Existence of IPBDs of Block Size Four

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Abstract. In this paper we prove that a $(v, u; \{4\}, 3)$-IPBD exists when $v, u \equiv 2$ or $3 \pmod 4$ and $v \geq 3u+1$, and then solve the problem of the existence of $(v, u; \{4\}, \lambda)$-IPBD completely, which generalizes the result of [7].

1. Introduction

We assume that the reader is familiar with the basic concepts in design theory such as pairwise balanced design (PBD), group divisible design (GDD), transversal design (TD), parallel classes of blocks, resolvability, etc. For general information and notatons see [1] and [8]. We begin with the definition of PBD missing one sub-PBD, called an incomplete PBD, adapted from [7].

An *incomplete PBD* (or IPBD) is a triple $(X, Y, \mathcal{A})$, where X is a set of points, $Y \subset X$ and $\mathcal{A}$ is a set of blocks which satisfies the following properties:

(1) for any $A \in \mathcal{A}$, $|A \cap Y| \leq 1$; and
(2) every pair of points $\{x, y\}$, where $\{x, y\} \in X \times X \backslash Y \times Y$, occurs in exactly λ blocks.

We say that $(X, Y, \mathcal{A})$ is a $(v, u; K, \lambda)$-IPBD if $|X| = v$, $|Y| = u$ and $|A| \in K$ for every $A \in \mathcal{A}$.

In this paper, we study the existence of $(v, u; \{4\}, \lambda)$-IPBD. The necessary conditions of the existence of such designs are easily seen to be as follows:

(1) $\lambda(v-1) \equiv 0 \pmod 3$;
(2) $\lambda(v-u) \equiv 0 \pmod 3$;
(3) $\lambda(v^2 - u^2 - v + u) \equiv 0 \pmod{12}$; and (*)
(4) $v \geq 3u + 1$.

The interesting problem is whether the necessary conditions (*) are sufficient. When $\lambda = 1, 2$ and 6 this problem is solved. The following three theorems are from Rees and Stinson [7], Rees and Rodger [6] and Kong and Zhu [3], respectively.

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Theorem 1.1. *A* $(v,u;\{4\},1)$*-IPBD exists if and only if* $u,v\equiv 1,4 \pmod{12}$, $u,v\equiv 7$ *or* $10 \pmod{12}$, *and* $v\geq 3u+1$.

Theorem 1.2. *A* $(v,u;\{4\},2)$*-IPBD exists if and only if* $v\equiv u\equiv 1 \pmod 3$ *and* $v\geq 3u+1$.

Theorem 1.3. *A* $(v,u;\{4\},6)$*-IPBD exists if and only if* $u\geq 1$ *and* $v\geq 3u+1$.

When $\lambda=3$, (*) reduces to $v,u\equiv 0$ or $1 \pmod 4$ or $v,u\equiv 2$ or $3 \pmod 4$ and $v\geq 3u+1$. Wei [8] proved the following theorem.

Theorem 1.4. *A* $(v,u;\{4\},3)$*-IPBD exists if* $v,u\equiv 0$ *or* $1 \pmod 4$ *and* $v\geq 3u+1$.

In this paper we discuss the case $\lambda=3$, $v,u\equiv 2$ or $3 \pmod 4$ and $v\geq 3u+1$. We will first prove the following theorem and then obtain the min theorem, Theorem 1.6.

Theorem 1.5. *A* $(v,u;\{4\},3)$*-IPBD exists if* $v,u\equiv 2$ *or* $3 \pmod 4$ *and* $v\geq 3u+1$.

Theorem 1.6. *A* $(v,u,\{4\},\lambda)$*-IPBD exists if and only if the triple* (v,u,λ) *satisfies (*).*

We use the following notations in this paper.

$$B_3=\{(v,u): v,u\equiv 2 \text{ or } 3 \pmod 4 \text{ and } v\geq 3u+1\}.$$
$$A_3=\{(v,u): \text{ there exists a } (v,u;\{4\},3)-\text{IPBD}\}.$$

2. Constructions use incomplete designs

A design with a missing subdesign is called an incomplete design. In this paper we need some other incomplete designs besides IPBD.

