A Generalized Steiner Distance for Graphs[1]

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Abstract. For a nonempty subset S of vertices of a k-connected graph G and for $1 \leq i \leq k$, the Steiner i-distance $d_i(S)$ of S is the minimum size among all i-connected subgraphs containing S. Relationships between Steiner i-distance and the connectivity and hamiltonian properties of a graph are discussed. For a k-connected graph G of order p and integers i and n with $1 \leq i \leq k$ and $1 \leq n \leq p$, the (i, n)-eccentricity of a vertex v of G is the maximum Steiner i-distance $d_i(S)$ of a set S containing v with $|S| = n$. The (i, n)-center $C_{i,n}(G)$ of G is the subgraph induced by those vertices with minimum (i, n)-eccentricity. It is proved that for every graph H and integers $i, n \geq 2$, there exists an i-connected graph G such that $C_{i,n}(G) \cong H$.

1. Introduction

The *distance* between two vertices u and v in a connected graph G is the length of a shortest path in G connecting u and v. Equivalently, it is the smallest size (number of edges) in a connected subgraph containing u and v. From this point of view, the standard distance between two vertices was extended in [2] to the Steiner distance of a *set* of vertices, namely, if S is a set of vertices in G, then the *Steiner distance* $d(S)$ of S is the smallest size of a connected subgraph containing the vertices of S. If the graph G is k-connected for some integer $k \geq 1$, then for each integer i with $1 \leq i \leq k$, there is always an i-connected subgraph of G containing the vertices of S. This observation suggests a generalization of the Steiner distance on graphs, which we present in this paper.

Let G be a k-connected graph where $k \geq 1$, and let S be a nonempty set of vertices of G. For $1 \leq i \leq k$, we define the *Steiner i-distance* $d_i(S)$ of S as the minimum size among all i-connected subgraphs containing S. Therefore, the Steiner 1-distance $d_1(S)$ of S is simply the Steiner distance $d(S)$ of S. A subgraph H of G is called a *Steiner i-subgraph* of S if H is i-connected, $S \subseteq V(H)$, and $d_i(S) = |E(H)|$. For the graph G of Figure 1 and $S = \{u, v, w\}$, we have $d_1(S) = 3$, $d_2(S) = 5$, $d_3(S) = 10$, and $d_4(S) = 17$. Steiner i-subgraphs for $i = 1, 2, 3$, and 4 are also shown in Figure 1.

The Steiner distance satisfies an extended triangle inequality, which we now describe. Let G be a connected graph, and let S, S_1, and S_2 be subsets of $V(G)$ such that $\emptyset \neq S \subseteq S_1 \cup S_2$ and $|S_1 \cap S_2| \geq 1$. Then $d(S) \leq d(S_1) + d(S_2)$. There

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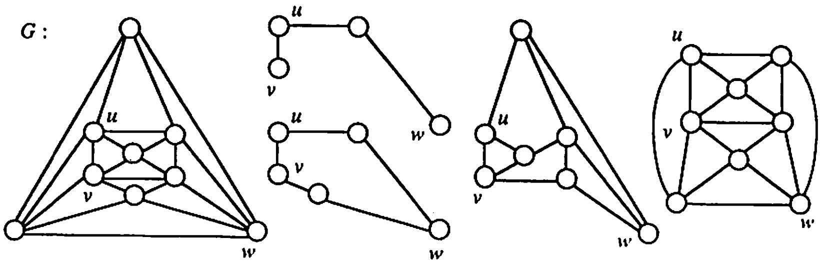

Figure 1

is an extension of this property to the Steiner i-distance. Let G be a k-connected graph and let S, S_1, and S_2 be subsets of $V(G)$ such that $\emptyset \neq S \subseteq S_1 \cup S_2$ and $|S_1 \cap S_2| \geq i$, where $1 \leq i \leq k$. Then $d_i(S) \leq d_i(S_1) + d_i(S_2)$. To see this, let H_i $(i = 1, 2)$ be a Steiner i-subgraph of $d_i(S_i)$ such that $S_i \subseteq V(H_i)$. Let H be the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. Since H_1 and H_2 are i-connected and $|V(H_1) \cap V(H_2)| \geq |S_1 \cap S_2| \geq i$, the graph H is i-connected. Since $S \subseteq V(H)$, it follows that $d_i(S) \leq |E(H)| \leq d_i(S_1) + d_i(S_2)$.

