On 3-Uniform Hypergraphic Sequences

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Abstract. This paper discusses new Erdös - Gallai type necessary conditions for a sequence Π of integers to be 3-hypergraphic. Further, we show that some of the known necessary conditions for 3-hypergraphic sequences are not sufficient.

1. Introduction

An *r-uniform hypergraph* H is a pair $(V, \mathcal{E})$ where V is a finite non-empty set and $\mathcal{E}$ is a family of subsets with exactly r elements of V. The elements of V and $\mathcal{E}$ are called *vertices* and *edges* respectively. Note that the elements of $\mathcal{E}$ need not be distinct. An r-uniform hypergraph $H = (V, \mathcal{E})$ is called *simple* if all the elements in $\mathcal{E}$ are distinct. The *degree* $\deg_H(v)$ of a vertex v in an r-uniform hypergraph H is the number of edges containing the vertex v. The sequence $\Pi = (\deg_H(v_1), \ldots, \deg_H(v_p))$ where $V = \{v_1, v_2, \ldots, v_p\}$ is called the *degree sequence* of $H = (V, \mathcal{E})$

In this paper $\Pi = (d_1, d_2, \ldots, d_p)$ denotes a non-increasing sequence of non-negative integers. Π is said to be *r-uniform hypergraphic* if there is a simple r-uniform hypergraph H on p vertices $v_1, v_2, \ldots, v_p$ such that $\deg_H(v_i) = d_i$ for every i, $1 \leq i \leq p$. In what follows, an r-uniform hypergraphic sequence will be simply referred to as an r-hypergraphic sequence.

Note that a simple 2-uniform hypergraph is a simple graph. For general definitions and notation we refer to Berge [1] and Bondy and Murty [3]. In this paper we provide new Erdös and Gallai type necessary conditions for 3-hypergraphic sequences. Further we show that some of the known necessary conditions for 3-hypergraphic sequences are not sufficient. Some of these results appeared in Simanihuruk [8].

For a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Further x^+ denotes $\max(0, x)$. We denote $(x_1^+, x_2^+, \ldots, x_p^+)$ by $(x_1, x_2, \ldots, x_p)^+$.

We state some of the known results used in the later sections.

Theorem 1.1: (Dewdney [6]). *Let* $\Pi = (d_1, d_2, \ldots, d_p)$ *be a non-increasing sequence of non-negative integers. Then* Π *is an* r*-hypergraphic sequence if and only if there exists a non-increasing sequence* $\Pi' = (d_2', d_3', \ldots, d_p')$ *of non-negative integers such that*

(i) Π' *is an* $(r-1)$*-hypergraphic sequence,*
(ii) $\sum_{i=2}^{p} d_i' = (r-1)d_1$*, and*
(iii) $\Pi'' = (d_2 - d_2', d_3 - d_3', \ldots, d_p - d_p')$ *is an* r*-hypergraphic sequence.* ∎

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Theorem 1.2: (Erdös and Gallai [7]). *A sequence* $\Pi = (d_1, d_2, \ldots, d_p)$ *is 2-hyperraphic iff* $\sum_{i=1}^{p} d_i$ *is even and*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{p} \min(d_j, k), \ 1 \leq k \leq p.$$

∎

There are no known necessary and sufficient conditions, generalising Theorem 1.2 to 3-hypergraphic sequences (refer Colbourn et. al. [5], Billington [2]). Further there is no known polynomial time algorithm to construct a 3-uniform hypergraph realizing a given Π. Billington [2] established the following Erdös and Gallai type necessary conditions for 3-hypergraphic sequences.

