Perfect Domination in Random Graphs

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Abstract. For a wide range of p, we show that almost every graph $G\in\mathcal{G}(n,p)$ has no perfect dominating set and for almost every graph $G\in\mathcal{G}(n,p)$ we bound the cardinality of a set of vertices which can be perfectly dominated. We also show that almost every tree $T\in\mathcal{T}(n)$ has no perfect dominating set.

1. Introduction.

For a graph G, the **neighborhood** $N_G(v)$ of a vertex v is the set of vertices adjacent to v and the **closed neighborhood** $N_G[v]$ of a vertex v is the set $N_G(v)\cup\{v\}$. A set S of vertices in a graph G is **independent** in G provided $N_G(v)\cap S=\emptyset$ for all $v\in S$ while S **dominates** G provided $N_G[v]\cap S\neq\emptyset$ for all $v\in V(G)$.

A set S of vertices in a graph G **perfectly dominates** G provided $|N_G[v]\cap S| = 1$ for all $v\in V(G)$. We note that S perfectly dominates G if and only if S is independent in G and $\{N_G(v) : v\in S\}$ partitions $V(G)-S$ (where we allow the empty set). The **perfect dominating number** $p(G)$ of G is the minimum cardinality of a perfect dominating set in G, should one exist.

We note that [5] contains a bibliography of the numerous domination parameters of a graph and [1], [2], [3] contain results regarding perfect domination of graphs (therein referred to as efficient domination of graphs).

The probability space $\mathcal{G}(n,p)$ consists of all graphs G with vertex set $[n]=\{1,\ldots,n\}$ in which the edges are chosen independently with probability $p=p(n)$ so that $P(G)=p^m q^{N-m}$ when G has the size m where $q=1-p$ and $N=\binom{n}{2}$.

The probability space $\mathcal{T}(n)$ consists of all trees with vertex set $[n]$ where each tree is chosen randomly according to a uniform distribution so that $P(T)=n^{-(n-2)}$ by Cayley's theorem.

A class of graphs which is closed under isomorphism is called a **property of graphs**. For a model $\Omega(n)$ of random graphs of order n, we say **almost every** graph in $\Omega(n)$ has a property Q provided $P(G\in\Omega(n)$ has $Q)\to 1$ as $n\to\infty$.

Although little is known about domination in random graphs, Weber [6] has essentially determined the domination number and the independent domination number of almost every graph $G\in\mathcal{G}(n,p)$ for constant p.

For a wide range of p, we show that almost every graph $G\in\mathcal{G}(n,p)$ has no perfect dominating set and for almost every graph $G\in\mathcal{G}(n,p)$ we bound the cardinality of a set of vertices which can be perfectly dominated. We also show that almost every tree $T\in\mathcal{T}(n)$ has no perfect dominating set.

We use the notation and terminology of [4]. In particular, we use the following standard inequalities.

$$1 - x \le e^{-x-x^2/2} \le e^{-x} \qquad \text{for } x \in [0,1], \qquad (1)$$
$$e^{-2x} \le 1 - x \qquad \text{for } x \in [0,.68] \text{ and} \qquad (2)$$
$$e^{x/2} \le 1 + x \le e^{x} \qquad \text{for } x \in [0,1]. \qquad (3)$$

For $1 \le t \le n$, let $(n)_t = n(n-1)\dots(n-t+1)$. Then (1) implies

$$(n)_t \le n^t e^{-\binom{t}{2}/n - \binom{t}{2}(2t-1)/6n^2} \le n^t e^{-\binom{t}{2}/n} \le n^t \qquad (4)$$

while (4) implies

$$\binom{n}{t} \le \left(\frac{en}{t}\right)^t e^{-\binom{t}{2}/n} \le \left(\frac{en}{t}\right)^t, \qquad (5)$$

since $t! \ge (t/e)^t$. Throughout the paper all logarithms are natural logarithms.

2. Results.

For a wide range of p, we first show that almost every $G \in \mathcal{G}(n,p)$ has no perfect dominating set. Our proof shows the probability that a graph has a perfect dominating set is exponentially small.