2. (i, n, p)-Graphs

Let G be a k-connected graph. For $1 \leq i \leq k$, every Steiner i-subgraph H of a subset S of $V(G)$ is necessarily minimally i-connected, that is, the removal of any edge of H results in a graph that is not i-connected. Since H is i-conencted, $\delta(H) \geq i$ so that $d_i(S) = |E(H)| \geq \lceil i|S|/2 \rceil$. An i-connected graph G of order p is called an *(i, n, p)-graph*, where $2 \leq i < n \leq p$, if $d_i(S) = \lceil i|S|/2 \rceil$ for *every* set S of n vertices of G. A graph G is *(i, n)-connected*, $1 \leq i < n$ if the induced subgraph $\langle S \rangle$ is i-connected for every set S of n vertices of G. Certainly, every (i, n, p)-graph is (i, n)-connected.

Lemma 1. *Let i, n, and p be integers with $1 \leq i < n \leq p$. A graph G of order p is (i, n)-connected if and only if G is $(p - n + i)$-connected.*

Proof: Suppose, to the contrary, that there exists an (i, n)-connected graph that is not $(p-n+i)$-connected. Then there exists a cut set X with $|X| = p-n+i-1$ such that $G - X$ is disconnected. Let $S = V(G) - X$ and let $X' = \{x_1, x_2, \ldots, x_{i-1}\}$ be an arbitrary subset of $i - 1$ vertices in X. Then X' is a cut set of $\langle S \cup X' \rangle$, so that $\langle S \cup X' \rangle$ is not i-connected. Since $|S \cup X'| = n$, it follows that G is not (i, n)-connected, a contradiction.

We now prove the sufficiency. Let G be a $(p - n + i)$-connected graph and let S be a subset of $V(G)$ with $|S| = n$. Suppose that $\langle S \rangle$ is not i-connected. Then there exists a subset X of S with $|X| = i - 1$ such that $\langle S \rangle - X$ is disconnected. Therefore, $G-(X \cup (V(G) - S))$ is disconnected. However, since $|X \cup (V(G) - S)| = p - n + i - 1$, it follows that G is not $(p - n + i)$-connected, which is a contradiction. ∎

Theorem 2. *Every (i, n, p)-graph, $2 \leq i < n \leq p$, is $(p - n + i)$-connected.*

Proof: Since every (i, n, p)-graph is (i, n)-connected, the theorem follows immediately from Lemma 1. ∎

If $n = i + 1$ in Theorem 2, then it follows that G is $(p - 1)$-connected, which yields the following result.

Corollary 3. *Let i and p be integers with $2 \leq i < p$. Then every $(i, i + 1, p)$-graph is complete.*

It is important to note that the converse of Theorem 2 is not true in general. For example, consider the complete bipartite graph $K_{3,3}$ and let $i = 2$ and $n = 5$. Thus, $p - n + i = 3$, and $K_{3,3}$ is $(p - n + i)$-connected. The graph $K_{3,3}$ is not a $(2, 5, 6)$-graph, however, since every 2-connected subgraph of order 5 in $K_{3,3}$ has size 6, not 5 as is required. In particular, a graph G is a $(2, n, p)$-graph if and only if G has order p and $\langle S \rangle$ is hamiltonian for every set S of n vertices of G. Therefore, G is a $(2, p, p)$-graph if and only if G is hamiltonian. From this point of view, the (i, n, p)-graphs are generalized hamiltonian graphs. Consequently, the problem of determining whether a graph is a $(2, n, p)$-graph is NP-complete.

Chartrand, Kapoor, and Lick [1] introduced the concept of n-hamiltonian graphs. A graph G is n-hamiltonian if the removal of any set of n vertices from G results in a hamiltonian subgraph. Therefore, a graph G is a $(2, n, p)$-graph if and only if G is $(p - n)$-hamiltonian. Wong and Wong [8] studied the minimum size of n-hamiltonian graphs (or $(2, p - n, p)$-graphs). The extremal graphs constructed by Wong and Wong are hamiltonian. We ask the following question: Does there exist a nonhamiltonian $(2, n, p)$-graph for each n with $3 \leq n < p$? The *circumference* $c(G)$ of a graph G is the length of a longest cycle in G. With the aid of the following lemma, we can give upper and lower bounds for $c(G)$ for a $(2, n, p)$-graph G.

Lemma 4. *Let G be a $(2, n, p)$-graph with $3 \leq n \leq p$. If S is a subset of $V(G)$ with $|S| \geq n$, then $\langle S \rangle$ is a $(2, n, |S|)$-graph.*

We now establish an upper bound for the circumference of nonhamiltonian $(2, n, p)$-graphs.

