The Existence of Incomplete Room Squares[1]

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Abstract. It has been conjectured by D. R. Stinson that an incomplete Room square (n, s)-IRS exists if and only if n and s are both odd and $n \geq 3s + 2$, except for the nonexistent case $(n, s) = (5, 1)$. In this paper we shall improve the known results and show that the conjecture is true except for 45 pairs (n, s) for which the existence of an (n, s)-IRS remains undecided.

1. Introduction

Let S be a set of $n+1$ elements called symbols. A *Room square* of side n (on symbol set S) is an $n \times n$ array, F, which satisfies the following properties:

1. every cell of F either is empty or contains an unordered pair of symbols from S
2. each symbol of S occur once in each row and column of F
3. every unordered pair of symbols occurs in precisely one cell of F.

It is immediate that n must be odd for a Room square of side n to exist. The spectrum of Room squares was determined in 1975 by Mullin and Wallis [7].

Theorem 1.1. *A Room square of side n exists if and only if n is an odd positive integer, and* $n \neq 3$ *or* 5.

A further problem, which has attracted much research interest in recent years, is the existence of incomplete Room squares. Here is the definition.

Let S be a set of $n+1$ symbols and let T be a subset of S of cardinality $s+1$. An (n, s)-IRS, called an *incomplete Room square*, is an $n \times n$ square array F which satisfies the following:

1. every cell of F either is empty or contains an unordered pair of symbols of S
2. there is an empty $s \times s$ subarray G of F
3. each symbol of $S \backslash T$ occurs once in each row and column of F
4. each symbol of T occurs once in each row and column not meeting G, but not in any row or column meeting G
5. the pairs occurring in F are precisely those $\{x, y\}$ where $(x, y) \in (S \times S) \backslash (T \times T)$.

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We refer to the subarray G as the *hole*. An $(n, 0)$-IRS is simply a Room square of side n. We suppose that $s > 0$ from now on. Simple counting shows that the existence of an (n, s)-IRS implies that n and s are both odd, $n \geq 3s + 2$, and $(n, s) \neq (5, 1)$. The following conjecture was first presented in [9].

Existence Conjecture. *An (n, s)-IRS exists if and only if n and s are both odd positive integers, $n \geq 3s + 2$, and $(n, s) \neq (5, 1)$.*

Many papers over the years have studied the existence of incomplete Room squares [3-4], [9-11], [13-15]. The best known general existence results for (n, s)-IRS are summarized in the following theorem.

Theorem 1.2.

1). *[7] For odd $n \geq 7$, there is an $(n, 1)$-IRS.*
2). *[15] For odd $s \geq 3$, there is an $(3s + 2, s)$-IRS.*
3). *[2-3], [15] For odd s, $3 \leq s \leq 15$ and $s = 23, 33$, and for all odd $n \geq 3s + 2$, there is an (n, s)-IRS.*
4). *[3] For all odd $s \geq 37$ and all odd $n \geq (7s - 5)/2$, there is an (n, s)-IRS.*
5). *[13] For all odd $s \geq 393$ and all odd $n \geq 3s + 2$, there is an (n, s)-IRS.*
6). *[3] For any odd s, $17 \leq s \leq 35$ and for any odd $n \geq 3s + k$ there is an (n, s)-IRS, where k is shown in the table.*

s	17	19	21	23	25	27	29	31	33	35
k	16	12	8	4	20	16	12	8	4	10

In this paper we shall improve these results and prove the following theorem.

Theorem 1.3. *For any odd integer $s > 1$, an $(n, s) - IRS$ exists if and only if n is odd and $n \geq 3s + 2$, with 45 ordered pairs (n, s) as possible exceptions, as listed in Table 1.*

2. Preliminaries

In this section we shall state some known constructions to obtain incomplete Room squares. The first one involves the use of transversal designs. The second one needs the existence of a starter and adder, while the last uses frames. For general concepts and notation on designs the reader is referred to the book of Beth, Jungnickel and Lenz [1].