Theorem 1. For $0 < p = p(n) < 1$ with $\overline{\lim}_{n\to\infty} p < 1$ and $pn \to \infty$,

$$P(G \in \mathcal{G}(n,p) \text{ has a perfect dominating set}) = O(ne^{-\delta n})$$

where $4\delta = 1 - \overline{\lim}_{n\to\infty} p$.

Proof. For $S \subseteq [n]$ and $G \in \mathcal{G}(n,p)$, let

$$X_S(G) = \begin{cases} 1, & S \text{ is a perfect dominating set of } G, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$X = \sum_S X_S \quad (X \text{ counts the number of perfect dominating sets}).$$

For $S = \{i_1,\dots,i_s\}$ with $1 \le i_1 < \dots < i_s \le n$ and division (an ordered partition where we allow empty sets) $(A_1,\dots,A_s)$ of $[n] - S$,

$$P(G \in \mathcal{G}(n,p) \text{ has } N_G(i_1) = A_1,\dots,N_G(i_s) = A_s)$$
$$= p^{|A_1|+\dots+|A_s|} q^{n-s-|A_1|+\dots+n-s-|A_s|+\binom{s}{2}}$$

and

$$E(X_S) = \sum_{\substack{(A_1,\dots,A_s)\\ \text{division of } [n]-S}} p^{|A_1|+\cdots+|A_s|} q^{n-s-|A_1|+\cdots+n-s-|A_s|+\binom{s}{2}}$$

$$= \sum_{\substack{(A_1,\dots,A_s)\\ \text{division of } [n]-S}} p^{n-s} q^{(s-1)(n-s)+\binom{s}{2}}$$

$$= (ps)^{n-s} q^{(s-1)(n-s)+\binom{s}{2}}$$

so that (1) implies

$$E(X) = \sum_{s=1}^{n} \binom{n}{s} (ps)^{n-s} q^{(s-1)(n-s)+\binom{s}{2}}$$

$$\leq \sum_{s=1}^{n} \binom{n}{s} (ps)^{n-s} e^{-p[(s-1)(n-s)+\binom{s}{2}]}$$

$$\leq \sum_{s=1}^{n} \binom{n}{s} e^{(n-s)[\log ps - ps + p] - p\binom{s}{2}} .$$

Let $4\delta = 1 - \overline{\lim_{n\to\infty}}\, p$ and $c\log(e/c)/(1-2c) = \delta$ so that $c \in (0, 1/16)$ since $\delta \in (0, 1/4]$. Then for all large n (say $n \geq n_0$) we have $1 - p \geq 2\delta, pc^2 n \geq 2$ and $n - 2 \geq 2cn \geq 2$ so that

$$c\log\left(\frac{e}{c}\right) - (1-p)(1-c) \leq c\log\left(\frac{e}{c}\right) - 2\delta(1-c) = -\delta$$

and

$$\log 2 - \frac{pc^2 n}{2} \leq -(1 - \log 2) \leq -\delta.$$

If $t = \lceil cn \rceil$ then for $n \geq n_0$ we have $1 \leq t \leq n/2$ while (5) implies

$$\binom{n}{t} \leq \left(\frac{en}{t}\right)^t \leq \left(\frac{e}{c}\right)^{cn+1} .$$

For $1 \leq s \leq t$ and $n \geq n_0$,

$$e^{(n-s)[\log ps - ps + p]} \leq 3e^{-(1-p)(1-c)n}$$

since $\log x - x \leq -1$ on $(0, \infty)$ and

$$\sum_{s=1}^{t} \binom{n}{s} e^{(n-s)[\log ps - ps + p] - p\binom{s}{2}} \leq 3t\left(\frac{e}{c}\right)^{cn+1} e^{-(1-p)(1-c)n}$$