Theorem 5. *Let G be a $(2, n, p)$-graph with $3 \leq n < p$. If G is not hamiltonian, then $c(G) \leq 2n - 6$.*

Proof: Let C be a longest cycle in G and let $H = \langle V(C) \cup \{v\} \rangle$, where v is a vertex not on C. Since G is a $(2, n, p)$-graph, it follows that $c(G) \geq n$. By Lemma 4, H is a $(2, n, c(G) + 1)$-graph. It follows from Theorem 2 that H is $(c(G) + 1 - n + 2)$-connected. Therefore,

$$\delta(H) \geq c(G) + 1 - n + 2 = c(G) - n + 3.$$

If $\delta(H) \geq (c(G)+1)/2$, then it would follow from a well known theorem of Dirac [3] that H is hamiltonian. Consequently, $\delta(H) \leq c(G)/2$ so that $c(G) - n + 3 \leq c(G)/2$. implying that $c(G) \leq 2n-6$. ▮

A lower bound for the circumference of a nonhamiltonian $(2,n,p)$-graph is given in our next result.

Theorem 6. *If G is a $(2,n,p)$-graph with $3 \leq n \leq p$, then $c(G) \geq p - \lfloor n/2 \rfloor + 1$.*

Proof: If G is hamiltonian, then $c(G) = p > p - \lfloor n/2 \rfloor + 1$. Otherwise, let C be a longest cycle in G and let $X = V(G) - V(C)$. Then $X \neq \emptyset$. We prove that $|X| \leq \lfloor n/2 \rfloor - 1$. Suppose, to the contrary, that $|X| \geq \lfloor n/2 \rfloor$. Define $m = n-2$ if $|X| \geq n-2$, and $m = |X|$ otherwise. Let $V_1 \subseteq X$ be a subset of cardinality m. Let $P : u = v_1, v_2, \ldots, v_{n-m} = w$ be a subpath of C. Since G is a $(2,n,p)$-graph and $|V_1 \cup V(P)| = n$, the graph $\langle V_1 \cup V(P) \rangle$ is hamiltonian. Suppose C_1 is a hamiltonian cycle in $\langle V_1 \cup V(P) \rangle$. Since C_1 contains the vertices u and w, the cycle C_1 produces two edge-disjoint $u-w$ paths P_1 and P_2. Clearly, at least one of them, say P_1, has length at least $\lceil n/2 \rceil$. Therefore, by the choice of m,

$$\begin{aligned} |V(P_1) \cup (V(C) - V(P))| &= \lceil n/2 \rceil + 1 + c(G) - n + m \\ &= c(G) + m - \lfloor n/2 \rfloor + 1 \\ &\geq c(G) + 1. \end{aligned}$$

Since $V(P_1) \cap (V(C) - V(P)) = \emptyset$, the induced graph $\langle V(P_1) \cup (V(C) - V(P)) \rangle$ is hamiltonian. Therefore, the graph G contains a cycle of length exceeding $c(G)$, a contradiction. ▮

For integers $p \geq 3$, we define the parameter $f(p)$ to be the minimum n with $3 \leq n < p$ for which there exists a nonhamiltonian $(2,n,p)$-graph. The parameter $f(p) = \infty$ if no nonhamiltonian $(2,n,p)$-graph exists for all n with $3 \leq n < p$. A lower bound for the parameter $f(p)$ is given in the following corollary.

Corollary 7. *For all integers $p \geq 3$, $f(p) \geq \lceil \frac{2p+14}{5} \rceil$.*

Proof: Suppose $f(p)$ is finite. Let G be a nonhamiltonian $(2,n,p)$-graph with $3 \leq n < p$. Combining Theorems 5 and 6, we have

$$p - \lfloor n/2 \rfloor + 1 \leq c(G) \leq 2n - 6,$$

that is, $n \geq \lceil (2p-1)/5 \rceil + 3$. Therefore, $f(p) \geq \lceil (2p+14)/5 \rceil$ for all $p \geq 3$. ▮

A graph G is *hypohamiltonian* if G is not hamiltonian and $G - v$ is hamiltonian for all vertices v of G. Therefore, a nonhamiltonian $(2, p-1, p)$-graph is then a hypohamiltonian graph. Much study has been done on the existence of hypohamiltonian graphs. Thomassen [7] showed that there exists a hypohamiltonian graph of order p for all $p \geq 13$ except for $p = 14, 17, 19$. Thus, $f(p) \leq p-1$ for $p \geq 13$ and $p \neq 14, 17, 19$.

