shown in [9] that every graph without isolated vertices is a target graph. For a target graph H, the *target Ramsey number* $TR(H)$ of H is the minimum positive integer n such that for every red-blue coloring of K_n, there exists a Ramsey chain in K_n having H as its target link. In [9], it was shown that $TR(H)$ exists for every graph H without isolated vertices and $TR(H) \geq R(H)$. For every graph H that has been studied, it has been shown that $TR(H) = R(H)$, which gave rise to the following question that appeared in [9, 8].

Question 1.1. *Is there a graph H such that $TR(H) > R(H)$?*

If the answer to this question is no, then $TR(H) = R(H)$ for every graph H. Hence, not only does every red-blue coloring of the complete graph $K_{R(H)}$ of order $R(H)$ produce a monochromatic H but also $K_{R(H)}$ contains a Ramsey chain with target link H. This implies that there are greater expectations for diagonal Ramsey numbers of graphs. Here, we introduce a related Ramsey concept and explore the possibility that the diagonal Ramsey number $R(H)$ of a graph H may have a property that is stronger than $TR(H) = R(H)$.

A sequence $s = (G_1, G_2, \ldots, G_k)$ of pairwise edge-disjoint graphs without isolated vertices is called an *ascending sequence* if G_i has size i for $1 \leq i \leq k$ and G_i is a subgraph of G_{i+1} for $1 \leq i \leq k-1$. This concept is related to the Ascending Subgraph Decomposition Conjecture (see [1, 14]). An ascending sequence s is a *Ramsey chain* if there is a positive integer n such that s is a Ramsey chain in *every* red-blue coloring of K_n.

Proposition 1.2. *For every ascending sequence s, there is a positive integer n such that*

s is a Ramsey chain in every red-blue coloring of K_n; that is, every ascending sequence is a Ramsey chain.

Proof. Let $s = (G_1, G_2, \ldots, G_k)$ be an ascending sequence. For $1 \le i \le k$, let $R(G_i) = n_i$ and let $n = \sum_{i=1}^{k} n_i$. Let there be given a red-blue coloring of K_n. Since $n \ge n_k$, there is a monochromatic subgraph G_k in K_n. Deleting the n_k vertices of G_k from K_n results in K_{n-n_k}. Since $n - n_k \ge n_{k-1}$ and $E(K_{n-n_k}) \cap E(G_k) = \emptyset$, there is a monochromatic subgraph G_{k-1} in K_{n-n_k} that is edge-disjoint from G_k. Continuing in this manner, we obtain a Ramsey chain $(G_1, G_2, \ldots, G_k)$ and so s is a Ramsey chain. $\square$

The *target Ramsey number* $TR(s)$ of a Ramsey chain s is the minimum positive integer n such that s is a Ramsey chain in every red-blue coloring of K_n. The following is a consequence of Proposition 1.2.

Observation 1.3. *For every Ramsey chain s, the target Ramsey number $TR(s)$ exists. Furthermore, if the target link of s is H, then $TR(s) \ge TR(H) \ge R(H)$.*

We now investigate the following question:

> *Which graphs H have the property that for every red-blue coloring of the complete graph $K_{R(H)}$ of order $R(H)$ and every Ramsey chain s with target link H, it follows that $TR(s) = R(H)$?*

We refer to the book [6] for notation and terminology not defined here. Information on the Ramsey numbers of well-known classes of graphs including paths, cycles, stars, and matchings can be found in [3, 4, 11, 10, 12, 13, 17].

For a positive integer m, let $K_{1,m}$ denote the star of size m and mK_2 the matching of size m. If $s = (G_1, G_2, \ldots, G_m)$ is a Ramsey chain where either $G_m = K_{1,m}$ or $G_m = mK_2$, then $G_i = K_{1,i}$ in the first case and $G_i = iK_2$ in the second case for $1 \le i \le m$. Since $TR(K_{1,m}) = R(K_{1,m})$ for all $m \ge 2$, we have the following result on stars.

Theorem 1.4. *If s is a Ramsey chain with target link $K_{1,m}$ for a positive integer m, then*

$$TR(s) = R(K_{1,m}).$$

Since $TR(mK_2) = R(mK_2)$ for $2 \le m \le 5$, we have the following result for matchings of these sizes.

Theorem 1.5. *If s is a Ramsey chain with target link mK_2 where $2 \le m \le 5$, then*

$$TR(s) = R(mK_2).$$

We will show that many other graphs H have the property that $TR(s) = R(H)$ for every Ramsey chain s with target link H including all graphs of size 3 or 4.

2. Ramsey chains with target link of size 3

In this section, we show that for every Ramsey chain $s = (G_1, G_2, G_3)$ with target link of size 3, it follows that $TR(s) = R(G_3)$. If G_3 is a graph of size 3 without isolated vertices, then

$$G_3 \in \{P_4, K_3, P_3 + K_2, K_{1,3}, 3K_2\}.$$

By Theorems 1.4 and 1.5, if $G_3 = K_{1,3}$ or $G_3 = 3K_2$, then $TR(s) = R(G_3)$. Hence, it remains to consider $G_3 \in \{P_4, K_3, P_3 + K_2\}$. For these graphs, Ramsey numbers are $R(P_4) = 5$ and $R(K_3) = R(P_3 + K_2) = 6$ (see [4, 15, 17]). First, we present a known lemma (see [6]). The *matching (edge-independence) number* $\alpha'(G)$ of a graph G is the maximum number of independent edges in G.

Lemma 2.1. *If G is a graph of size m with maximum degree Δ, then $\alpha'(G) \geq \left\lceil \frac{m}{\Delta} \right\rceil$.*

Theorem 2.2. *If s is a Ramsey chain with target link G_3 of size 3, then $TR(s) = R(G_3)$.*

Proof. Let $s = (G_1, G_2, G_3)$ be a Ramsey chain where $G_3 \in \{P_4, K_3, P_3 + K_2\}$. By Observation 1.3, it suffices to show that $TS(s) \leq R(G_3)$. We consider two cases, according to whether $R(G_3) = 5$ or $R(G_3) = 6$.

Case 1. $R(G_3) = 5$. Then $G_3 = P_4$ and $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G = K_5$ with $V(G) = \{v_1, v_2, \ldots, v_5\}$. Then there is a monochromatic subgraph $G_3 = P_4$ of G, say $G_3 = (v_1, v_2, v_3, v_4)$. The graph $H = G - E(G_3)$ has order 5 and size 7. Since two of the four edges v_5v_i $(1 \leq i \leq 4)$ have the same color, there is a monochromatic subgraph $G_2 = P_3$ in H. Thus, we may assume that $G_2 = 2K_2$. Let H_r and H_b be the red and blue subgraphs of H with $|E(H_r)| \geq |E(H_b)|$, say. Thus, H_r has order 5 and size at least 4. If H_r does not contain $2K_2$, then $H_r = K_{1,4}$ with $E(H_r) = \{v_5v_i : 1 \leq i \leq 4\}$. If one of the two edges v_1v_3 and v_2v_4 of H is red, say v_1v_3 is red, then there is a red $G_2 = 2K_2$ with $E(G_2) = \{v_1v_3, v_2v_5\}$. Thus, we may assume that v_1v_3 and v_2v_4 are blue, producing a blue $G_2 = 2K_2$ in H. Since the size of G is 10 and $|E(G_3) \cup E(G_2)| = 5$, there is always there is an edge in $E(G) - [E(G_3) \cup E(G_2)]$ that is available for $G_1 = K_2$. Hence, there is a Ramsey chain s in G. Therefore, $TR(s) \leq 5$ and so $TR(s) = 5$.

Case 2. $R(G_3) = 6$. Then $G_3 \in \{K_3, P_3 + K_2\}$. If $G_3 = K_3$, then $G_2 = P_3$; while if $G_3 = P_3 + K_2$, then $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{v_1, v_2, \ldots, v_6\}$. Since $R(G_3) = 6$ where $G_3 \in \{K_3, P_3 + K_2\}$, there is a monochromatic subgraph G_3 of G. Let $H = G - E(G_3)$. We may assume that $v_6 \notin V(G_3)$. Since at least two of the edges v_6v_i $(1 \leq i \leq 5)$ have the same color, there is a monochromatic subgraph $G_2 = P_3$ in H. Thus, we may assume that $G_2 = 2K_2$. Let H_r and H_b be the red and blue subgraphs respectively of H where $|E(H_r)| \geq |E(H_b)|$, say. Thus, H_r has order 6 and size at least 6. Since $\Delta(H_r) \leq 5$, it follows by Lemma 2.1 that $\alpha'(H_r) \geq 2$ and so H_r contains the subgraph $G_2 = 2K_2$. Since the size of G is 15 and $|E(G_3) \cup E(G_2)| = 5$, there is an edge in $E(G) - [E(G_3) \cup E(G_2)]$ that is available for $G_1 = K_2$. Hence, there is a Ramsey chain s in G. Therefore, $TR(s) \leq 6$ and so

$TR(s) = 6$. □

3. Ramsey chains with target link of size 4

In this section, we establish the following result.