Theorem 1.3: (Billington [2]). *Let* $\Pi = (d_1, d_2, \ldots, d_p)$ *be a 3-hypergraphic sequence. Then* $\sum_{i=1}^{p} d_i \equiv 0 \pmod 3$, *and for every* k, $1 \leq k \leq p$, *we have*

$$\sum_{i=1}^{k} d_i \leq 3\binom{k}{3} - 3\left(\binom{k-1}{2} - d_k\right)^{+} + 2\sum_{i=k+1}^{p} \min\left(d_i, \binom{k}{2}\right) + \sum_{i=k+1}^{p} \min(d_i, (i-k-1)k) \tag{1.1}$$

∎

A k-multigraph is a loopless undirected graph with at most k edges joining a pair of vertices. Note that a k-multigraph is a 2-uniform hypergraph where an edge appears at most k times. Let $\mathcal{G}_k(\Pi) = \{G\colon G$ is a k-multigraph on p vertices $v_1, v_2, \ldots, v_p$ such that $\deg_G(v_i) \leq d_i$ for every i, $1 \leq i \leq p\}$. Let

$$M_k(\Pi) = \max\{|\varepsilon(G)| \colon G \in \mathcal{G}_k(\Pi)\}.$$

A sequence Π is said to be k-multigraphic if there is a k-multigraph whose degree sequence is Π. Choudum [4] established the following Erdös and Gallai type necessary conditions for 3-hypergraphic sequences.

Theorem 1.4: (Choudum [4]). *Let* $\Pi = (d_1, d_2, \ldots, d_p)$ *be a 3-hypergraphic sequence. Then*

$$\sum_{i=1}^{p} d_i \equiv 0 \pmod 3;$$

$$\sum_{i=1}^{k} d_i \leq k\binom{k-1}{2} + \sum_{j=k+1}^{p} 2\min\left(d_j, \binom{k}{2}\right) + M_k\left(\left(d_{k+1} - \binom{k}{2}, \ldots, d_p - \binom{k}{2}\right)^{+}\right), 1 \leq k \leq p. \tag{1.2}$$

∎

2. 3-Hypergraphic Sequences

Let $\Pi = (d_1, d_2, \ldots, d_p)$ be a non-increasing 3-hypergraphic sequence and $H = (V, \mathcal{E})$ be a 3-uniform hypergraph realising Π. In the following we introduce some notation related to H that will be assumed throughout the paper. We represent the vertex set V by $\{1, 2, \ldots, p\}$. Further, assume that $\deg_H(i) = d_i$, $1 \le i \le p$. Note that $|\mathcal{E}| = \frac{1}{3}\sum_{i=1}^{p} d_i$. Given an integer k such that $1 \le k \le p$, we partition the vertex set V into sets $S_k = \{1, 2, \ldots, k\}$ and $T_k = \{k+1, \ldots, p\}$. Further define subsets A_k, B_k and C_k of the edge set $\mathcal{E}$ as:

$$\begin{aligned} A_k &= \{E \in \mathcal{E}: |E \cap S_k| = 3\}, \\ B_k &= \{E \in \mathcal{E}: |E \cap S_k| = 2\}, \text{and} \\ C_k &= \{E \in \mathcal{E}: |E \cap S_k| = 1\}. \end{aligned} \tag{2.1}$$

Note that A_k, B_k and C_k are pairwise disjoint. Further, it is easy to see that every edge of A_k contributes exactly 3 to the sum $\sum_{i=1}^{k} d_i$. Similarly every edge of B_k (C_k) contributes exactly 2 (1) to the sum $\sum_{i=1}^{k} d_i$. Thus we have

$$\sum_{i=1}^{k} d_i = 3|A_k| + 2|B_k| + |C_k|, \; 1 \le k \le p. \tag{2.2}$$

Lemma 2.1. *Let $\Pi = (d_1, \ldots, d_p)$ be a non-increasing 3-hypergraphic sequence and $H = (V, \mathcal{E})$ be a 3-uniform hypergraph realising Π. Then for every k, $1 \le k \le p$, we have*

(i) $|A_k| \le \binom{k}{3} - \left(\binom{k-1}{2} - d_k\right)^+$,

(ii) $|B_k| \le \sum_{j=k+1}^{p} \min\left(d_j, \binom{k}{2}\right)$,

(iii) $|C_k| \le \sum_{j=k+2}^{p} \min(d_j, (j-k-1)k)$ *and*

(iv) $|B_k| + |C_k| \le \min\left(|\mathcal{E}|, \sum_{j=k+1}^{p} \min\left(d_j, (j-k-1)k + \binom{k}{2}\right)\right)$.