3. (i,n)-Eccentricity and (i,n)-Centers

Let G be a k-connected graph of order p, and let i and n be integers with $1 \leq i \leq k$ and $1 \leq n \leq p$. The *(i,n)-eccentricity* $e_{i,n}(v)$ of a vertex v of G is defined by

$$e_{i,n}(v) = \max\{d_i(S) \mid v \in S \subseteq V(G) \text{ and } |S| = n\}.$$

Observe that, for $v \in V(G)$,

(1) $e_{1,1}(v) = 0$;
(2) $e_{1,2}(v) = e(v)$, the standard eccentricity;
(3) $e_{1,n}(v) = e_n(v)$, the Steiner n-eccentricity (see [6]).

We call a nondecreasing sequence $S : a_1, a_2, \ldots, a_p$ of nonnegative integers an *(i,n)-eccentricity sequence* if there exists an i-connected graph G whose vertices can be labeled as $v_1, v_2, \ldots, v_p$ so that $e_{i,n}(v_j) = a_j$ for $1 \leq j \leq p$. The $(1,2)$-eccentricity of a connected graph is the standard eccentricity sequence. Lesniak [4] showed that a nondecreasing sequence $S : a_1, a_2, \ldots, a_p$ with m distinct values is the eccentricity sequence of a connected graph of order p if and only if some subsequence with m distinct values is eccentric. A $(2,2)$-eccentricity sequence may be characterized in an analogous fashion.

Theorem 8. *A nondecreasing sequence $S : a_1, a_2, \ldots, a_p$ with m distinct values is the $(2,2)$-eccentricity sequence of a graph if and only if some subsequence of S with m distinct values is the $(2,2)$-eccentricity sequence of some graph.*

Proof: If S is a sequence with m distinct values that is the $(2,2)$-eccentricity sequence of some graph, then S is a subsequence of itself, that is, S is the $(2,2)$-eccentricity sequence of a graph.

For the converse, suppose that S' is a subsequence of S that has the same m distinct values as S and suppose that S' is the $(2,2)$-eccentricity sequence of some graph G. Let $t_1, t_2, \ldots, t_m$ be the distinct values of S'. For each t_i, $1 \leq i \leq m$, select a vertex v_i of G whose $(2,2)$-eccentricity in G is t_i. Let n_i $(1 \leq i \leq m)$ be one more than the number of occurrences of t_i in S less the number of occurrences of t_i in S'. In G replace v_1 with a copy of K_{n_1} and join each vertex of K_{n_1} to all the vertices adjacent to v_1 in G. Denote this graph by G_1. In G_1, replace v_2 with a copy of K_{n_2} and join each vertex of K_{n_2} to all the vertices adjacent to v_2 in G_1. We continue in this fashion to obtain the graph G_m. Then S is the $(2,2)$-eccentricity sequence of G_m. ∎

The *(i,n)-center* $C_{i,n}(G)$ of an i-connected graph G is the subgraph induced by those vertices with minimum (i,n)-eccentricity. For $i = 1$, this is the Steiner n-center of H (see [5]). We now prove that for integers i and n with $i, n > 1$ and for a given graph H, there exists an i-connected graph G such that the (i,n)-center of G is isomorphic to H.

Theorem 9. *Let H be a graph and let $i, n \geq 2$ be integers. Then there exists an i-connected graph G such that $C_{i,n}(G) \cong H$.*

Proof: Let $k = \max\{\lceil i/2 \rceil, i - |V(H)|\}$ and let m (≥ 3) be an integer such that $i\lfloor i/2 \rfloor m > 2|E(H)| + 1$. We first define a preliminary graph G_1 by

$$V(G_1) = \{v\} \cup \{v_{r,t} \mid 1 \leq r \leq n+1, 1 \leq t \leq m\}$$

and

$$E(G_1) = \{v_{r,t}v_{r,t+1} \mid 1 \leq r \leq n+1, 1 \leq t \leq m-1\} \cup \{v_{n+1,m}v_{1,m}\} \cup \{v_{r,m}v_{r+1,m} \mid 1 \leq r \leq n\} \cup \{v\,v_{r,1} \mid 1 \leq r \leq n+1\}.$$

Let G be the graph obtained from G_1 by first replacing the vertex v by $V(H)$ and replacing the vertex $v_{r,t}$, where $1 \leq r \leq n+1$ and $1 \leq t \leq m$, by the set $V_{r,t} = \{v_{r,t}^1, v_{r,t}^2, \ldots, v_{r,t}^k\}$ of vertices. To define the edge set of G, we let $\varphi : V(G) \to V(G_1)$ be a mapping defined by $\varphi(w) = v$ for $w \in V(H)$ and $\varphi(v_{r,t}^s) = v_{r,t}$, where $1 \leq r \leq n+1$, $1 \leq t \leq m$ and $1 \leq s \leq k$. We then define the edge set of G by

$$E(G) = E(H) \cup \{xy \mid x, y \in V(G) \text{ and } \varphi(x)\varphi(y) \in E(G_1)\}$$

(see Figure 2).