Table 1
parameters s and k for which a $(3s + k, s) - IRS$ is unknown

s	k
21,37,51,57,121	4
31,35,41,49	4,6
45	4,8
19,25,55	4,6,8
29,89	4,8,10
105	4,10,12
17,47	4,8,10,12
27	4,8,12,14

A transversal design, denoted by $TD[k, m]$, is a triple $(X, \mathbf{G}, \mathbf{A})$, which satisfies the following properties:

1. X is a km-set
2. $\mathbf{G}$ is a partition of X into km-subsets called *groups*
3. $\mathbf{A}$ is a set of k-subsets of X (called *blocks*) such that a group and a block contain at most one common point, and
4. every pair of points from distinct groups occurs in a unique block.

We also require the idea of incomplete transversal designs. Informally, an incomplete TD, denoted by $TD[k, n] - TD[k, m]$, is a $TD[k, n]$ "missing" a sub-$TD[k, m]$. We observe that a $TD[k, n] - TD[k, 0]$ and a $TD[k, n] - TD[k, 1]$ exists if and only if a $TD[k, n]$ exists. We have the following known results on TDs and incomplete TDs.

Lemma 2.1. [12, Lemma 1.4] *For all integers* $m \geq 5$, $m \neq 6, 10, 14, 18, 22, 26, 30, 34$ *or* 42, *there is a* $TD[6, m]$.

Lemma 2.2. *[5] There exists a* $TD[4, n] - TD[4, m]$ *for any integer* $m \geq 2$ *and* $n \geq 3m$.

The first construction is as follows.

Lemma 2.3. [3, Theorem 4.7] *Suppose there is a* $TD[6, m]$, *and suppose there exists an* $(2r + a, a)$*-IRS for all* r *such that* $m \leq r \leq 2m$. *Let* $m \leq t \leq 2m$ *and let* $5m \leq w \leq 10m$. *Then there exists an* $(2w + 2t + a, 2t + a)$*-IRS.*

Remark: From the proof of [3, Theorem 4.7] one may easily find that if the condition $m \leq r \leq 2m$ in Lemma 2.3 is replaced by $m \leq m' \leq r \leq 2m$, then the conclusion still valid provided that $5m' \leq w \leq 10m$.

A *starter* in an abelian group G of order $2n - 1$ is a set of $n - 1$ unordered pairs $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{n-1}, y_{n-1}\}$ which satisfy the following properties:

1. $\{s_i : 1 \leq i \leq n - 1\} \cup \{t_i : 1 \leq i \leq n - 1\} = G \backslash \{0\}$

2. $\{\pm(s_i - t_i): 1 \le i \le n-1\} = G\backslash\{0\}$.

An *adder* for a starter $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{n-1}, y_{n-1}\}$ in G is a set of $n-1$ distinct non-zero elements $a_1, a_2, \ldots, a_{n-1}$ such that $\{x_i + a_i: 1 \le i \le n-1\} \cup \{y_i + a_i: 1 \le i \le n-1\} = G\backslash\{0\}$.

For existence of starter an adder, the following result is known.

Lemma 2.4. *[8] For odd $g, 7 \le g \le 47$, there is a starter and adder in Z_g.*

The following is the second known construction.

Lemma 2.5. [13, Lemma 4.4] *Suppose there exists a starter and adder in Z_g. Suppose $0 \le u \le 3(g-1)/2$ and $0 \le k \le 7[(g-1)/2 - \lceil u/3 \rceil]$. Further, suppose there is a $(6u+2k+11, 2u+3)$-IRS. Then there is a $(24g+6u+2k+11, 8g+2u+3)$-IRS.*

Let S be a set, and let $\{S_1, S_2, \ldots, S_n\}$ be a partition of S. An $\{S_1, S_2, \ldots, S_n\}$-*Room frame* is an $|S| \times |S|$ array, F, indexed by S, which satisfies the following properties:

1. every cell of F either is empty or contains an unordered pair of symbols of S
2. the subarrays $S_i \times S_i$ are empty, for $1 \le i \le n$ (these subarrays are referred to as holes)
3. each symbol $x \notin S_i$ occurs once in row (or column) s, for any $s \in S_i$, and
4. the pairs occurring in F are those $\{s, t\}$, where $(s, t) \in (S \times S) \setminus \cup_{1 \le i \le n} (S_i \times S_i)$.

As is usually done in the literature, we shall refer to a Room frame simply as a frame. The *type* of a frame F is defined to be the multiset $\{|S_i|: 1 \le i \le n\}$. We usually use an "exponential" notation to describe types: a type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \le i \le k$. The order of the frame is $|S|$.