A sub-GDD$(Y,\mathcal{G}',\mathcal{A}')$ of a GDD$(X,\mathcal{G},\mathcal{A})$ is a GDD whose points and blocks are respectively points and blocks of the GDD$(X,\mathcal{G},\mathcal{A})$ and whose every group is contained in some group of the latter. When the sub-GDD is missing, then it becomes an *incomplete* GDD, or IGDD, and denoted by $(X,Y,\mathcal{G},\mathcal{A}\backslash\mathcal{A}')$. Sometimes we denote it by IGDD$[K,\lambda]$ when $|A|\in K$ for every $A\in\mathcal{A}\backslash\mathcal{A}'$, and define its group type to be the multiset of ordered pairs $\{|G|,|G\cap Y|): G\in\mathcal{G}\}$.

A design which is obtained by deleting all blocks of a TD(k,u) from a TD(k,v) is called an *incomplete array* denoted by IA$_{k-2}(v,u)$.

We also employ PBDs which have two subdesigns deleted. These designs are referred to as $\diamondsuit$-IPBDs (see [7]). $\diamondsuit$-IPBD is a quadruple $(X,Y_1,Y_2,\mathcal{A})$, where $Y_1\subset X$, $Y_2\subset X$, and $\mathcal{A}$ is a set of blocks such that every pair of points $\{x,y\}$ occurs in exactly λ blocks, unless $\{x,y\}\subset Y_1$ or $\{x,y\}\subset Y_2$, in which case the pair occurs in no block. We denote it by $(v;w_1,w_2;w_3;K,\lambda)$-$\diamondsuit$-IPBD if $|X|=v$, $|Y_1|=w_1$, $|Y_2|=w_2$, $|Y_1\cap Y_2|=w_3$ and $|A|\in K$ for every $A\in\mathcal{A}$. When $Y_2=\emptyset$ the $\diamondsuit$-IPBD is an IPBD.

The following construction provides a way to obtain an IPBD from an IGDD and some ◇-IPBD (see [3], [7]).

Construction 2.1 (Filling in groups): Let K be a set of positive integers, and let $b \geq a \geq 0$. Suppose that the following designs exist:

1) an IGDD$[K, \lambda]$ of type $\{(s_1, t_1), (s_2, t_2), \ldots, (s_n, t_n)\}$;
2) an $(s_i + b; t_i + a, b; a; K, \lambda)$-◇-IPBD, for $1 \leq i \leq n-1$; and
3) an $(s_n + b, t_n + a; K, \lambda)$-IPBD.

Then there exists an $(s+b, t+a; K, \lambda)$-IPBD, where $s = \sum_{i=1}^{n} s_i$ and $t = \sum_{i=1}^{n} t_i$.

To employ this construction we need some IGDDs to start with, which can be obtained by weighting (see also [3], [7]).

Construction 2.2: Suppose $(X, \mathcal{G}, \mathcal{A})$ is a GDD with index λ, and let $s, t : X \rightarrow Z^+ \cup \{0\}$ be functions such that $t(x) \leq s(x)$, for every $x \in X$. For every block $A \in \mathcal{A}$, suppose that we have an IGDD$[K, \lambda]$ of type $\{(s(x), t(x)) : x \in A\}$. Then there exists a IGDD$[K, \lambda]$ of type $\{(\sum_{x \in G} s(x), \sum_{x \in G} t(x)) : G \in \mathcal{G}\}$.

The following two constructions are the special cases of Construction 2.1 which will be used in this paper.

Construction 2.3: Let K be a set of positive integers and $a \geq 0$. Suppose that the following designs exist:

1) an IGDD$[K, \lambda]$ of type $\{(s_1, t_1), (s_2, t_2), \ldots, (s_n, t_n)\}$; and
2) an $(s_i + a, t_i + a; K, \lambda)$-IPBD, for $1 \leq i \leq n$.

Then there exists an $(s+a, t+a; K, \lambda)$-IPBD, where $s = \sum_{i=1}^{n} s_i$ and $t = \sum_{i=1}^{n} t_i$.