Theorem 3.1. *If s is a Ramsey chain with target link G_4 of size 4, then $TR(s) = R(G_4)$.*

If G_4 is a graph of size 4 without isolated vertices, then

$$G_4 \in \{C_4, P_5, S_{3,2}, K_{1,3} + e, K_{1,4}, K_3 + K_2, K_{1,3} + K_2, 2P_3, P_4 + K_2, P_3 + 2K_2, 4K_2\}.$$

By Theorems 1.4 and 1.5, if $G_4 \in \{K_{1,4}, 4K_2\}$, then $TR(s) = R(G_4)$. Thus, we may assume that $G_4 \notin \{K_{1,4}, 4K_2\}$. The Ramsey numbers of the other graphs of size 4 without isolated vertices are $R(C_4) = R(P_5) = R(S_{2,3}) = 6$, $R(K_{1,3} + e) = R(K_3 + K_2) = R(K_{1,3} + K_2) = R(2P_3) = 7$, $R(P_4 + K_2) = 8$, and $R(P_3 + 2K_2) = 9$ (see [4, 17]).

3.1. Target links of size 4 and Ramsey number 6

We now consider all Ramsey chains whose target link $G_4 \in \{C_4, P_5, S_{2,3}\}$ has Ramsey number 6. First, the following lemma is presented.

Lemma 3.2. *Every red-blue coloring of a graph of order 6 and size 8 results in a monochromatic P_3.*

Proof. Since every graph of order 6 and size 8 has a vertex v of degree 3 or more, every red-blue coloring of such a graph results in two edges of the same color incident with v, producing a monochromatic P_3. □

Theorem 3.3. *If s is a Ramsey chain with target link C_4, then $TR(s) = R(C_4) = 6$.*

Proof. By Observation 1.3, it suffices to show that $TR(s) \leq 6$. Let $s = (G_1, G_2, G_3, G_4)$ be a Ramsey chain where $G_4 = C_4$. Then $G_3 = P_4$ and $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{v_1, v_2, \ldots, v_6\}$. Since $R(C_4) = 6$, there is a monochromatic subgraph $G_4 = C_4$ in G, say $C_4 = (v_1, v_2, v_3, v_4, v_1)$. We show that $F = K_6 - E(G_4)$ contains a monochromatic P_4. Since at least two of the four edges v_5v_i $(i = 1, 2, 3, 4)$ have the same color, we may assume that v_5v_a and v_5v_b are red where $\{a, b\} = \{1, 2\}$ or $\{a, b\} = \{1, 3\}$. We consider these two cases.

Case 1. *The edges v_1v_5 and v_2v_5 are red.* If one of v_1v_6 and v_2v_6 is red, there is a red P_4 in F. Thus, we may assume that v_1v_6 and v_2v_6 are blue. If v_1v_3 is red, then (v_3, v_1, v_5, v_2) is a red $G_3 = P_4$ in F. If v_1v_3 is blue, then (v_3, v_1, v_6, v_2) is a blue $G_3 = P_4$ in F.

Case 2. *The edges v_1v_5 and v_3v_5 are red.* By Case 1, we may assume that v_2v_5 and v_4v_5 are blue. By the argument in Case 1, we may further assume that v_1v_6 and v_3v_6 are blue and v_2v_6 and v_4v_6 are red; for otherwise, there is a monochromatic P_4. Then regardless of the color of v_5v_6, there is a monochromatic subgraph $G_3 = P_4$.

Next, let $H = G - [E(C_4) \cup E(P_4)]$. Since H has order 6 and size 8, it follows by Lemma 3.2 that H contains a monochromatic subgraph $G_2 = P_3$. Thus, it remains to show that H contains a monochromatic $G_2 = 2K_2$. Assume, to the contrary, that H contains no monochromatic $2K_2$. We verify the following claim.

Claim: If H contains no monochromatic $2K_2$, then the red subgraph H_r of H and the blue subgraph H_b of H are both stars.

Suppose that the claim is false. The only graphs (without isolated vertices) that contain no $2K_2$ are stars and the triangle K_3. Because the size of H is 8, one of H_r and H_b is a star and the other is K_3, say $H_r = K_{1,5}$ centered at a vertex v_p and $H_b = K_3$. Since $\deg_H v_i \le 3$ for $1 \le i \le 4$, it follows that $v_p \in \{v_5, v_6\}$, say $v_p = v_6$. Then $E(H_r) = \{v_6v_i : 1 \le i \le 5\}$. Since $E(C_4) \cap E(P_4) = \emptyset$ and $C_4 = (v_1, v_2, v_3, v_4, v_1)$, it follows that either $v_1v_3 \notin E(P_4)$ or $v_2v_4 \notin E(P_4)$, say the former. Thus, $v_1v_3 \in E(H_b)$ and so $H_b = (v_1, v_3, v_5, v_1)$. Then $E(G) - [E(C_4) \cup E(H)] = \{v_2v_4, v_4v_5, v_5v_2\}$, which forms the triangle (v_2, v_4, v_5, v_2). On the other hand, $E(P_4) = E(G) - [E(C_4) \cup E(H)]$, which is impossible. This verifies the claim.

Next, suppose that $H_r = K_{1,s}$ is centered at a vertex v_p and $H_b = K_{1,t}$ is centered at a vertex v_q where $s + t = 8$. We may assume that $s \ge t$ and so $s \in \{4, 5\}$ and $t \ge 3$. By the argument used in verifying the claim, we may assume that (1) $v_p \in \{v_5, v_6\}$, say $v_p = v_6$, and (2) $v_1v_3 \in E(H)$ and so $v_q \in \{v_1, v_3\}$, say $v_q = v_3$. Since $3 \le t = \deg_{H_b} v_3 \le \deg_H v_3 = 3$, it follows that $t = 3$ and $s = 5$. Then $v_6v_i \in E(H_r)$ for $1 \le i \le 5$. However then, v_3 is adjacent only to v_1 and v_5 in H_b and so $\deg_{H_b} v_3 = 2$, which is a contradiction. Thus, H contains a monochromatic $2K_2$. Since $E(G) - [E(G_4) \cup E(G_3) \cup E(G_2)] \ne \emptyset$, there is a monochromatic subgraph $G_1 = K_2$ that is edge-disjoint from G_i for $i = 2, 3, 4$. Therefore, s is a Ramsey chain in G and so $TR(s) \le 6$. Therefore, $TR(s) = R(C_4) = 6$. □

To show that $TR(s) = R(P_5)$ for every Ramsey chain s with target link P_5, we first present some preliminary results.

Lemma 3.4. *If A and B are two edge-disjoint monochromatic subgraphs of a red-blue coloring of $G = K_6$ where $A = P_5$ and B has size 3 without isolated vertices such that $G - [E(A) \cup E(B)]$ has no monochromatic $2K_2$, then $B = P_4$.*

Proof. Let $G = K_6$ with $V(G) = \{v_1, v_2, \ldots, v_6\}$ such that $H = G - [E(A) \cup E(B)]$ has no monochromatic $2K_2$. We may assume that $A = P_5 = (v_1, v_2, \ldots, v_5)$. Let H_r and H_b be the red and blue subgraphs of H where $|E(H_r)| \ge |E(H_b)|$, say. Since the size of H is $\binom{6}{2} - 7 = 8$ and H has no monochromatic $2K_2$, it follows that either (1) $H_r = K_{1,5}$ and $H_b = K_3$ or (2) H_r and H_b are stars. We consider these two cases.