$$\leq 6en e^{cn\log(e/c) - (1-p)(1-c)n}$$

$$\leq 6en e^{-\delta n}.$$

For $t+1 \leq s \leq n$ and $n \geq n_0$,

$$e^{-p\binom{s}{2}} \leq e^{-pc^2n^2/2}$$

and

$$\sum_{s=t+1}^{n} \binom{n}{s} e^{(n-s)[\log ps - ps + p] - p\binom{s}{2}} \leq 2^n e^{-pc^2n^2/2}$$

$$\leq e^{n\log 2 - pc^2n^2/2}$$

$$\leq e^{-\delta n}.$$

Consequently,

$$E(X) = O(ne^{-\delta n})$$

and Markov's Inequality implies

$$P(G \in \mathcal{G}(n,p) \text{ has a perfect dominating set}) = O(ne^{-\delta n}). \qquad \blacksquare$$

For a graph G with $S \subseteq T \subseteq V(G)$, we say S **perfectly dominates** T in G provided $|N_G[v] \cap S| = 1$ for all $v \in T$. Note that S perfectly dominates T in G if and only if S is independent in G and $\{N_G(v) \cap (T-S) : v \in S\}$ partitions $T-S$. Observe that if S perfectly dominates T in G and $S \subseteq U \subseteq T$ then S perfectly dominates U in G.

For a graph G, let $\epsilon(G) = \max\{|T| : \exists S$ which perfectly dominates T in $G\}$. For a wide range of p, we next show that for some constant $d \in (0,1)$, $\epsilon(G) \leq \lceil dn \rceil$ for almost every $G \in \mathcal{G}(n,p)$.

Theorem 2. For $0 < p = p(n) < 1$ with $\overline{\lim\limits_{n\to\infty}}\, p < 1/2$ and $pn \to \infty$, there exists $d \in (0,1)$ so that

$$P(G \in \mathcal{G}(n,p) \text{ has } \epsilon(G) \geq \lceil dn \rceil) = O\big((pn)^{-4}\big).$$

Proof. First, for $S \subseteq T \subseteq [n]$ and $G \in \mathcal{G}(n,p)$, let

$$X_{(S,T)}(G) = \begin{cases} 1, & S \text{ perfectly dominates } T \text{ in } G, \\ 0, & \text{otherwise,} \end{cases}$$

and for $1 \leq r, t \leq n$

$$X_{(r,t)} = \sum_{\substack{1 \leq |S| \leq r \\ |T-S| = t}} X_{(S,T)}.$$

For $S \subseteq T \subseteq [n]$ with $|S| = s$ and $|T-S| = t$,

$$E(X_{(S,T)}) = (ps)^t q^{(s-1)t + \binom{s}{2}}$$

so that (1) implies

$$E(X_{(r,t)}) = \sum_{s=1}^{r} \binom{n}{s}\binom{n-s}{t}(ps)^t q^{(s-1)t+\binom{s}{2}}$$
$$\leq \sum_{s=1}^{r} \binom{n}{s}\binom{n-s}{t} e^{t[\log ps - ps + p]}$$
$$\leq \sum_{s=1}^{r} \binom{n}{s}\binom{n-s}{t} e^{-(1-p)t}.$$

Next, for $S \subseteq [n]$ and $G \in \mathcal{G}(n,p)$, let

$$Y_S(G) = \begin{cases} 1, & S \text{ is independent in } G, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$Y = \sum_{|S|=r} Y_S$$

so that (1) implies

$$E(Y) = \binom{n}{r} q^{\binom{r}{2}}$$
$$\leq \binom{n}{r} e^{-p\binom{r}{2}}.$$