We observe that existence of a Room square of side n is equivalent to existence of a frame of type 1^n, and existence of an (n, s)-IRS is equivalent to existence of a frame of type $1^{n-s} s^1$.

For existence of frames, the following results are known.

Lemma 2.6. *[3],[9] There exist frames of type $4^4, 2^5 4^1$ and $4^4 6^1$.*

Lemma 2.7. *[9] Suppose there is a frame of type T, and suppose m is a positive integer, $m \ne 2$ or 6. Then there is a frame of type mT.*

Using frames and the Filling in Holes techniques we have the third construction to obtain incomplete Room squares.

Lemma 2.8. *[9] Suppose there is a frame of type $\{s_i: 1 \le i \le n\}$, and let $a \ge 0$ be an integer. For $1 \le i \le n-1$, suppose there is an $(s_i + a, a)$-IRS. Then there is an $(s + a, s_n + a)$-IRS, where $s = \Sigma_{1 \le i \le n} s_i$.*

We now give generalization of Filling in Holes [9] which starts with a Room square and uses an incomplete transversal design.

Lemma 2.9. *[6] Suppose there is a Room square of side u, and suppose there exist a $(v+1, w)$-IRS and a $TD[4, v] - TD[4, w]$. Then there exists a $(uv + 1, uw)$-IRS.*

3. Main result

In this section we shall prove our main result, namely Theorem 1.3. Define $S = \{s\text{: there exists an } (n, s)\text{-IRS for all odd } n \geq 3s + 2\}$.

Lemma 3.1. *$s \in S$ for any odd integer s, $107 \leq s \leq 391$, and $s \not\equiv 9 \pmod{16}$.*

Proof: For any such s, s can be uniquely written in the form $s = 8g + 2u + 3$ such that g is an odd integer, $13 \leq g \leq 47$ and $0 \leq u \leq 6$. Since there exists by Lemma 2.4 a starter and adder in Z_g, we may apply Lemma 2.5 to obtain incomplete Room squares. By Theorem 1.2 3), $2u + 3 \in S$ for $0 \leq u \leq 6$. We obtain a $(3s + 2 + 2k, s)$-IRS for $0 \leq k \leq 7[(g-1)/2 - \lceil u/3 \rceil]$. On the other hand, there exists from Theorem 1. 2 4) an (n, s)-IRS for all odd $n \geq (7s-5)/2$. It is readily checked that

$$3s + 2 + 14[(g-1)/2 - \lceil u/3 \rceil] \geq (7s-5)/2.$$

Then the conclusion follows immediately.

Lemma 3.2. *$s \in S$ for any odd integer $s \equiv 9 \pmod{16}$ and $281 \leq s \leq 391$.*

Proof: Since $33 \in S$ from Theorem 1.2 3), we take $u = 15$ in Lemma 2.7. Write $s = 8g + 2u + 3$, where $g \geq 31$. A similar proof to that of Lemma 3.1 shows that $s \in S$.

In what follows we shall discuss the remaining cases, namely $s \leq 105$ and $s \equiv 9 \pmod{16}$ for $107 \leq s \leq 279$. For each s we have from Theorem 1.2 4) that an (n, s)-IRS exists if $s \geq 37$ and $n \geq (7s-5)/2$. Similar results for $17 \leq s \leq 35$ are also known from Theorem 1.2 6). These results provided a bound $k_0 = k_0(s)$ such that an (n, s)-IRS exists whenever $n \geq 3s + k_0$. In the next lemma we shall improve the bound $n \geq 3s + k_0$ to $n \geq 3s + k_1$ by using Lemma 2.3.

Lemma 3.3. *For any odd integer s, $37 \leq s \leq 105$, and for any $s \equiv 9 \pmod{16}$, $107 \leq s \leq 279$, an (n, s)-IRS exists whenever $n \geq 3s + k_1$ is odd, where the value k_1 is given in Table 2.*