Case 1. $H_r = K_{1,5}$ and $H_b = K_3$. Then H_r is centered at v_6 and $E(H_r) = \{v_6v_i : 1 \le i \le 5\}$. Thus, $H_b = K_3 = (v_1, v_3, v_5, v_1)$. Hence, $E(B) = \{v_1v_4, v_4v_2, v_2v_5\}$ and so $B = (v_1, v_4, v_2, v_5) = P_4$. The subgraph induced by $E(A) \cup E(H)$ is illustrated in Figure 1, where an edge in A is indicated by a dotted line, a red edge in H_r is indicated by a bold line, and a blue edge in H_b is indicated by a thin line, and $E(B) = E(G) - [E(A) \cup E(H)]$.

Case 2. H_r and H_b are stars. Let $H_r = K_{1,s}$ centered at a vertex v_p and $H_b = K_{1,t}$ centered at a vertex v_q where $s \ge t$ and $s + t = 8$. Thus, $s \in \{4, 5\}$ and $t \ge 3$.

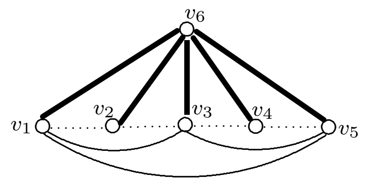

Fig. 1. The subgraph induced by $E(A) \cup E(H)$ in Case 1 of Lemma 3.4

⋆ If $s > t$, then $s = 5$ and $t = 3$. Then $v_p = v_6$ and $E(H_r) = \{v_6v_i : 1 \leq i \leq 5\}$. This implies that $v_q \in \{v_1, v_5\}$, say $v_q = v_1$. Then $E(H_b) = \{v_1v_i : \ i = 3, 4, 5\}$. Hence, $E(B) = \{v_4v_2, v_2v_5, v_5v_3\}$ and so $B = (v_4, v_2, v_5, v_3) = P_4$. The subgraph induced by $E(A) \cup E(H)$ is illustrated in Figure 1 when $v_q = v_1$ and $E(B) = E(G) - [E(A) \cup E(H)]$.

⋆ If $s = t$, then $s = t = 4$. Since $\deg_H v_i \leq 3$ for $i = 2, 3, 4$, it follows that $v_p, v_q \in \{v_1, v_5, v_6\}$. If $\{v_p, v_q\} \subseteq \{v_1, v_5\}$, then $v_pv_q \in E(H_r) \cap E(H_b)$, which is a contradiction. Thus, we may assume that $v_p = v_6$ and $v_q = v_1$. Then $E(H_b) = \{v_1v_i : \ i = 3, 4, 5, 6\}$ and $E(H_r) = \{v_6v_i : 2 \leq i \leq 5\}$. Hence, $E(B) = \{v_4v_2, v_2v_5, v_5v_3\}$ and so $B = (v_4, v_2, v_5, v_3) = P_4$. The subgraph induced by $E(A) \cup E(H)$ is illustrated in Figure 1 when $v_q = v_1$ and $E(B) = E(G) - [E(A) \cup E(H)]$. □

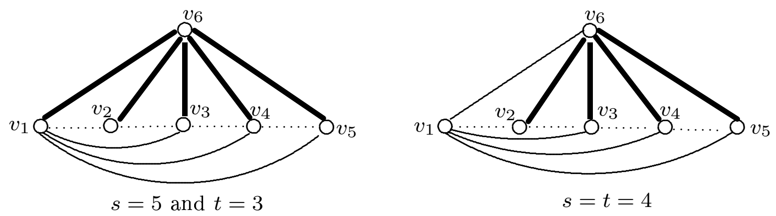

Fig. 2. Subgraphs induced by $E(A) \cup E(H)$ in Case 2 of Lemma 3.4

The following is a consequence of the proof of Lemma 3.4.

Corollary 3.5. *For the graph $G = K_6$ with a red-blue coloring where $V(G) = \{v_1, v_2, \ldots, v_6\}$, let $A = P_5 = (v_1, v_2, \ldots, v_5)$ and B be two edge-disjoint monochromatic subgraphs of G where B has size 3 without isolated vertices. Furthermore, let $H = G - [E(A) \cup E(B)]$ with red subgraph H_r and blue subgraph H_b where $|E(H_r)| \geq |E(H_b)|$. If H has no monochromatic $2K_2$, then $B = P_4$ and one of the following occurs:*

(1) *$H_r = K_{1,5}$ is centered at v_6 and $H_b = K_3$,*

(2) *$H_r = K_{1,5}$ is centered at v_6 and $H_b = K_{1,3}$ is centered at $v_q \in \{v_1, v_5\}$, and*

(3) *$H_r = H_b = K_{1,4}$ where H_r is centered at v_6 and H_b is centered at $v_q \in \{v_1, v_5\}$.*

If (1) occurs, then $B = (v_1, v_4, v_2, v_5)$; while if (2) or (3) occurs, then $B = (v_4, v_2, v_5, v_3)$ if $v_q = v_1$ or $B = (v_3, v_1, v_4, v_2)$ if $v_q = v_5$.

A graph G is *F-free* for some graph F if G contains no subgraph isomorphic to F.

Lemma 3.6. *If G is a graph of order 6 and size at least 6 that is $(P_3 + K_2)$-free, then $G = K_4 + 2K_1$.*

Proof. Let G be a graph of order 6 and size $m \geq 6$ that is (P_3+K_2)-free. Since the order of G is 6 and $m \geq 6$, it follows that $2 \leq \Delta(G) \leq 5$.

$\star$ If $\Delta(G)=2$, then $G = 2K_3$ or $G = C_6$. Since $P_3+K_2 \subseteq 2K_3$ and $P_3+K_2 \subseteq C_4$, it follows that $P_3+K_2 \subseteq G$.

$\star$ If $\Delta(G)=3$, let v be a vertex of degree 3 and $N(v)=\{v_1,v_2,v_3\}$. If there is an edge $e \in E(G)-\{vv_i : i=1,2,3\}$ that is not incident with two vertices in $\{v_1,v_2,v_3\}$, say e is not incident with v_1 and v_2, then $P_3+K_2 \subseteq G$, where $P_3=(v_1,v,v_2)$ and $E(K_2)=\{e\}$. If every edge in $E(G)-\{vv_i : i=1,2,3\}$ joins two vertices in $\{v_1,v_2,v_3\}$, then $m=6$ and $G=K_4+2K_1$ where $V(K_4)=\{v,v_1,v_2,v_3\}$.

$\star$ If $\Delta(G) \in \{4,5\}$, let v be a vertex of degree $\Delta(G)$. For $\Delta(G)=4$, let $N(v)=\{v_1,v_2,v_3,v_4\}$; while for $\Delta(G)=5$, let $N(v)=\{v_1,v_2,v_3,v_4,v_5\}$. If there is an edge joining two vertices in $N(v)$, then G contains $K_{1,4}+e$ as a subgraph. If $N(v)$ is an independent set of vertices of G, then $\Delta(G)=4$ and some vertex in $N(v)$ is adjacent to v_5. Thus, G contains the double star $S_{2,4}$ as a subgraph. Since $P_3+K_2 \subseteq K_{1,4}+e$ and $P_3+K_2 \subseteq S_{2,4}$, it follows that $P_3+K_2 \subseteq G$. $\square$

Theorem 3.7. *If s is a Ramsey chain with target link P_5, then $TR(s)=R(P_5)=6$.*

Proof. By Observation 1.3, it suffices to show that $TR(s) \leq 6$. Let $s=(G_1,G_2,G_3,G_4)$ be a Ramsey chain where $G_4=P_5$ and $G_3 \in \{P_4, P_3+K_2\}$. Then $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G=K_6$ with $V(G)=\{v_1,v_2,\ldots,v_6\}$. Since $R(P_5)=6$, there is a monochromatic subgraph $G_4=P_5$ of G where $P_5=(v_1,v_2,v_3,v_4,v_5)$ say, and let $F=K_6-E(G_4)$ with red subgraph F_r and blue subgraph F_b. We may assume that $|E(F_r)| \geq |E(F_b)|$. We show that F contains a monochromatic subgraph $G_3 \in \{P_4, P_3+K_2\}$. There are two possibilities.

Case 1. $G_3=P_4$. Since at least two of the three edges v_6v_1, v_6v_2, v_6v_3 have the same color, we may assume that v_iv_6 and v_jv_6 are red where $1 \leq i < j \leq 3$. If one of v_iv_5 and v_jv_5 is red, then there is a red P_4 in F. Thus, we may assume that v_iv_5 and v_jv_5 are blue. Then regardless of the color of v_iv_4, there is a monochromatic subgraph $G_3=P_4$ in F.