Proof: We notice that $|A_k|$ is at most $\binom{k}{3}$ and every vertex of S_k is contained in at most $\binom{k-1}{2}$ edges of A_k. Thus if $d_k < \binom{k-1}{2}$ then the above upper bound for $|A_k|$ can be further reduced by $\binom{k-1}{2} - d_k$. Thus

$$|A_k| \le \binom{k}{3} - \max\left(0, \binom{k-1}{2} - d_k\right).$$

This proves (i).

Consider j such that $k+1 \le j \le p$. Let $B'_j = \{E: j \in E \in B_k\}$. Then $\{B'_j: k+1 \le j \le p\}$ provides a partition of B_k. Thus $|B_k| = \sum_{j=k+1}^{p} |B'_j|$. Further it is easy to see that

$$|B'_j| \le \min\left(d_j, \binom{k}{2}\right). \tag{2.3}$$

Therefore $|B_k| \le \sum_{j=k+1}^{p} \min\left(d_j, \binom{k}{2}\right)$ and this establishes (ii).

Consider integers i and j such that $k+1 \le i < j \le p$. Let

$$C'_{ij} = \{E: i, j \in E \in C_k\}.$$

Clearly $\{C'_{ij}: k+1 \le i < j \le p\}$ is a partition of C_k. Thus

$$|C_k| = \sum_{j=k+2}^{p} \sum_{i=k+1}^{j-1} |C'_{ij}|. \tag{2.4}$$

For a fixed j, $k+2 \le j \le p$, an edge E of $\bigcup_{i=k+1}^{j-1} C'_{ij}$ consists of the vertices j, i and ℓ where $k+1 \le i \le j-1$ and $1 \le \ell \le k$. Therefore

$$|\bigcup_{i=k+1}^{j-1} C'_{ij}| \le (j-1-k)k$$

for a fixed j, $k+2 \le j \le p$. Also $|\bigcup_{i=k+1}^{j-1} C'_{ij}| \le d_j$, since $\deg_H(j) = d_j$.

Hence

$$|\bigcup_{i=k+1}^{j-1} C'_{ij}| \le \min(d_j, (j-k-1)k). \tag{2.5}$$

Combining (2.4) and (2.5) we have

$$|C_k| \le \sum_{j=k+2}^{p} \min(d_j, (j-k-1)k)$$

and this proves (iii).

From the definitions of B'_j and C'_{ij} we have

$$\begin{aligned} B_k \cup C_k &= \left(\bigcup_{j=k+1}^{p} B'_j\right) \cup \left(\bigcup_{j=k+2}^{p} \bigcup_{i=k+1}^{j-1} C'_{ij}\right) \\ &= (B'_{k+1}) \cup \left(\bigcup_{j=k+2}^{p} B'_j\right) \cup \left(\bigcup_{j=k+2}^{p} \bigcup_{i=k+1}^{j-1} C'_{ij}\right) \\ &= (B'_{k+1}) \cup \left(\bigcup_{j=k+2}^{p} \left(B'_j \cup \left(\bigcup_{i=k+1}^{j-1} C'_{ij}\right)\right)\right). \end{aligned}$$

Note that $B'_j \cap C'_{ij} = \phi$.

Using (2.3) and (2.5) we have

$$|B'_j \cup \left(\bigcup_{i=k+1}^{j-1} C'_{ij} \right)| \leq (j-1-k)k + \binom{k}{2}.$$

Also

$$|B'_j \cup \left(\bigcup_{i=k+1}^{j-1} C'_{ij} \right)| \leq d_j \text{ since } \deg_H(j) = d_j.$$

Therefore, for $k+2 \leq j \leq p$, we have,

$$|B'_j| + \sum_{i=k+1}^{j-1} |C'_{ij}| \leq \min\left(d_j, (j-k-1)k + \binom{k}{2}\right).$$