Construction 2.4: Let K be a set of positive integers and $a \geq 0$. Suppose that the following designs exist:

1) a GDD$[K, \lambda]$ of type $\{t_1, t_2, \ldots, t_n\}$; and
2) a $(t_i + a, a; K, \lambda)$-IPBD, for $1 \leq i \leq n-1$.

Then there exists a $(t + a, t_n + a; K, \lambda)$-IPBD, where $t = \sum_{i=1}^{n} t_i$.

3. Constructions of IPBDs

In this section we list the constructions of IPBDs used in this paper. The main recursive constructions of this paper are the following two lemmas, Lemma 3.1 and Lemma 3.3.

Lemma 3.1. *Let m, n, r, a and b be positive integers, where $2 \leq n \leq m$, $3n \leq r \leq 3m + n$, $m \notin \{2, 3, 6, 10\}$. Suppose there exist a $(3m + b; m + a, b; a; \{4\}, 3)$-◇-IPBD and an $(r + b, n + a; \{4\}, 3)$-IPBD. Then there exists a $(v, u; \{4\}, 3)$-IPBD, where $v = 12m + r + b$, $u = 4m + n + a$.*

Proof: For $m \notin \{2, 3, 6, 10\}$ there exists a TD$(5, m)$ (see [2]). Let X be the points set of the TD. Partition one group of this TD into $Y_1 \cup Y_2$ such that $|Y_1| = n$,

$|Y_2| = m - n$. Define $s, t : X \to Z^+ \cup \{0\}$ such that:

$$(s(x), t(x)) = \begin{cases} (3,1) \text{ or } (4,1) & \text{if } x \in Y_1, \\ (3,0) \text{ or } (0,0) & \text{if } x \in Y_2, \\ (3,1) & \text{otherwise.} \end{cases}$$

Applying Construction 2.2 by filling in blocks of IGDD[{4},3] of type $(3,1)^5$, $(3,1)^4(4,1)^1$, $(3,1)^4$ or $(3,1)^4(3,0)^1$ (see [8, Lemma 2.9]), we obtain an IGDD[{4},3] of type $(3m, m)^4(r, n)^1$. Further applying Construction 2.1 we obtain a $(v, u; \{4\}, 3)$-IPBD, where $v = 12m + r + b$, $u = 4m + n + a$.

We need some GDDs for use which we listed in the following lemma.

Lemma 3.2. *There exist GDD[{4},3] of the following types:* $1^8, 1^9, 1^8 2^1, 1^7 3^1, 1^8 3^1, 1^7 3^1 2^1, 1^7 3^2$.

Proof: The designs of type $1^8, 1^9$ come from (8,4,3)-BIBD and (9,4,3)-BIBD. The designs of type $1^8 2^1$, $1^8 3^1$, $1^7 3^1$ come from Appendix A. The designs of types $1^7 3^1 2^1$, $1^7 3^2$ are constructed in Appendix B.

Lemma 3.3. (see [8]) *Suppose there exist* $(w + t_1, t_1; \{4\}, 1)$*-IPBD and* $(w + t_2, t_2; \{4\}, 2)$*-IPBD, where* $t_1 > t_2$ *and* $t_1 - t_2 \equiv 0 \pmod 3$. *Then there exists a* $(w + t_2 + (t_1 - t_2)/3, t_2 + (t_1 - t_2)/3; \{4\}, 3)$*-IPBD.*

Lemma 3.4. *If* $(v, u) \in B_3$ *and* $v \leq 3u + 5$, *then* $(v, u) \in A_3$.

Proof: For $(v, u) \in \{(11,2), (11,3)\}$, see [8]. For $(v, u) \in \{(10,2), (23,6)\}$, see Appendix C. The other designs are listed in Appendix A.

The proof of the following lemma is trivial and is omitted.

Lemma 3.5. *If* $\{(v, w), (w, u)\} \subset A_3$, *then* $(v, u) \in A_3$.

Lemma 3.6. *If* $(v, 2) \in B_3$ *and* $v \leq 23$, *then* $(v, 2) \in A_3$.