Case 2. $G_3=P_3+K_2$. Since $|E(F_r)| \geq |E(F_b)|$ and $|E(F)|=11$, it follows that F_r is a graph of order 6 and size at least 6. If $P_3+K_2 \subseteq F_r$, then there is a red $G_3=P_3+K_2$. If F_r does not contain P_3+K_2, then $F_r=K_4+2K_1$ by Lemma 3.6. This implies that $V(K_4)=\{v_1,v_3,v_5,v_6\}$. Thus, F_b contains $G_3=P_3+K_2$ where $P_3=(v_1,v_4,v_6)$ and $K_2=(v_2,v_5)$.

Next, let $H=G-[E(G_4)\cup E(G_3)]$. It then follows by Lemma 3.2 that H contains a monochromatic P_3. Thus, if $G_2=P_3$, then s is a Ramsey chain in G. Hence, we may assume that $G_2=2K_2$. By Lemma 3.4, if $G_3=P_3+K_2$, then H contains a monochromatic subgraph $G_2=2K_2$ and so s is a Ramsey chain in G. Hence, we may assume that $G_3=P_4$. By Corollary 3.5, there are three possibilities:

(1) $H_r=K_{1,5}$ is centered at v_6 and $H_b=K_3$.

(2) $H_r=K_{1,5}$ is centered at v_6 and $H_b=K_{1,3}$ is centered at $v_q \in \{v_1,v_5\}$.

(3) $H_r=H_b=K_{1,4}$ where H_r is centered at v_6 and H_b is centered at $v_q \in \{v_1,v_5\}$.

Furthermore, if (1) occurs, then $G_3=P_4=(v_1,v_4,v_2,v_5)$ is monochromatic; while if

(2) or (3) occurs, then we may assume that $v_q = v_1$ and so $G_3 = P_4 = (v_4, v_2, v_5, v_3)$ is monochromatic. Depending on whether $G_4 = P_5$ is red or blue, we construct a new Ramsey chain $A_1 = K_2, A_2 = 2K_2, A_3 = P_4, A_4 = P_5$ as follows.

$\star$ If (1) occurs, then $G_3 = P_4 = (v_1, v_4, v_2, v_5)$ is monochromatic. If $P_5 = (v_1, v_2, v_3, v_4, v_5)$ is red, let $A_4 = (v_1, v_2, v_6, v_4, v_5)$ be a red P_5 and $A_3 = (v_2, v_3, v_6, v_5)$ a red P_4. If $P_5 = (v_1, v_2, v_3, v_4, v_5)$ is blue, let $A_4 = (v_1, v_2, v_3, v_5, v_4)$ be a blue P_5 and $A_3 = (v_4, v_3, v_1, v_5)$ a blue $G_3 = P_4$. This is illustrated in Figure 3 where an edge in A_i is labeled i for $i \in \{3, 4\}$. Thus, A_4 and A_3 are edge-disjoint.

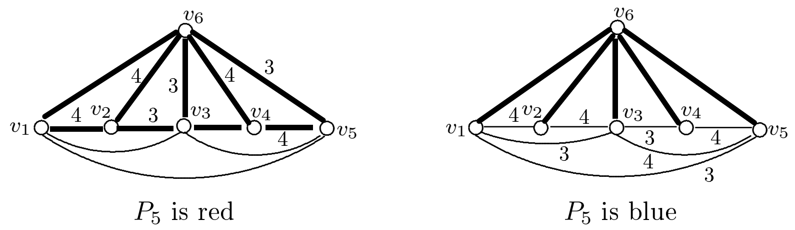

Fig. 3. The edges of A_3 and A_4 when (1) occurs

$\star$ If (2) or (3) occurs, then $G_3 = P_4 = (v_4, v_2, v_5, v_3)$ is monochromatic. First, suppose that (2) occurs. If $P_5 = (v_1, v_2, v_3, v_4, v_5)$ is red, let $A_4 = (v_1, v_2, v_6, v_4, v_5)$ be a red P_5 and $A_3 = (v_2, v_3, v_6, v_5)$ a red P_4. If $P_5 = (v_1, v_2, v_3, v_4, v_5)$ is blue, let $A_4 = (v_2, v_1, v_5, v_4, v_3)$ be a blue P_5 and $A_3 = (v_2, v_3, v_1, v_4)$ a blue $G_3 = P_4$. This is illustrated in Figure 4 and so A_4 and A_3 are edge-disjoint.

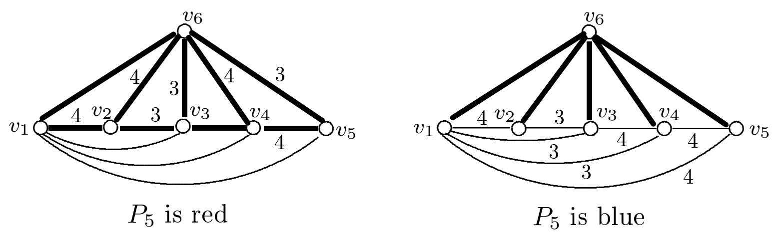

Fig. 4. The edges of A_3 and A_4 when (2) occurs

Next, suppose that (3) occurs. If $P_5 = (v_1, v_2, v_3, v_4, v_5)$ is red, let $A_4 = (v_1, v_2, v_6, v_4, v_5)$ be a red P_5 and $A_3 = (v_2, v_3, v_6, v_5)$ a red P_4. If $P_5 = (v_1, v_2, v_3, v_4, v_5)$ is blue, let $A_4 = (v_6, v_1, v_5, v_4, v_3)$ be a blue P_5 and $A_3 = (v_3, v_2, v_1, v_4)$ a blue $G_3 = P_4$. This is illustrated in Figure 5 and so A_4 and A_3 are edge-disjoint.

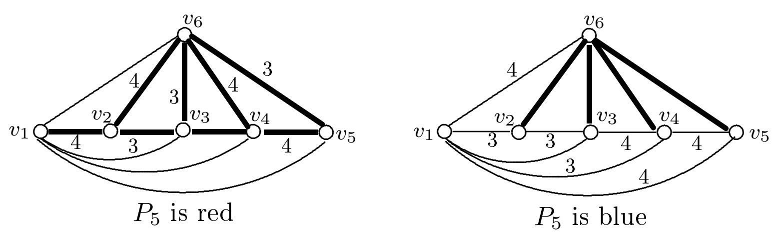

Fig. 5. The edges of A_3 and A_4 when (3) occurs

In either situation, the monochromatic subgraph $G_3 = P_4 = (v_4, v_2, v_5, v_3)$ is edge-disjoint from A_4 and A_3 and can be decomposed into $A_2 = 2K_2$ and $A_1 = K_2$. Therefore, A_1, A_2, A_3, A_4 are pairwise edge-disjoint. Thus, s is a Ramsey chain in G. $\square$

Next, we show that $TR(s) = R(S_{2,3})$ for every Ramsey chain s with target link $S_{2,3}$. First, we present a lemma.

Lemma 3.8. *If $A = S_{2,3}$ and B are two edge-disjoint monochromatic subgraphs of the graph $G = K_6$ with a red-blue coloring where B has size 3 without isolated vertices such that $G - [E(A) \cup E(B)]$ has no monochromatic $2K_2$, then $B \in \{P_3 + K_2, K_3, P_4\}$.*

Proof. Let $G = K_6$ with $V(G) = \{v_1, v_2, \ldots, v_6\}$. Since $R(S_{2,3}) = 6$, there is a monochromatic subgraph $A = S_{2,3}$. We may assume that A is obtained from the 4-path (v_2, v_3, v_4, v_5) by adding the edge v_1v_3. Suppose that $H = G - [E(A) \cup E(B)]$ does not contain monochromatic $2K_2$. Let H_r and H_b be the red and blue subgraphs of H where $|E(H_r)| \geq |E(H_b)|$, say. Since the size of H is $\binom{6}{2} - 7 = 8$ and H has no monochromatic $2K_2$, it follows that either (1) $H_r = K_{1,5}$ and $H_b = K_3$ or (2) H_r and H_b are stars. We consider these two cases.