Let $8\delta = 1/2 - \overline{\lim}_{n\to\infty} p$ and $c/2 + \log c = 1/2 - 2\delta$ so that $c \in (.9, 1)$ since $\delta \in (0, 1/16]$. Then for all large n (say $n \geq n_0$) we have $p \leq 1/2 - 4\delta$, $\delta pn \geq 5 \log^2 pn$ and $n \geq 1/(1-c)$ so that $pn \geq 6000$ and $pn \geq 5 \log pn \geq 40$. If $r = \lceil 2 \log pn/p \rceil$ and $t = \lceil cn \rceil$ then for $n \geq n_0$ we have

$$32 \leq 4 \log pn \leq r \leq \min\left\{\frac{n}{2}, (pn)^r\right\}$$

and

$$\frac{2r}{t} \leq \frac{\delta}{\log pn}$$

while (5) implies

$$\binom{n}{r} \leq \left(\frac{en}{r}\right)^r \leq \left(\frac{epn}{2 \log pn}\right)^r \leq (pn)^r$$

and

$$\binom{n}{t} \leq \left(\frac{en}{t}\right)^t e^{-\binom{t}{2}/n} \leq 2\left(\frac{e}{c}\right)^t e^{-tc/2}.$$

For $n \geq n_0$,

$$\begin{aligned} E(X_{(r,t)}) &\leq r\binom{n}{r}\binom{n}{t}e^{-(1-p)t} \\ &\leq 2\,e^{t[(2r/t)\log pn + \log(e/c) - c/2 - (1-p)]} \\ &\leq 2\,e^{-\delta t} \\ &\leq 2\,e^{-\delta cn} \end{aligned}$$

while

$$\begin{aligned} E(Y) &\leq e^{r[\log(epn/2\log pn) - \log pn + 1/2]} \\ &\leq e^{r[1-\log\log pn]} \\ &\leq e^{-r} \\ &\leq (pn)^{-4}. \end{aligned}$$

Then

$$E(X_{(r,t)} + Y) = O\big((pn)^{-4}\big)$$

and Markov's Inequality implies

$$P(X_{(r,t)} \geq 1) + P(Y \geq 1) = O\big((pn)^{-4}\big).$$

Let $d \in (c, 1)$. Then for all large n (say $n \geq n_1 \geq n_0$) we have $r + t \leq \lceil dn \rceil$. For $n \geq n_1$,

$$\begin{aligned} &P(G \in \mathcal{G}(n,p) \text{ has } \epsilon(G) \geq \lceil dn \rceil) \\ &\qquad \leq P(\exists S \text{ perfectly dominates } T \text{ in } G \text{ with } |S| \leq r \text{ and } |T - S| \geq t) \\ &\qquad\qquad + P(\exists S \text{ perfectly dominates } T \text{ in } G \text{ with } |S| \geq r) \\ &\qquad \leq P(X_{(r,t)} \geq 1) + P(Y \geq 1). \end{aligned}$$

Consequently,

$$P(G \in \mathcal{G}(n,p) \text{ has } \epsilon(G) \geq \lceil dn \rceil) = O\big((pn)^{-4}\big). \qquad \blacksquare$$

Remark. As a consequence of our last result, there exist constants $0 < c_1 < c_2 < 1$ so that $c_1 n \leq \epsilon(G) \leq c_2 n$ for almost every $G \in \mathcal{G}(n,p)$ with p constant, since $\epsilon(G) \geq \Delta(G)$.

We finally show that almost every tree $T \in \mathcal{T}(n)$ has no perfect dominating set. Again, our proof shows the probability that a tree has a perfect dominating set is exponentially small.

Theorem 3. $P(T \in \mathcal{T}(n)$ has a perfect dominating set$) = O(ne^{-n/125})$.

Proof. Fix $S = \{i_1, \ldots, i_s\} \subseteq [n]$ with $1 \leq i_1 < \cdots < i_s \leq n$ and division $(A_1, \ldots, A_s)$ of $[n] - S$. For $T \in \mathcal{T}(n)$ with perfect dominating set S and $N_T(i_j) = A_j$ $(1 \leq j \leq s)$ we have, (1) $S, A_1, \ldots, A_s$ are independent sets in T, (2) $|A_1|, \ldots, |A_s| \geq 1$ since T is connected and (3) there is at most one edge between A_i and A_j for $1 \leq i \neq j \leq s$ since T is acyclic. (We note that necessarily $s \leq n/2$ by (2)).