Proof: For $v = 7, 10$ and 11, see Lemma 3.4. For $v = 14$, see Table 1. For $v = 15$ or 19, see Appendix C. Delete two points of one group of a TD(5,4) to obtain a GDD[{4,5},1] of type $4^4 2^1$ and use Construction 2.4 with $a = 0$, we obtain $(18, 2) \in A_3$. Since $(v, 7) \in A_3$ for $v = 22$ and 23 (see Lemma 3.4) and $(7,2) \in A_3$, $(v, 2) \in A_3$ by Lemma 3.5.

v	u	w	t_1	t_2	v	u	w	t_1	t_2
14	2	12	4	1	$12n+3, n \geq 2$	6	$12n-3$	10	4
18	3	15	7	1	$12n+6, n \geq 2$	6	$12n$	10	4
27	3	24	7	1	50	11	39	19	7
30	3	27	7	1	62	11	51	19	7
47	11	36	13	10	198	27	171	31	25

Table 1. Applications of Lemma 3.3

In Table 1, the existence of $(w + t_1, t_1; \{4\}, 1)$-IPBD and $(w + t_2, t_2; \{4\}, 2)$-IPBD comes from Theorem 1.1 and 1.2.

Lemma 3.7. *If* $(v,3) \in B_3$ *and* $v \leq 19$, *then* $(v,3) \in A_3$.

Proof: For $v = 10, 11$ and 14, see Lemma 3.4. For $v = 15$, see Appendix C. For $v = 18$, see Table 1. For $v = 19$, see [8].

Lemma 3.8. *Suppose that there exist a* TD$(9,m)$, *an* $(m+a,a;\{4\},3)$*-IPBD and an* $(r+a,a;\{4\},3)$*-IPBD where* $a \geq 0$ *and* $0 \leq r \leq 3m$. *Then* $(8m+2n+r+a, m+2n+a) \in A_3$, *where* $0 \leq n \leq m$. *Moreover, if there also exists an* $(m+2n+a,a;\{4\},3)$*-IPBD, then* $\{(8m+2n+r+a, m+a), (8m+2n+r+a, r+a)\} \subset A_3$.

Proof: Construct a TD$(9,m)$ on point set X with groups $G_1, G_2, \ldots, G_9$. Partition group G_8 into $Y_1 \cup Y_2$ such that $|Y_1| = n$, $|Y_2| = m - n$. Define $s : X \to Z^+ \cup \{0\}$ such that:

$$s(x) = \begin{cases} 3 & \text{if } x \in Y_1, \\ 0,1,2 \text{ or } 3 & \text{if } x \in G_9, \\ 1 & \text{otherwise.} \end{cases}$$

Apply Construction 2.2 by letting $t(x) = 0$. Filling in blocks of the GDDs given in Lemma 3.2, we obtain a GDD$[\{4\},3]$ of type $m^7 t^1 r^1$, where $t = m + 2n$. Further applying Construction 2.4, we complete the proof.

$u \equiv 2 \pmod 4$, $v-u \equiv 0,1 \pmod 4$				$u \equiv 3 \pmod 4$, $v-u \equiv 0,3 \pmod 4$			
$v-u$	u	m	a	$v-u$	u	m	a
56–80	10–24	8	0	56–80	11–23	8	1
61–77	10–26	8	2	63–90	11–27	9	0
77–110	14–34	11	1	72–90	15–31	9	4
88–110	18–38	11	5	91–130	15–39	13	0
112–160	18–46	16	0	100–130	19–43	13	4
133–190	22–58	19	1	119–170	19–51	17	0
144–190	26–62	19	5	128–170	23–55	17	4
161–230	26–70	23	1	136–170	27–59	17	8
189–270	30–82	27	1	175–250	27–75	25	0
217–310	34–94	31	1	203–290	31–87	29	0
224–320	34–96	32	0	224–320	35–95	32	1
241–320	42–104	32	8	231–330	35–99	33	0
301–430	46–130	43	1	259–370	39–111	37	0
329–470	50–142	47	1	287–410	43–123	41	0
$56k$–$80k$	$8k$–$24k$	$8k$	0	343–490	51–147	49	0
	$(k \geq 7)$			371–530	55–159	53	0
				$56k$–$80k$	$8k+1$–$24k+1$	$8k$	1
					$(k \geq 7)$		

Table 2. Applications of Lemma 3.8

In Table 2, the existence of $(m+a,a;\{4\},3)$-IPBD and $(r+a,a;\{4\},3)$-IPBD come from Theorem 1.4 and Lemma 3.6.