Case 1. *$H_r = K_{1,5}$ and $H_b = K_3$.* Then H_r is centered at v_6 and $E(H_r) = \{v_6v_i : 1 \leq i \leq 5\}$. Thus, $H_b = K_3 = (v_1, v_2, v_5, v_1)$. Hence, $E(B) = \{v_1v_4, v_4v_2, v_3v_5\}$ and so $B = P_3 + K_2$.

Case 2. *H_r and H_b are stars.* Suppose that $H_r = K_{1,s}$ is centered at a vertex v_p and $H_b = K_{1,t}$ is centered at a vertex v_q where $s + t = 8$, $s \geq t$, $s \in \{4, 5\}$, and $t \geq 3$.

$\star$ If $s > t$, then $s = 5$ and $t = 3$. Then $v_p = v_6$ and $E(H_r) = \{v_6v_i : 1 \leq i \leq 5\}$. This implies that $v_q \in \{v_1, v_2, v_5\}$. Thus, $v_q = v_1$ or $v_q = v_5$. If $v_q = v_1$, then $E(H_b) = \{v_1v_i : \ i = 2, 4, 5\}$. Hence, $E(B) = \{v_4v_2, v_2v_5, v_5v_3\}$ and so $B = (v_4, v_2, v_5, v_3) = P_4$. If $v_q = v_5$, then $E(H_b) = \{v_5v_i : \ i = 1, 2, 3\}$. Hence, $E(B) = \{v_1v_2, v_2v_4, v_4v_1\}$ and so $B = (v_1, v_2, v_4, v_1) = K_3$.

$\star$ If $s = t$, then $s = t = 4$. Since $\deg_H v_i \leq 3$ for $i = 3, 4$, it follows that $v_p, v_q \in \{v_1, v_2, v_5, v_6\}$. If $\{v_p, v_q\} \subseteq \{v_1, v_2, v_5\}$, then $v_pv_q \in E(H_r) \cap E(H_b)$, which is a contradiction. Thus, we may assume that $v_p = v_6$ and $v_q \in \{v_1, v_5\}$. If $v_q = v_1$, then $E(H_b) = \{v_1v_i : i = 2, 4, 5, 6\}$ and $E(H_r) = \{v_6v_i : 2 \leq i \leq 5\}$. Hence, $E(B) = \{v_4v_2, v_2v_5, v_5v_3\}$ and so $B = (v_4, v_2, v_5, v_3) = P_4$. If $v_q = v_5$, then $E(H_b) = \{v_5v_i : \ i = 1, 2, 3, 6\}$ and $E(H_r) = \{v_6v_i : 1 \leq i \leq 4\}$. Hence, $E(B) = \{v_1v_2, v_2v_4, v_4v_1\}$ and so $B = (v_1, v_2, v_4, v_1) = K_3$. $\square$

The following is a consequence of Lemma 3.8.

Corollary 3.9. *Let $G = K_6$ with a red-blue coloring where $V(G) = \{v_1, v_2, \ldots, v_6\}$, let $A = S_{2,3}$ and B be two edge-disjoint monochromatic subgraphs of G where $E(A) = \{v_1v_3, v_2v_3, v_3v_4, v_4v_5\}$ and B has size 3 without isolated vertices, and let $H = G - [E(A) \cup E(B)]$ with red subgraph H_r and blue subgraph H_b where $|E(H_r)| \geq |E(H_b)|$. If H has no monochromatic $2K_2$, then $B \in \{P_3 + K_2, K_3, P_4\}$. Furthermore*

(a) $B = P_3 + K_2$ only when $H_r = K_{1,5}$ is centered at v_6, $H_b = K_3 = (v_1, v_2, v_5, v_1)$, and $E(B) = \{v_1v_4, v_4v_2, v_3v_5\}$ and

(b) $B = P_4$ *only when one of the following occurs*

(1) $H_r = K_{1,5}$ *is centered at* v_6 *and* $H_b = K_{1,3}$ *is centered at* $v_q \in \{v_1, v_2\}$.

(2) $H_r = H_b = K_{1,4}$ *where* H_r *is centered at* v_6 *and* H_b *is centered at* $v_q \in \{v_1, v_2\}$.

Theorem 3.10. *If* s *is a Ramsey chain with target link* $S_{2,3}$*, then* $TR(s) = R(S_{2,3}) = 6$.

Proof. By Observation 1.3, it suffices to show that $TS(s) \leq 6$. Let $s = (G_1, G_2, G_3, G_4)$ be a Ramsey chain where $G_4 = S_{2,3}$. Then $G_3 \in \{P_4, K_{1,3}, P_3 + K_2\}$ and $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G = K_6$ with $V(G) = \{v_1, v_2, \ldots, v_6\}$. Since $R(S_{2,3}) = 6$, in the red-blue coloring of G, there is a monochromatic subgraph $G_4 = S_{2,3}$ obtained, say, from the 4-path (v_2, v_3, v_4, v_5) by adding the edge v_1v_3. We show that $F = K_6 - E(G_4)$ contains a monochromatic subgraph $G_3 \in \{K_{1,3}, P_4, P_3 + K_2\}$. There are three possibilities.

Case 1. $G_3 = K_{1,3}$. Since at least three of the five edges v_6v_i $(1 \leq i \leq 5)$ have the same color, there is a monochromatic subgraph $G_3 = K_{1,3}$ in F.

Case 2. $G_3 = P_4$. Since at least two of the three edges v_6v_1, v_6v_2, v_6v_3 have the same color, we may assume that v_6v_i and v_6v_j are red where $1 \leq i < j \leq 3$, where either $\{i, j\} = \{1, 2\}$ or $\{i, j\} = \{1, 3\}$. If one of v_5v_i and v_5v_j is red, then there is a red P_4 in F. Thus, we may assume that v_5v_i and v_5v_j are blue. Regardless of the color of v_1v_4, there is a monochromatic subgraph $G_3 = P_4$ in F.

Case 3. $G_3 = P_3 + K_2$. Since at least two of the three edges v_6v_1, v_6v_2, v_6v_3 have the same color, we may assume that v_iv_6 and v_jv_6 are red where $1 \leq i < j \leq 3$ and either $\{i, j\} = \{1, 2\}$ or $\{i, j\} = \{1, 3\}$.

Subcase 3.1. v_6v_1 *and* v_6v_2 *are red.* Then v_3v_5 is blue, for otherwise, $(v_1, v_6, v_2) + (v_3, v_5)$ is a red $P_3 + K_2$. The edge v_2v_4 is either red or blue.

⋆ If v_2v_4 is red, then v_1v_5 is blue, for otherwise $(v_6, v_2, v_4) + (v_1, v_5)$ is a red $P_3 + K_2$. Then v_6v_4 is red, for otherwise $(v_1, v_5, v_3) + (v_6, v_4)$ is a blue $P_3 + K_2$.

∘ If v_6v_3 is red, then $(v_1, v_6, v_3) + (v_2, v_4)$ is a red $P_3 + K_2$.

∘ If v_6v_3 is blue, then v_1v_2 is red, for otherwise $(v_6, v_3, v_5) + (v_1, v_2)$ is a blue $P_3 + K_2$. This implies that v_6v_5 is blue, for otherwise, $(v_1, v_2, v_4) + (v_6, v_5)$ is a red $P_3 + K_2$. Regardless of the color of v_1v_4, there is a monochromatic $P_3 + K_2$.

⋆ If v_2v_4 is blue, then v_6v_3, v_6v_4, v_6v_5 are red, for otherwise there is a blue $P_3 + K_2$. Regardless of the color of v_1v_2, there is a monochromatic $P_3 + K_2$.

Subcase 3.2. v_6v_1 *and* v_6v_3 *are red.* Then v_2v_4 and v_2v_5 are blue, for otherwise there is a red $P_3 + K_2$. Consider the color of the edge v_1v_2.

⋆ If v_1v_2 is red, then v_3v_5 is blue, for otherwise $(v_6, v_1, v_2) + (v_3, v_5)$ is a red $P_3 + K_2$. Then v_6v_4 is red, for otherwise $(v_3, v_5, v_2) + (v_6, v_4)$ is a blue $P_3 + K_2$. Then $(v_3, v_6, v_4) + (v_1, v_2)$ is a red $P_3 + K_2$.