We recall that $|B'_{k+1}| \leq \min\left(d_{k+1}, \binom{k}{2}\right)$. Now

$$\begin{aligned} |B_k| + |C_k| &= |B'_{k+1}| + \sum_{j=k+2}^{p} \left(|B'_j| + \sum_{i=k+1}^{j-1} |C'_{ij}| \right) \\ &\leq \sum_{j=k+1}^{p} \min\left(d_j, (j-k-1)k + \binom{k}{2}\right). \end{aligned}$$

It is obvious that $|B_k| + |C_k| \leq |\mathcal{E}|$. Thus we have established (iv) and this completes the proof of Lemma 2.1. ∎

Using Lemma 2.1 in (2.2) we have the following:

Theorem 2.2. *Let* $\Pi = (d_1, d_2, \ldots, d_p)$ *be a non-increasing 3-hypergraphic sequence. Then*

$$\sum_{i=1}^{p} d_i \equiv 0 \pmod 3;$$

$$\sum_{i=1}^{k} d_i \leq 3\binom{k}{3} - 3\left(\binom{k-1}{2} - d_k\right)^{+} + \sum_{j=k+1}^{p} \min\left(d_j, \binom{k}{2}\right) \tag{2.6}$$

$$+ \sum_{j=k+1}^{p} \min\left(d_j, (j-k-1)k + \binom{k}{2}\right), \quad 1 \leq k \leq p.$$

∎

Let U_1, and U_2 be the upperbounds for $\sum_{i=1}^{k} d_i$ from Theorems 1.3 and 2.2 respectively. We have the following remark.

Remark 2.3. *$U_2 \leq U_1$. Further $U_2 < U_1$ if there exists an integer j, $k+1 \leq j \leq p$ such that $d_j < (j-k-1)k + \binom{k}{2}$.*

Proof: Let

$$\alpha = 3\binom{k}{3} - 3\left(\binom{k-1}{2} - d_k\right)^+ + \sum_{j=k+1}^{p} \min\left(d_j, \binom{k}{2}\right).$$

Then,

$$U_1 = \alpha + \sum_{j=k+1}^{p} \min\left(d_j, \binom{k}{2}\right) + \sum_{j=k+1}^{p} \min(d_j, (j-k-1)k),$$

$$U_2 = \alpha + \sum_{j=k+1}^{p} \min\left(d_j, (j-k-1)k) + \binom{k}{2}\right).$$

Note that when $k = 1$, $U_1 = U_2$. Therefore let $k > 1$. Now observe the following inequalities for $k+1 < j \leq p$. If $d_j < (j-k-1)k + \binom{k}{2}$, then

$$\min\left(d_j, (j-k-1)k + \binom{k}{2}\right) < \min\left(d_j, \binom{k}{2}\right) + \min(d_j, (j-k-1)k). \tag{2.7}$$

If $d_j \geq (j-k-1)k + \binom{k}{2}$, then

$$\min\left(d_j, (j-k-1)k + \binom{k}{2}\right) = \min\left(d_j, \binom{k}{2}\right) + \min(d_j, (j-k-1)k). \tag{2.8}$$

Next note that when $j = k+1$, (2.8) is obviously true. Now using (2.7) and (2.8) it is easy to see that $U_2 \leq U_1$. Further we have $U_2 < U_1$ if there exists an integer j such that $d_j < (j-k-1)k + \binom{k}{2}$. This completes the proof. ∎

Remark 2.4. *The conditions of Theorems 1.3, 1.4 or 2.2 are not sufficient for $\Pi = (d_1, \ldots, d_p)$ to be a 3-hypergraphic sequence.*

Proof: Consider the sequence $\Pi = (7,5,5,3,3,1)$. It is easy to verify that Π satisfies the conditions (1.2) and (2.6) of Theorems 1.4 and 2.2 respectively. Using Remark 2.3, note that Π satisfies conditions (1.1) of Theorem 1.3. Now we show that Π is not a 3-hypergraphic sequence. Suppose that Π is a 3-hypergraphic sequence. Then by Theorem 1.1, there exists a non-increasing sequence $\Pi' = (d'_2, d'_3, d'_4, d'_5, d'_6)$ of non-negative integers such that

(i) Π' is a 2-hypergraphic sequence,
(ii) $\sum_{i=2}^{6} d'_i = 2d_1 = 14$ and
(iii) $\Pi'' = (d_2 - d'_2, d_3 - d'_3, d_4 - d'_4, d_5 - d'_5, d_6 - d'_6)$ is a 3-hypergraphic sequence.