Table 2

s	k_0	k_1	m	t	a	x-y
37	16	6	8	15	7	6–86
39	17	2	8	16	7	2–82
41	18	8	9	17	7	8–98
43	19	4	9	18	7	4–94
45	20	10	9	18	9	10–90
47	21	16	11	19	9	16–126
49	22	12	11	20	9	12–122
51	23	8	11	21	9	8–118
53	24	4	11	22	9	4–114
55	25	10	12	22	11	10–130
57	26	6	12	23	11	6–126
59	27	2	12	24	11	2–122
61	28	8	13	25	11	8–138
63	29	4	13	26	11	4–134
65	30	20	15	26	13	20–170
67	31	16	15	27	13	16–166
69	32	12	15	28	13	12–162
71	33	8	15	29	13	8–158
73	34	4	15	30	13	4–154
75	35	10	16	30	15	10–170
77	36	6	16	31	15	6–166
79	37	2	16	32	15	2–162
81	38	8	17	33	15	8–178
83	39	4	17	34	15	4–174
85	40	20	19	35	15	20–210
87	41	16	19	36	15	16–206
89	42	12	19	37	15	12–202
91	43	8	19	38	15	8–198
93	44	14	20	39	15	14–214
95	45	10	20	40	15	10–210
97	46	16	21	41	15	16–226
99	47	12	21	42	15	12–222
101	48	28	23	43	15	28–258
103	49	24	23	44	15	24–254
105	50	20	23	45	15	20–250
121	58	8	25	49	23	8–258
137	66	16	29	57	23	16–306
153	74	24	33	65	23	24–354
169	82	32	37	73	23	32–402

Table 2 (cont'd)

s	k_0	k_1	m	t	a	x-y
185	90	40	41	81	23	40–450
201	98	48	45	89	23	48–498
217	106	56	49	97	23	56–546
233	114	64	53	105	23	64–594
249	122	72	57	113	23	72–642
265	130	80	61	121	23	80–690

Proof: In Table 2, we give the integer k_0 where the bound $n \geq 3s + k_0$ comes from Theorem 1.2 4). By applying Lemma 2.3 we have the improved bound $n \geq 3s + k_1$ where the integer k_1 is also given. The parameters m, t, a in Lemma 2.3 are listed. Instead of giving a resultant interval $[3s + x, 3s + y]$, we simply write "x - y", where $x = 10m - 4t - 2a$ and $y = x + 10m$. The fifth line in Table 2 comes from the Remark of Lemma 2.3 where $m' = 10$, $x = 10m' - 4t - 2a$ and $y = 20m - 4t - 2a$.

We shall further lower the bound $n \geq 3s + k_1$ whenever possible.

Lemma 3.4. *$s \in S$ if $s \equiv 9 \pmod{16}$ and $137 \leq s \leq 265$.*

Proof: Apply Lemma 2.5 with $u = 7$ and $g = (s - 17)/8$. Since $15 \leq g \leq 31$ and g is odd, there exists from Lemma 2.4 a starter and adder in Z_g. We then obtain a $(3s + 2 + 2k, s)$-IRS for $7 \leq k \leq 7[(g - 1)/2 - 3]$ since an $(n, 17)$-IRS exists for $n \geq 67$. Combining this with the bound $n \geq 3s + k_1$ in Table 2 gives a new bound $n \geq 3s + k_2$ where $k_2 = 16$.

Again using Lemma 2.5 with $u = 15$ and $g = (s - 33)/8$. We obtain a $(3s + 2 + 2k, s)$-IRS for $0 \leq k \leq 7[(g - 1)/2 - 5]$ since $33 \in S$. To show that $s \in S$ we need to have $(g - 1)/2 - 5 \geq 1$, which is implied by $g = (s - 33)/8$ and $s \geq 137$. The proof is complete.

Lemma 3.5. *$s \in S$ if $s \geq 59$ except possibly if $s \in \{89, 105, 121\}$.*

Proof: Take $g = 7$ and $0 \leq u \leq 6$ in Lemma 2.5. We obtain a $(3s + 2 + 2k, s)$-IRS for $0 \leq k \leq 7[3 - \lceil u/3 \rceil]$, where $s = 8g + 2u + 3$, Combining this with the bound $n \geq 3s + k_1$ in Table 2 gives $s \in S$ for $59 \leq s \leq 71$. From Table 2 we also have $73 \in S$. Further take $g = 9, 11$ and $0 \leq u \leq 6$ in Lemma 2.5. We obtain that $s \in S$ for $75 \leq s \leq 87$ and $91 \leq s \leq 103$. The conclusion then follows from Therem 1.2 5) and Lemmas 3.1 – 3.4.