⋆ If v_1v_2 is blue, then v_6v_4 is red, for otherwise $(v_1, v_2, v_5) + (v_6, v_4)$ is a blue $P_3 + K_2$. Then v_1v_5 is blue, for otherwise $(v_3, v_6, v_4) + (v_1, v_5)$ is a red $P_3 + K_2$. Then v_3v_5 is red, for otherwise $(v_4, v_2, v_1) + (v_3, v_5)$ is a blue $P_3 + K_2$. There is a red $P_3 + K_2 = (v_1, v_6, v_4) + (v_3, v_5)$.

Next, let $H = G - [E(G_4) \cup E(G_3)]$. It then follows by Lemma 3.2 that H contains a monochromatic P_3. Thus, if $G_2 = P_3$, then s is a Ramsey chain in G. Hence, we may assume that $G_2 = 2K_2$. Then $G_3 \in \{P_3 + K_2, P_4\}$.

• If $G_3 = P_3 + K_2$, then it follows by Corollary 3.9 that $H_r = K_{1,5}$ is centered at v_6, $H_b = K_3 = (v_1, v_2, v_5, v_1)$, and $G_3 = P_3 + K_2 = (v_1, v_4, v_2) + (v_3, v_5)$. Depending on whether $G_4 = S_{2,3}$ is red or blue, we construct a new Ramsey chain

$$A_1 = K_2, A_2 = 2K_2, A_3 = P_3 + K_2, A_4 = S_{2,3},$$

as follows.

⋆ First, suppose that $G_4 = S_{2,3}$ is red. Then $A_4 = S_{2,3}$ is red obtained from $P_4 = (v_2, v_3, v_6, v_4)$ by adding the edge v_6v_5 and $A_3 = P_3 + K_2 = (v_3, v_4, v_5) + (v_1, v_6)$ is red. The monochromatic subgraph $G_3 = P_3 + K_2$ is edge-disjoint from A_4 and A_3 and can be decomposed into $A_2 = 2K_2$ and $A_1 = K_2$.

⋆ Next, suppose that $G_4 = S_{2,3}$ is blue.

∘ If $G_3 = P_3 + K_2 = (v_1, v_4, v_2) + (v_3, v_5)$ is red, then $A_4 = S_{2,3}$ is red obtained from $P_4 = (v_2, v_4, v_6, v_3)$ by adding the edge v_6v_5, $A_3 = P_3 + K_2 = (v_6, v_1, v_4) + (v_2, v_5)$ is red, $A_2 = 2K_2$ is blue with $E(A_2) = \{v_1v_2, v_3v_4\}$, and $A_1 = K_2$.

∘ If $G_3 = P_3 + K_2 = (v_1, v_4, v_2) + (v_3, v_5)$ is blue, then $A_4 = S_{2,3}$ is blue obtained from $P_4 = (v_5, v_2, v_1, v_3)$ by adding the edge v_1v_4, $A_3 = P_3 + K_2 = (v_2, v_4, v_3) + (v_1, v_5)$ is blue, $A_2 = 2K_2$ is blue with $E(A_2) = \{v_2v_3, v_4v_5\}$, and $A_1 = K_2$.

• If $G_3 = P_4$, then there are two possibilities by Corollary 3.9, namely,

(1) $H_r = K_{1,5}$ centered at $v_p = v_6$ and $H_b = K_{1,3}$ centered at $v_q \in \{v_1, v_2\}$, or

(2) $H_r = K_{1,4}$ centered at $v_p = v_6$ and $H_b = K_{1,4}$ centered at $v_q \in \{v_1, v_2\}$.

In either (1) or (2) occurs, we may assume that $v_q = v_1$. Depending on whether $G_4 = S_{2,3}$ is red or blue, we construct a new Ramsey chain

$$A_1 = K_2, A_2 = 2K_2, A_3 = P_4, A_4 = S_{2,3},$$

as follows.

⋆ First, suppose that (1) occurs. Then $E(H_r) = \{v_6v_i : 1 \le i \le 5\}$ and $E(H_b) = \{v_1v_i : i = 2, 4, 5\}$.

∘ If G_4 is red, then $A_4 = S_{2,3}$ is red obtained from $P_4 = (v_1, v_3, v_4, v_5)$ by adding the edge v_4v_6 and $A_3 = P_4 = (v_3, v_2, v_6, v_5)$ is red.

∘ If G_4 is blue, then then $A_4 = S_{2,3}$ is blue obtained from $P_4 = (v_3, v_2, v_1, v_4)$ by adding the edge v_1v_5 and $A_3 = P_4 = (v_1, v_3, v_4, v_5)$ is blue.

⋆ Next, suppose that (2) occurs. Then $E(H_r) = \{v_6v_i : 2 \le i \le 5\}$ and $E(H_b) = \{v_1v_i : i = 2, 4, 5, 6\}$. For A_4 and A_3 in situation (1), the edge v_1v_6 is not used. Thus, regardless of the color of G_4, there are edge-disjoint monochromatic subgraphs $A_4 = S_{2,3}$ and $A_3 = P_4$ in H.

In either situation, the monochromatic subgraph $G_3 = P_4 = (v_3, v_5, v_2, v_4)$ is edge-disjoint from A_4 and A_3 and can be decomposed into $A_2 = 2K_2$ and $A_1 = K_2$. Thus, s is a Ramsey chain in G. □

In summary, we have the following.

Theorem 3.11. *If s is a Ramsey chain with target link G_4 with $R(G_4) = 6$, then $TR(s) = R(G_4)$.*

3.2. *Target links of size 4 and Ramsey Number 7*

We now consider Ramsey chains of length 4 whose target link G_4 has Ramsey number 7. Thus, $G_4 \in \{K_{1,3} + e, K_{1,4}, K_3 + K_2, K_{1,3} + K_2, 2P_3\}$. We show that if $R(G_4) = 7$, then $TR(s) = R(G_4)$. First, we present two lemmas.

Lemma 3.12. *If G is a graph of order $n \geq 7$ and size $m \geq 2n$, then every red-blue coloring of G results in a monochromatic subgraph $G_3 = P_3 + K_2$. Consequently, every red-blue coloring of G results in a monochromatic subgraph $G_2 \in \{P_3, 2K_2\}$.*

Proof. For a graph G of order $n \geq 7$ and size $m \geq 2n$ with a red-blue coloring, let G_r be the red subgraph and let G_b be the blue subgraph. We may assume that $|E(G_r)| \geq |E(G_b)|$. Thus G_r has order $n \geq 7$ and size at least n. Since a forest (each component is a tree) of order n with $k \geq 1$ components has exactly $n - k$ edge and K_4 has six edges, it follows that G is neither a forest nor K_4. In particular, G is neither a star nor a matching. This implies that G_r contains $P_3 + K_2$ as a subgraph. Becuase each of P_3 and $2K_2$ is a subgraph of $P_3 + K_2$, it follows that G_r contains P_3 or $2K_2$. □

Lemma 3.13. [5] *Every red-blue coloring of K_7 results in two edge-disjoint monochromatic triangles.*

Theorem 3.14. *If s is a Ramsey chain with target link $K_{1,3} + e$, then $TR(s) = R(K_{1,3} + e) = 7$.*

Proof. By Observation 1.3, it suffices to show that $TR(s) \leq 7$. Let $s = (G_1, G_2, G_3, G_4)$ be a Ramsey chain where $G_4 = K_{1,3} + e$. Then $G_3 \in \{K_{1,3}, P_4, K_3\}$ and $G_2 \in \{P_3, 2K_2\}$. For a red-blue coloring of $G = K_7$ with $V(G) = \{v_1, v_2, \ldots, v_7\}$, we first show that G contains edge-disjoint monochromatic subgraphs $G_4 = K_{1,3} + e$ and $G_3 \in \{K_{1,3}, P_4, K_3\}$.

⋆ First, suppose that $G_3 \in \{K_{1,3}, P_4\}$. Since $R(K_{1,3} + e) = 7$, there is a monochromatic subgraph $G_4 = K_{1,3} + e$ with $E(G_4) = \{v_i v_4 : i = 1, 2, 3\} \cup \{v_1 v_2\}$, say. Let $F = G - E(G_4)$. Since at least three of the five edges $v_6 v_i$ $(1 \leq i \leq 5)$ of F have the same color, there is a monochromatic subgraph $G_3 = K_{1,3}$ in F. Since $G[\{v_1, v_3, v_5, v_6, v_7\}] \cong K_5 \subseteq F$ and $R(P_4) = 5$, there is a monochromatic subgraph $G_3 = P_4$ in F.