The various non-trivial (2-hypergraphic) candidates for Π' satisfying condition (ii) are:

(a) $\Pi' = (4,3,3,3,1)$
(b) $\Pi' = (4,3,3,2,2)$
(c) $\Pi' = (3,3,3,3,2)$ and
(d) $\Pi' = (4,4,2,2,2)$.

Now, $\Pi' = (4,3,3,3,1)$, results in $\Pi'' = (1,2,0,0,0)$ which is clearly not 3-hypergraphic, contradicting (iii). Similarly $\Pi' = (4,3,3,2,2)$ or $(3,3,3,3,2)$ or $(4,4,2,2,2)$ results in a contradiction to (iii). Thus $\Pi = (7,5,5,3,3,1)$ is not a 3-hypergraphic sequence. This completes the proof of Remark 2.4. ∎

Billington [2] suggested that his Erdős - Gallai type conditions are not likely to be sufficient for 3-hypergraphic sequences since they take no account of the interactions between the edges from the sets A_k, B_k and C_k. In the following lemma we incorporate some quantification of these interactions between A_k, B_k and C_k in the form of certain lower bounds on their sizes.

Lemma 2.5. *Let $\Pi = (d_1, \ldots, d_p)$ be a non-increasing 3-hypergraphic sequence and $H = (V, \mathcal{E})$ be a 3-uniform hypergraph realising Π. Then for every k, $1 \le k \le p$, we have*

(i) $|B_k| \ge \sum_{j=k+1}^{p} \left(d_j - \binom{p-k-1}{2} - \sum_{i=1}^{k} \min(d_i, (p-k-1))\right)^+$

(ii) $|C_k| \ge \sum_{i=1}^{k} \left(d_i - \binom{k-1}{2} - \sum_{j=k+1}^{p} \min(d_j, (k-1))\right)^+$ *and*

(iii) $|B_k| + |C_k| \ge \sum_{j=k+1}^{p} \left(d_j - \binom{p-k-1}{2} - \sum_{i=j+1}^{p} \min(d_i, k)\right)^+$.

Proof: Consider an integer j such that $k+1 \le j \le p$. We partition the edges containing the vertex j as

$$\overline{D}_j = \{E : |E \cap T_k| = 3 \text{ and } j \in E\};$$
$$\overline{C}_{ij} = \{E : i, j \in E \in C_k\},\ 1 \le i \le k; \textit{and}$$
$$B'_j = \{E : j \in E \in B_k\}.$$

Clearly

$$d_j = |\overline{D}_j| + |B'_j| + \sum_{i=1}^{k} |\overline{C}_{ij}|.$$

Now using the facts that

$$|\overline{D}_j| \le \binom{p-k-1}{2} \text{ and } |\overline{C}_{ij}| \le \min(d_i, (p-k-1))$$

we get

$$d_j \le |B'_j| + \binom{p-k-1}{2} + \sum_{i=1}^{k} \min(d_i, (p-k-1)).$$

Combining this with the fact that $|B'_j| \ge 0$ we have

$$|B'_j| \ge \left(d_j - \binom{p-k-1}{2} - \sum_{i=1}^{k} \min(d_i, (p-k-1)) \right)^+.$$

Using this inequality and the fact that $|B_k| = \sum_{j=k+1}^{p} |B'_j|$, we have,

$$|B_k| \ge \sum_{j=k+1}^{p} \left(d_j - \binom{p-k-1}{2} - \sum_{i=1}^{k} \min(d_i, (p-k-1)) \right)^+.$$

This establishes (i). Now (ii) can be proved using similar arguments.