Consequently, in $[n] - S$ if we contract each A_i to a vertex i we obtain a simple graph (no loops nor multiple edges) T^* which must be a tree with vertex set $[s]$. Conversely, any tree T^* with vertex set $[s]$ and degree sequence $(d_1, \ldots, d_s)$ where $d_{T^*}(i) = |N_{T^*}(i)| = d_i$ $(1 \leq i \leq s)$ comes from at most

$$|A_1|^{d_1} \ldots |A_s|^{d_s}$$

such trees $T \in \mathcal{T}(n)$.

The number of trees T^* with vertex set $[s]$ and degree sequence $(d_1, \ldots, d_s)$ where $d_{T^*}(i) = d_i \in \mathrm{N} = \{1, 2, 3, \ldots\}$ $(1 \leq i \leq s)$ is

$$\binom{s-2}{d_1 - 1, \ldots, d_s - 1}$$

so that the Multinomial Theorem gives

$$P(T \in \mathcal{T}(n) \text{ has perfect dominating set } S \text{ with } N_T(i_1) = A_1, \ldots, N_T(i_s) = A_s)$$
$$\leq \sum_{\substack{(d_1, \ldots, d_s) \in \mathrm{N}^s \\ d_1 + \cdots + d_s = 2s-2}} \binom{s-2}{d_1 - 1, \ldots, d_s - 1} \prod_{j=1}^{s} |A_j|^{d_j} n^{-(n-2)}$$
$$= \prod_{j=1}^{s} |A_j| (n-s)^{s-2} n^{-(n-2)}.$$

First, going from fixed neighborhoods to fixed degrees we have

$$P(T \in \mathcal{T}(n) \text{ has perfect dominating set } S \text{ with } d_T(i_1) = \ell_1, \ldots, d_T(i_s) = \ell_s)$$
$$\leq \binom{n-s}{\ell_1, \ldots, \ell_s} \prod_{j=1}^{s} \ell_j (n-s)^{s-2} n^{-(n-2)}$$
$$= \binom{n-2s}{\ell_1 - 1, \ldots, \ell_s - 1} (n-s)_s (n-s)^{s-2} n^{-(n-2)}$$

so that upon summing over all such degree sequences, the Multinomial Theorem gives

$$\begin{aligned}
&P(T \in \mathcal{T}(n) \text{ has perfect dominating set } S) \\
&\le \sum_{\substack{(\ell_1,\dots,\ell_s)\in \mathbb{N}^s \\ \ell_1+\cdots+\ell_s=n-s}} \binom{n-2s}{\ell_1-1,\dots,\ell_s-1}(n-s)_s(n-s)^{s-2}n^{-(n-2)} \\
&= s^{n-2s}(n-s)_s(n-s)^{s-2}n^{-(n-2)}
\end{aligned}$$

and, finally, going from fixed set to fixed cardinality we have

$$\begin{aligned}
&P(T \in \mathcal{T}(n) \text{ has perfect dominating set of cardinality } s) \\
&\le \binom{n}{s} s^{n-2s}(n-s)_s(n-s)^{s-2}n^{-(n-2)} \\
&= \frac{(n)_{2s}(n-s)^{s-2}s^{n-2s}}{s!\, n^{n-2}}.
\end{aligned}$$

Let

$$f(s) = \frac{(n)_{2s}(n-s)^{s-2}s^{n-2s}}{s!\, n^{n-2}}$$

and

$$g(s) = \frac{(n-s-1)(n-2s)(n-2s-1)}{(s+1)^3}$$

and note that $g(s)$ is a nonnegative, decreasing function of s on $[1,(n-1)/2]$.