⋆ Next, suppose that $G_3 = K_3$. By Lemma 3.13, $G = K_7$ contains two edge-disjoint monochromatic triangles T_1 and T_2. We may assume that either $V(T_1) = \{v_1, v_2, v_3\}$ and $V(T_2) = \{v_3, v_4, v_5\}$ or $V(T_1) = \{v_1, v_2, v_3\}$ and $V(T_2) = \{v_4, v_5, v_6\}$. If T_1 and T_2 have different colors, then regardless of the color of $v_1 v_3$, there are edge-disjoint monochromatic subgraphs $G_4 = K_{1,3} + e$ and $G_3 = K_3$. Thus, we may assume that T_1 and T_2 have the same color, say red. If there is a red edge that joins either (1) a vertex of T_1 and a vertex of T_2 or (2) a vertex of $V(T_1) \cup V(T_2)$ and the vertex v_7, then there are edge-disjoint red subgraphs $G_4 = K_{1,3} + e$ and $G_3 = K_3$. If every edge between $V(T_1)$ and $V(T_2)$ and every edge between $V(T_1) \cup V(T_2)$ and v_7 is blue, then there are edge-disjoint blue subgraphs $G_4 = K_{1,3} + e$ and $G_3 = K_3$.

The graph $H = G - [E(G_4) \cup E(G_3)]$ has order 7 and size 14. By Lemma 3.12, H contains a monochromatic subgraph $G_2 \in \{P_3, 2K_2\}$. Since the size of G is 21 and $|E(G_4) \cup E(G_3) \cup E(G_2)| = 9$, there is an edge in $E(G) - [E(G_4) \cup E(G_3) \cup E(G_2)]$ that is available for $G_1 = K_2$. Therefore, s is a Ramsey chain in G. □

Theorem 3.15. *If s is a Ramsey chain with target link $K_3 + K_2$, then $TR(s) = R(K_3 + K_2) = 7$.*

Proof. By Observation 1.3, it suffices to show that $TR(s) \leq 7$. Let $s = (G_1, G_2, G_3, G_4)$ be a Ramsey chain where $G_4 = K_3 + K_2$. Then $G_3 \in \{K_3, P_3 + K_2\}$ and $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G = K_7$ with $V(G) = \{v_1, v_2, \ldots, v_7\}$. First, we show that G contains edge-disjoint monochromatic subgraphs $G_4 = K_3 + K_2$ and $G_3 \in \{K_3, P_3 + K_2\}$.

⋆ First, suppose that $G_3 = P_3 + K_2$. Since $R(K_3 + K_2) = 7$, there is a monochromatic subgraph $G_4 = K_3 + K_2$ where $E(G_4) = \{v_1v_2, v_2v_3, v_3v_1\} \cup \{v_4v_5\}$, say. Since $F = G - E(G_4)$ is a graph of order 7 and size $\binom{7}{2} - 4 = 17 \geq 14$, it follows by Lemma 3.12 that F contains a monochromatic subgraph $G_3 = P_3 + K_2$.

⋆ Next, suppose that $G_3 = K_3$. It follows by Lemma 3.13 that $G = K_7$ contains two edge-disjoint monochromatic triangles T_1 and T_2. First, suppose that T_1 and T_2 have a vertex v_3 in common, say $V(T_1) = \{v_1, v_2, v_3\}$ and $V(T_2) = \{v_3, v_4, v_5\}$. See Figure 6. If T_1 and T_2 are of different colors, then regardless of the color of v_6v_7, there are edge-disjoint monochromatic subgraphs $G_4 = K_3 + K_2$ and $G_3 = K_3$. Thus, we may assume that T_1 and T_2 have the same color, say red. If there is a red edge joining a vertex in $\{v_6, v_7\}$ and a vertex in $V(T_1) \cup V(T_2)$, then there are edge-disjoint red subgraphs $G_4 = K_3 + K_2$ and $G_3 = K_3$. Thus, we may assume that every edge joining a vertex in $\{v_6, v_7\}$ and a vertex in $V(T_1) \cup V(T_2)$ is blue. Regardless of the color of v_6v_7, there are edge-disjoint monochromatic subgraphs $G_4 = K_3 + K_2$ and $G_3 = K_3$.

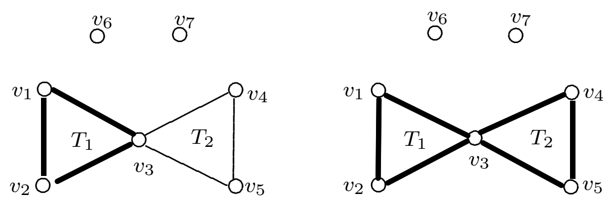

Fig. 6. The two edge-disjoint triangles T_1 and T_2 having a vertex in commen

Next, suppose that T_1 and T_2 are vertex-disjoint, say $V(T_1) = \{v_1, v_2, v_3\}$ and $V(T_2) = \{v_4, v_5, v_6\}$. See Figure 7. If T_1 and T_2 are of different color, say T_1 is red and T_2 is blue, then we may assume that v_7v_i is red for $i = 1, 2, 3$ and v_7v_j is blue for $j = 4, 5, 6$; for otherwise, there are edge-disjoint monochromatic subgraphs $G_4 = K_3 + K_2$ and $G_3 = K_3$. Then regardless of the color of v_1v_4, there are edge-disjoint monochromatic subgraphs $G_4 = K_3 + K_2$ and $G_3 = K_3$. Thus, we may assume that T_1 and T_2 have the same color, say red. Then v_7v_i is blue for each integer i with $1 \le i \le 6$, for otherwise, there are edge-disjoint red subgraphs $G_4 = K_3+K_2$ and $G_3 = K_3$. If there is a blue edge joining a vertex of T_1 and a vertex of T_2, say v_1v_4 is blue, then there are a red $G_4 = K_3+K_2 = T_1+(v_4, v_5)$ and a blue $G_3 = K_3 = (v_1, v_4, v_7, v_1)$. Thus, we may assume that every edge between T_1 and T_2 are red. Then there are edge-disjoint red subgraphs $G_4 = K_3 + K_2$ and $G_3 = K_3$. For example, $G_4 = K_3 + K_2 = T_1 + (v_4, v_5)$ and $G_3 = (v_1, v_5, v_6, v_1)$.

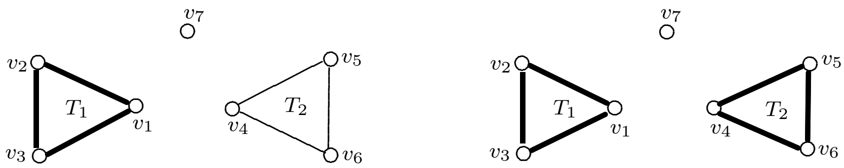

Fig. 7. The two vertex-disjoint triangles T_1 and T_2

The graph $H = G - [E(G_4) \cup E(G_3)]$ has order 7 and size 14. By Lemma 3.12, H contains a monochromatic subgraph $G_2 \in \{P_3, 2K_2\}$. Since the size of G is 21 and $|E(G_4) \cup E(G_3) \cup E(G_2)| = 9$, there is an edge in $E(G) - [E(G_4) \cup E(G_3) \cup E(G_2)]$ that is available for $G_1 = K_2$. Therefore, s is a Ramsey chain in G. □

Theorem 3.16. *If s is a Ramsey chain with target link $K_{1,3}+K_2$, then $TR(s) = R(K_{1,3}+K_2) = 7$.*

Proof. By Observation 1.3, it suffices to show that $TR(s) \le 7$. Let $s = (G_1, G_2, G_3, G_4)$ be a Ramsey chain where $G_4 = K_{1,3} + K_2$. Then $G_3 \in \{K_{1,3}, P_3 + K_2\}$ and $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G = K_7$ with $V(G) = \{v_1, v_2, \ldots, v_7\}$. Since $R(K_{1,3} + K_2) = 7$, there is a monochromatic subgraph $G_4 = K_{1,3} + K_2$ of G where $E(G_4) = \{v_1v_i : i = 2, 3, 4\} \cup \{v_5v_6\}$, say. Let $F = G - E(G_4)$. Since at least three of the six edges v_7v_i $(1 \le i \le 6)$ of F have the same color, there is a monochromatic subgraph $G_3 = K_{1,3}$ in F. Furthermore, since F has order 7 and size 17, it follows by Lemma 3.12 that F contains a monochromatic subgraph $G_3 = P_3 + K_2$. Next, let $H = G - [E(G_4) \cup E(G_3)]$. Then H is a graph of order 7 and size 14. By Lemma 3.12 again, H contains a monochromatic subgraph $G_2 \in \{P_3, 2K_2\}$. Since the size of G is 21 and $|E(G_4) \cup E(G_3) \cup E(G_2)| = 9$, there is an edge in $E(G) - [E(G_4) \cup E(G_3) \cup E(G_2)]$ that is available for $G_1 = K_2$. Therefore, s is a Ramsey chain in G. □