Lemma 3.6. *A $(3s + k, s)$-IRS exists for $s = 105$ and for all even $k \geq 14$.*

Proof: Apply Lemma 2.5 with $g = 11$ and $u = 7$. We obtain a $(3s + k, s)$-IRS for $s = 105$ and $14 \leq k \leq 20$. Then the conclusion follows from Lemma 3.3.

We now treat some sporadic pairs (n, s).

Lemma 3.7. *There exists an* (n, s)*-IRS for* $(n, s) \in E$*, where*

$$E = \{(3u + 6, u), (3v + 10, v), (3w + 14, w):$$
$$u = 17, 21, 27, 29, 45, 47, 51;$$
$$v = 19, 27, 49; w = 17, 47\}$$

Proof: Apply Lemma 2.8 with the parameters shown in Table 3. The required frames of type mT come from the frames of type T in Lemma 2.6 and Lemma 2.7 with suitable m. The required $(s_i + a, a)$-IRS are all known from Theorem 1.2 3).

Table 3

(n, s)	T	m	$(s_i + a, a)$-IRS
(57, 17)	$2^5 4^1$	4	(8+1,1)
(65, 17)	4^4	4	(16+1,1)
(67, 19)	4^4	4	(16+3,3)
(69, 21)	4^4	4	(16+5,5)
(87, 27)	4^4	5	(20+7,7)
(91, 27)	$4^4 6^1$	4	(16+3,3)
(93, 29)	$4^4 6^1$	4	(16+5,5)
(141, 45)	4^4	8	(32+13,13)
(147, 47)	$2^5 4^1$	10	(20+7,7)
(155, 47)	4^4	9	(36+11,11)
(157, 49)	4^4	9	(36+13,13)
(159, 51)	4^4	9	(36+15,15)

Lemma 3.8. *There exists an* (n, s)*-IRS for* $(n, s) = (113, 35), (155, 49)$ *and* $(323, 105)$.

Proof: The conclusion follows fom Lemma 2.9 and the following expressions:

$$113 = 7 \times (11 + 5) + 1,$$
$$155 = 7 \times (15 + 7) + 1,$$
$$323 = 7 \times (31 + 15) + 1.$$

The required $TD[4, v] - TD[4, w]$ and IRS come from Lemma 2.2 and Theorem 1.2, respectively.

Lemma 3.9. *There exists an* $(3s + 6, s)$*-IRS for* $s = 89, 105$ *and* 121.

Proof: In Lemma 2.5, let $k = 2$ and $(g, u) \in \{(9, 7), (11, 7), (13, 7)\}$. Since a $(57, 17)$-IRS exists from Lemma 3.7, the conclusion then follows.

Lemma 3.10. *A* $(3s+k, s)$*-IRS exists for* $s = 25$ *and for all even* $k \geq 10$.

Proof: Apply the Remark of Lemma 2.3 with $m = 5$, $m' = 6$, $t = 10$ and $a = 5$. We obtain a $(3s+k, s)$-IRS for $s = 25$ and $10 \leq k \leq 50$. Then the conclusion follows from Theorem 1.2 6).

We are now in a position to prove our main result.

Proof of Theorem 1.3: By Lemma 3.3, Lemmas 3.5–3.6 and Lemmas 3.8–3.9 we have proved our conclusion for $s \geq 59$. By Lemma 3.3 and Theorem 1.2 we have $s \in S$ for those s not in Table 1. For those s in Table 1 and $s < 59$ we compare the bound $n \geq 3s + k_1$ shown in Table 2, Theorem 1.2 6) and Lemma 3.10 with the exceptional pairs $(3s+k, s)$ in Table 1. The gap between them is just filled by Lemmas 3.7 –3.8, shown in Table 4, where the third column lists those k for which a $(3s+k, s)$-IRS is known from Lemmas 3.7–3.8 and the last column lists the unknown cases. The proof is complete.

Table 4

s	k_1	k	remaining cases
17	16	6, 14	4,8,10,12
19	12	10	4,6,8
21	8	6	4
25	10	–	4,6,8
27	16	6, 10	4,8,12,14
29	12	6	4,8,10
31	8	–	4,6
35	10	8	4,6
37	6	–	4
41	8	–	4,6
45	10	6	4,8
47	16	6, 14	4,8,10,12
49	12	8, 10	4,6
51	8	6	4
55	10	–	4,6,8
57	6	–	4

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