To prove (iii), once again consider an integer j such that $k+1 \le j \le p$. Now partition the edges containing the vertex j as follows:

$$\begin{aligned} \overline{D}_j &= \{E: |E \cap T_k| = 3 \text{ and } j \in E\}; \\ C'_{ij} &= \{E: i, j \in E \in C_k\},\ k+1 \le i < j; \\ C'_{ji} &= \{E: i, j \in E \in C_k\},\ j < i \le p; \text{ and} \\ B'_j &= \{E: j \in E \in B_k\}. \end{aligned}$$

Clearly

$$d_j = |\overline{D}_j| + \sum_{i=k+1}^{j-1} |C'_{ij}| + \sum_{i=j+1}^{p} |C'_{ji}| + |B'_j|.$$

Using the facts that $|\overline{D}_j| \le \binom{p-k-1}{2}$ and $|C'_{ji}| \le \min(k, d_i)$ for $k+1 \le j < i \le p$ we get

$$|B'_j| + \sum_{i=k+1}^{j-1} |C'_{ij}| \ge \left(d_j - \binom{p-k-1}{2} - \sum_{i=j+1}^{p} \min(d_i, k) \right)^+.$$

Now

$$\begin{aligned} |B_k| + |C_k| &= \sum_{j=k+1}^{p} \left(|B'_j| + \sum_{i=k+1}^{j-1} |C'_{ij}| \right) \\ &\ge \sum_{j=k+1}^{p} \left(d_j - \binom{p-k-1}{2} - \sum_{i=j+1}^{p} \min(d_i, k) \right)^+. \end{aligned}$$

This establishes (iii) and completes the proof of Lemma 2.5. ∎

Using Choudum's [4] proof of Theorem 1.4 one can easily establish the following for a 3-hypergraphic sequence $\Pi = (d_1, \ldots, d_p)$:

$$|B_k| + |C_k| \le \sum_{j=k+1}^{p} \min\left(d_j, \binom{k}{2}\right) + M_k\left(\left(d_{k+1} - \binom{k}{2}, \ldots, d_p - \binom{k}{2}\right)^+\right), \; 1 \le k \le p.$$

Now combining this and (iv) of Lemma 2.1 we have the following remark.

Remark 2.6. *Let $\Pi = (d_1, \ldots, d_p)$ be a non-increasing 3-hypergraphic sequence and $H = (V, \mathcal{E})$ be a 3-uniform hypergraph realising Π. Then for every k, $1 \le k \le p$, we have*

$$|B_k| + |C_k| \le \min(|\mathcal{E}|, \alpha, \beta)$$

where

$$\alpha = \sum_{j=k+1}^{p} \min\left(d_j, (j-k-1)k + \binom{k}{2}\right)$$

and

$$\beta = \sum_{j=k+1}^{p} \min\left(d_j, \binom{k}{2}\right) + M_k\left(\left(d_{k+1} - \binom{k}{2}, d_{k+2} - \binom{k}{2}, \ldots, d_p - \binom{k}{2}\right)^+\right).$$

∎

Next we prove our main theorem.

Theorem 2.7. *Let $\Pi = (d_1, d_2, \ldots, d_p)$ be a non-increasing 3-hypergraphic sequence. For a fixed k, $1 \le k \le p$, let f_k be the maximum value of $3x + 2y + z$ subject to the constraints:*

(i) $x \le \binom{k}{3} - \left(\binom{k-1}{2} - d_k\right)^+$,

(ii) $\sum_{j=k+1}^{p} \left(d_j - \binom{p-k-1}{2} - \sum_{i=1}^{k} \min(d_i, (p-k-1))\right)^+ \le y$
$\le \sum_{j=k+1}^{p} \min\left(d_j, \binom{k}{2}\right)$,

(iii) $\sum_{i=1}^{k} \left(d_i - \binom{k-1}{2} - \sum_{j=k+1}^{p} \min(d_j, (k-1))\right)^+ \le z$
$\le \sum_{i=k+2}^{p} \min(d_i, (i-k-1)k)$,