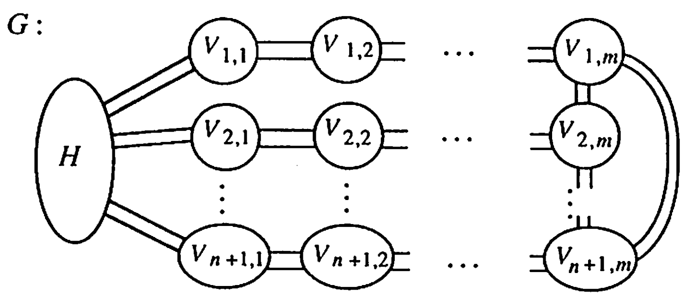

Figure 2

Now we show that all vertices of H have the same (i,n)-eccentricity in G. Let $v \in V(H)$. Suppose that $S \subseteq V(G) - \{v\}$ is a set of $n-1$ vertices and F is an i-connected subgraph of G with $\{v\} \cup S \subseteq V(F)$ such that $e_{i,n}(v) = d_i(\{v\} \cup S) = |E(F)|$. We claim that $S \cap V(H) = \emptyset$. Suppose, to the contrary, that $w \in S \cap V(H)$. Since $|S| = n-1$, there exists V_r such that $S \cap V_r = \emptyset$.

Let $S' = (S - \{w\}) \cup \{v_{r,2}^1\}$. Let F' be an i-connected subgraph of G with $\{v\} \cup S' \subseteq V(F')$ such that $d_i(\{v\} \cup S') = |E(F')|$. Since F' is i-connected, it contains at least $\lfloor i/2 \rfloor$ vertices of $V_{r,j}$ for each j with $1 \leq j \leq m$. Then,

$$\begin{aligned} d_i(\{v\} \cup S') &\geq d_i(\{v\} \cup S) + \lfloor \frac{i\lfloor i/2 \rfloor m}{2} \rfloor - |E(H)| \\ &> d_i(\{v\} \cup S) = e_{i,n}(v), \end{aligned}$$

a contradiction. Therefore, $S \subseteq \sum_{r=1}^{n+1} V_r$. Assume, without loss of generality, that $v_{r,1}^s \in V(F)$. Then $v_{r,1}^s$ is adjacent to at least $i - k$ vertices of $V(H)$ in F. Therefore, $|V(F) \cap V(H)| \geq i - k$. Since F is a minimal i-connected graph containing $\{v\} \cup S$, we have $|V(F) \cap V(H)| = i - k$. Further, the induced subgraph $\langle V(F) \cap V(H) \rangle$ of F is empty. Therefore, $e_{i,n}(v)$ in G is independent of the choice of v from H. Hence, $e_{i,n}(v) = e_{i,n}(w)$ for all $v, w \in V(H)$.

We now prove that $e_{i,n}(v) < e_{i,n}(w)$ for all $v \in V(H)$ and $w \in V(G) - V(H)$, from which it will follow that $C_{i,n}(H) \cong G$. Consider again a vertex $v \in V(H)$. Then, by the above, $e_{i,n}(v) = d_i(\{v\} \cup S)$, where S is a subset of $\{v'_{j,2} \mid 1 \leq j \leq n + 1\}$ of cardinality $n - 1$. Consider a vertex $v_{r,t}^s \in V(G) - V(H)$. Let S' be a subset of $\{v'_{j,2} \mid 1 \leq j \leq n + 1, j \neq r\}$ of cardinality $n - 1$. Then

$$e_{i,n}(v_{r,t}^s) \geq d_i(\{v_{r,t}^s\} \cup S').$$

Let F' be an i-connected subgraph containing $\{v_{r,t}^s\} \cup S'$ such that $d_i(\{v_{r,t}^s\} \cup S') = |E(F')|$. Then, clearly F' contains $i - k$ (> 1) vertices of H. Suppose that v is such a vertex. Therefore,

$$d_i(\{v_{r,t}^s\} \cup S') = d_i(\{v, v_{r,t}^s\} \cup S').$$

Clearly, $d_i(\{v, v_{r,t}^s\} \cup S') > d_i(\{v\} \cup S')$. Therefore, $e_{i,n}(v_{r,t}^s) > e_{i,n}(v)$. This completes the proof. ∎

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