Theorem 3.17. *If s is a Ramsey chain with target link $2P_3$, then $TR(s) = R(2P_3) = 7$.*

Proof. By Observation 1.3, it suffices to show that $TR(s) \le R(2P_3) = 7$. Let $s = (G_1, G_2, G_3, G_4)$ be a Ramsey chain where $G_4 = 2P_3$. Then $G_3 = P_3 + K_2$ and $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G = K_7$ with $V(G) = \{v_1, v_2, \ldots, v_7\}$. Since $R(2P_3) = 7$, there is a monochromatic subgraph $G_4 = 2P_3$ of G where $V(G_4) = \{v_i : 1 \le i \le 6\}$, say. Let $F = G - E(G_4)$. Since F has order 7 and size 17, it follows by Lemma 3.12 that F contains a monochromatic subgraph $G_3 = P_3 + K_2$. The graph $H = G - [E(G_4) \cup E(G_3)]$ has order 7 and size 14. By Lemma 3.12 again, H contains a monochromatic subgraph $G_2 \in \{P_3, 2K_2\}$. Since the size of G is 21 and $|E(G_4) \cup E(G_3) \cup E(G_2)| = 9$, there is an edge in $E(G) - [E(G_4) \cup E(G_3) \cup E(G_2)]$ that is available for $G_1 = K_2$. Therefore, s is a Ramsey chain in G. □

In summary, we then have the following.

Theorem 3.18. *If s is a Ramsey chain with target link G_4 with $R(G_4) = 7$, then $TR(s) = R(G_4)$.*

3.3. Target links of size 4 and Ramsey Number 8 or 9

Finally, we consider Ramsey chains whose target link $G_4 \in \{P_4 + K_2, P_3 + 2K_2\}$ has Ramsey number 8 or 9. We show here that $TR(s) = R(G_4)$ for every such Ramsey chain s, beginning with $G_4 = P_4 + K_2$.

Theorem 3.19. *If s is a Ramsey chain with target link $P_4 + K_2$, then $TR(s) = R(P_4 + K_2) = 8$.*

Proof. By Observation 1.3, it suffices to show that $TR(s) \leq 8$. Let $s = (G_1, G_2, G_3, G_4)$ be a Ramsey chain where $G_4 = P_4 + K_2$. Then $G_3 \in \{P_4, P_3 + K_2\}$ and $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G = K_8$ with $V(G) = \{v_1, v_2, \ldots, v_8\}$. Since $R(P_4 + K_2) = 8$, there is a monochromatic subgraph $G_4 = P_4 + K_2$ of G where $G_4 = (v_1, v_2, v_3, v_4) + (v_5, v_6)$, say.

Let $F = G - E(G_4)$. Since F has order 8 and size $\binom{8}{2} - 4 = 24 > 16$, it follows by Lemma 3.12 that F contains a monochromatic subgraph $G_3 = P_3 + K_2$. Furthermore, $G[\{v_1, v_3, v_5, v_7, v_8\}] = K_5 \subseteq F$. Since $R(P_4) = 5$, there is a monochromatic subgraph $G_3 = P_4$. Next, the graph $H = G - [E(G_4) \cup E(G_3)]$ has order 8 and size 21. By Lemma 3.12 again, H contains a monochromatic subgraph $G_2 \in \{P_3, 2K_2\}$. Furthermore, H contains an edge not in G_2 that is available for $G_1 = K_2$. Therefore, s is a Ramsey chain in G. □

To show that $TR(s) = R(G_4)$ for every such Ramsey chain s with target link $P_3 + 2K_2$, we first present a lemma.

Lemma 3.20. [16] *Every red-blue coloring of K_8 produces four pairwise edge-disjoint monochromatic copies of $3K_2$.*

Theorem 3.21. *If s is a Ramsey chain with target link $P_3 + 2K_2$, then $TR(s) = R(P_3 + 2K_2) = 9$.*

Proof. By Observation 1.3, it suffices to show that $TR(s) \leq 9$. Let $s = (G_1, G_2, G_3, G_4)$ be a Ramsey chain where $G_4 = P_3 + 2K_2$. Then $G_3 \in \{P_3 + K_2, 3K_2\}$ and $G_2 \in \{P_3, 2K_2\}$. Let there be given a red-blue coloring of $G = K_9$ with $V(G) = \{v_1, v_2, \ldots, v_9\}$. We show that G contains edge-disjoint monochromatic subgraphs $G_4 = P_3 + 2K_2$ and $G_3 \in$

$\{P_3 + K_2, 3K_2\}$.

⋆ First, suppose that $G_3 = P_3 + K_2$. Since $R(P_3 + 2K_2) = 9$, there is a monochromatic subgraph $G_4 = P_3 + 2K_2$. Furthermore, since $G - E(G_4)$ has order 9 and size $\binom{9}{2} - 4 = 32 > 18$, it follows by Lemma 3.12 that $G - E(G_4)$ contains a monochromatic subgraph $G_3 = P_3 + K_2$.

⋆ Next, suppose that $G_3 = 3K_2$. Let $F = G - v_9 \cong K_8$. By Lemma 3.20, F contains four pairwise edge-disjoint monochromatic copies F_1, F_2, F_3, F_4 of $3K_2$. If three of the subgraphs F_1, F_2, F_3, F_4 have the same color, say F_1, F_2, F_3 are red $3K_2$, then there is a red $G_4 = P_3+2K_2$ in the subgraph $G[E(F_1)\cup E(F_2)\cup E(F_3)]$ induced by $E(F_1)\cup E(F_2)\cup E(F_3)$. Let $G_3 = F_4 = 3K_2$. Thus, we may assume that F_1 and F_2 are red and F_3 and F_4 are blue. Let $A = G[E(F_1) \cup E(F_2)]$ and $B = G[E(F_3) \cup E(F_4)]$. If $A \neq C_6$ or $B \neq C_6$, then there is a red $G_4 = P_3 + 2K_2$ in A or B. Thus, we may assume that$A = C_6$ is red and $B = C_6$ is blue. If v_9 is joined to a vertex in A by a red edge, then there is a red $G_4 = P_3 + 2K_2$. Let $G_3 = F_3 = 3K_2$ in F. Thus, we may assume that v_9 is joined to every vertex in A by a blue edge. Since $V(A) \cap V(B) \neq \emptyset$, it follows that v_9 is joined to a vertex in B by a blue edge, producing a blue $G_4 = P_3 + 2K_2$. Let $G_3 = F_1 = 3K_2$ in F.

The graph $H = G - [E(G_4) \cup E(G_3)]$ has order 9 and size 29. By Lemma 3.12, H contains a monochromatic subgraph $G_2 \in \{P_3, 2K_2\}$. Furthermore, H contains an edge not in G_2 that is available for $G_1 = K_2$. Therefore, s is a Ramsey chain in G. □

In summary, we have the following result.

Theorem 3.22. *If s is a Ramsey chain with target link G_4 where $R(G_4) = 8$ or $R(G_4) = 9$, then $TR(s) = R(G_4)$.*

4. Closing Remarks

We have seen several graphs H for which if s is a Ramsey chain with target link H, then $TR(s) = R(H)$. In particular, all graphs H of size 3 or 4 without isolated vertices have the property that $TR(s) = R(H)$ for every Ramsey chain s with target link H. In fact, it can be verified that there are other well-known graphs with this property. For example, if $H \in \{K_4, C_5, P_6\}$, then $TR(s) = R(H)$ for every Ramsey chain s with target link H. Furthermore, we know of no graph H and a Ramsey chain s with target link H such that $TR(s) > R(H)$. Therefore, we close with the following question.

Question 4.1. *Is there a graph H without isolated vertices and a Ramsey chain s with target link H for which $TR(s) > R(H)$?*

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