(iv) $\sum_{j=k+1}^{p}\left(d_j - \binom{p-k-1}{2} - \sum_{i=j+1}^{p}\min(d_i, k)\right)^+ \le y + z$
$$\le \min\left\{\begin{array}{c}\frac{1}{3}\sum_{i=1}^{p} d_i, \sum_{j=k+1}^{p}\min\left(d_j, (j-k-1)k + \binom{k}{2}\right),\\ \sum_{j=k+1}^{p}\min\left(d_j, \binom{k}{2}\right) + M_k\left(\left(d_{k+1} - \binom{k}{2}, \ldots, d_p - \binom{k}{2}\right)^+\right)\end{array}\right\},$$
(v) $x + y + z \le \frac{1}{3}\sum_{i=1}^{p} d_i$, *and*
(vi) x, y *and* z *are non-negative integers.*

Then

$$\sum_{i=1}^{p} d_i \equiv 0 \pmod 3;$$

and

$$\sum_{i=1}^{k} d_i \le f_k,\ 1 \le k \le p.$$

Proof: Let $H = (V, \mathcal{E})$ be a 3-uniform hypergraph realising $\Pi = (d_1, d_2, \ldots, d_p)$. Then $\sum_{i=1}^{p} d_i \equiv 0 \pmod 3$ since the number of edges in H is $\frac{1}{3}\sum_{i=1}^{p} d_i$. Define $x = |A_k|$, $y = |B_k|$ and $z = |C_k|$ for a fixed k, $1 \le k \le p$. From Lemmas 2.1, 2.5 and Remark 2.6 it follows that (x, y, z) satisfy the constraints (i) to (iv). Since $|A_k| + |B_k| + |C_k|$ is at most, the size of $\mathcal{E}$, we have the constraint (v). Now from (2.2) and the definition of f_k we have

$$\sum_{i=1}^{k} d_i = 3|A_k| + 2|B_k| + |C_k| \le f_k,\ 1 \le k \le p.$$

This establishes Theorem 2.7. ∎

Remark 2.8. *The sequence $\Pi = (7, 5, 5, 3, 3, 1)$ which formed a counter example for the conditions of Theorems 1.3, 1.4 and 2.2 to be sufficient, does not satisfy the second condition of Theorem 2.7.*

Let $k = 3$. Then $\sum_{i=1}^{3} d_i = 17$. We determine f_3 as follows:

$$f_3 = \max\{3x + 2y + z\}$$

where

(i) $x \le 1$
(ii) $0 \le y \le 7$
(iii) $1 \le z \le 4$
(iv) $1 \le y + z \le 7$
(v) $x + y + z \le 8$.

It is easy to check that the optimal solution is given by $x^* = 1$, $y^* = 6$ and $z^* = 1$ yielding $f_3 = 16$. Thus the condition $\sum_{i=1}^{3} d_i \leq f_3$ is violated. This proves Remark 2.8. ∎

We conclude this paper by emphasising that we are unable to prove or disprove the sufficiency of the conditions of Theorem 2.7 for a sequence to be 3-hypergraphic.

References

1. C. Berge, "Graphs and Hypergraphs", North-Holland, Amsterdam, 1973.
2. D. Billington, *Conditions for degree sequences to be realisable by 3-uniform hypergraphs*, J. Combinatorial Mathematics and Combinatorial Computing 3 (1988), 71–91.
3. J.A. Bondy and U.S.R. Murty, "Graph Theory with Applications", The MacMillan Press, London, 1977.
4. S.A. Choudum, *On graphic and 3-hypergraphic sequences*, Discrete Mathematics **87** (1991), 91–95.
5. C.J. Colbourn, W.L. Kocay and D.R. Stinson, *Some NP-complete problems for hypergraph degree sequences*, Discrete Appl. Math. 14 (1986), 239–254.
6. A. Dewdney, *Degree sequences in complexes and hypergraphs*, Proc. Amer. Math. Soc. **53** (1975), 535–540.
7. P. Erdős and T. Gallai, *Graphs with prescribed degrees of vertices (Hungarian)*, Mat. Lapok **11** (1960), 264–274.
8. M. Simanihuruk, *Graphs and their degree sequences*, Masters Thesis, Curtin University of Technology, Perth, Australia (1990).