For $2 \le s \le (n-2)/2$,

$$\frac{f(s+1)}{f(s)} = g(s)\left(1-\frac{1}{n-s}\right)^{s-2}\left(1+\frac{1}{s}\right)^{n-2s}$$

so that (2), (3) imply

$$\frac{f(s+1)}{f(s)} \ge g(s)e^{-2s/(n-s)+(n-2s)/2s} := g(s)h_1(s)$$

while (1), (3) imply

$$\frac{f(s+1)}{f(s)} \le g(s)e^{-(s-2)/(n-s)+(n-2s)/s} := g(s)h_2(s)$$

where $h_1(s), h_2(s)$ are decreasing functions of s on $[2, n/2]$. Hence $g(s)h_1(s) \ge g((n-3)/3)h_1(n/3) \ge 2e^{-1/2} > 1$ for $2 \le s \le (n-3)/3$ while $g(s)h_2(s) \le g(3n/8)h_2(3n/8) \le (20/27)e^{1/4} < 1$ for $3n/8 \le s \le (n-2)/2$ provided

$n \geq 18$. Then $f(s)$ is an increasing sequence on $\{2, \ldots, \lfloor n/3 \rfloor\}$ and a decreasing sequence on $\{\lceil 3n/8 \rceil, \ldots, \lfloor n/2 \rfloor\}$ provided $n \geq 18$.

For $1 \leq s \leq n/2$, (1) implies

$$(1 - \frac{s}{n})^s \leq e^{-s^2/n - s^3/2n^2}$$

while (4) implies

$$(n)_{2s} \leq 3n^{2s} e^{-2s^2/n - 4s^3/3n^2}$$

so that

$$f(s) \leq 12 e^{s - 3s^2/n - 11s^3/6n^2 + (3s-n)\log(n/s)}$$

where $s - 3s^2/n - 11s^3/6n^2$ is a decreasing function of s on $[n/12, n/2]$ while $(3s - n)\log(n/s)$ is an increasing function of s on $[n/12, n/2]$. Then

$$f(s) \leq 12 e^{n\{(1-3c_2)\log c_2 + c_1 - 3c_1^2 - 11c_1^3/6\}}$$

for $n/12 \leq c_1 n \leq s \leq c_2 n \leq n/2$.

Finally, let

$$h(c, d) = (1 - 3d)\log d + c - 3c^2 - \frac{11c^3}{6}$$

with $c_1 = .33$, $c_2 = .35$, $c_3 = .362$, $c_4 = .37$, $c_5 = .376$, and $c_6 = .38$ so that $h(c_i, c_{i+1}) \leq -.008$ for $1 \leq i \leq 5$ and

$$f(s) \leq 12 e^{-n/125}$$

on $\{\lceil .33n \rceil, \ldots, \lfloor .38n \rfloor\}$. For $n \geq 400$,

$$f(\lceil .33n \rceil) \leq f(\lfloor n/3 \rfloor) \leq 12 e^{h(.33, 1/3)n} \leq 12 e^{-n/16}$$

since $n/12 \leq \lceil .33n \rceil \leq \lfloor n/3 \rfloor \leq n/2$ while

$$f(\lfloor .38n \rfloor) \leq f(\lceil 3n/8 \rceil) \leq 12 e^{h(3/8, .38)n} \leq 12 e^{-n/125}$$

since $n/12 \leq \lceil 3n/8 \rceil \leq \lfloor .38n \rfloor \leq n/2$. Note that $f(1) = n^{-(n-3)} = O(ne^{-n/125})$.

Consequently,

$$\begin{aligned} &P(T \in \mathcal{T}(n) \text{ has perfect dominating set}) \\ &\leq \sum_{s=1}^{n/2} P(T \in \mathcal{T}(n) \text{ has a perfect dominating set of cardinality } s) \\ &= O(ne^{-n/125}). \qquad \blacksquare \end{aligned}$$

3. Conclusion.

For a wide range of p we have shown that almost every graph $G \in \mathcal{G}(n, p)$ has no perfect dominating set and we have determined the order of magnitude of $\epsilon(G)$ for almost every graph $G \in \mathcal{G}(n, p)$ provided p is constant. Determination of the order of magnitude of $\epsilon(G)$ for other ranges of p could be of interest. We have also shown that even for the set of labelled trees, almost every tree has no perfect dominating set. Again, determination of the order of magnitude of $\epsilon(T)$ for almost every tree $T \in \mathcal{T}(n)$ could be of interest.

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