The following construction depends on the existence of RBIBD of block size 4 which is proved in [4].

Lemma 3.9. *If* $v - u \equiv 4 \pmod{12}$ *and* $v \geq 4u + 1$, *then* $(v, u) \in A_3$.

Proof: Add u new points into a $(v - u, 4, 1)$-RBIBD such that each new point is added to all the blocks of one parallel class of the RBIBD. Then we obtain a $(v, u; \{4, 5\}, 1)$-IPBD. As $(4, 4, 3)$-BIBD and $(5, 4, 3)$-BIBD exist, we know that $(v, u) \in A_3$.

Lemma 3.10. *Let* $m \geq 4$ *and* $a \geq 0$, *where* $m \notin \{6, 10\}$. *If* $(m+a, a) \in A_3$, *then* $(4m + r + a, r + a) \notin A_3$ *for* $0 \leq r \leq m$. *Moreover, if* $(r + a, a) \in A_3$, *then* $\{(4m + r + a, m + a), (4m + r + a, a)\} \subseteq A_3$.

Proof: For each m there exists a TD$(5, m)$. Delete some points of one group of the TD to obtain a GDD$[\{4, 5\}, 1]$ of type $m^4 r^1$. Then the concluslon follows from Construction 2.4.

v	u	m	r	a	v	u	m	r	a
18	2	4	2	0	59	11	12	11	0
38	6	8	6	0	55	11	11	10	1
50	6	11	5	1	54	14	11	7	3
27	7	5	5	2	55–62	14	12	5–12	2
39	7	8	7	0	50	14	12	0	2
51	7	11	6	1	54	15	13	0	2
75	7	17	7	0	58	15	12	7	3
38	10	7	7	3	59–67	15	13	5–13	2
39	10	8	5	2	70–71	18	15	7–8	3
42	10	8	8	2	70	19	17	0	2
54	10	11	9	1	74	19	16	7	3
42	11	8	7	3	86	26	20	0	6
43	11	8	8	3	87	27	20	0	7

Table 3. Applications of Lemma 3.10

In Table 3, the existence of $(m + a, a; \{4\}, 3)$-IPBDs comes from Appendix C ($m = 20$ and $a = 6$), Theorem 1.4 ($a = 0$ or 1), Lemma 3.6 and 3.7 ($a = 2$ or 3) and this table ($m = 20$ and $a = 7$).

4. Existence of IPBDs for $u = 2, 3, 6, 7$ and 10

In this section we prove the existence of $(v, u; \{4\}, 3)$-IPBDs for $u = 2, 3, 6, 7$ and 10. The main results of this paper are based on these designs by recursive constructions.

Lemma 4.1. *Let* $(v, u) \in B_3$. *If* $u \equiv 2 \pmod{4}$ *and* $(v, u) \in A_3$ *for* $3u + 1 \leq v \leq 3u + 12$, *then* $(v, u) \in A_3$ *for* $v \geq 12u + 47$. *If* $u \equiv 3 \pmod{4}$ *and* $(v, u) \in A_3$ *for* $3u + 1 \leq v \leq 3u + 10$, *then* $(v, u) \in A_3$ *for* $v \geq 12u + 38$.

Proof: For a given value of v, we select a suitable w such that $v - w \equiv 4 \pmod{12}$, $v \geq 4w + 1$ and $(w, u) \in A_3$, then the conclusion comes from Lemma 3.5 and 3.9. We list the details in Table 4, where $m \geq 0$.

u	v	w	u	v	w
4t+2	48t+35+12m	12t+7	4t+3	48t+50+12m	12t+10
	48t+50+12m	12t+10		48t+51+12m	12t+11
	48t+51+12m	12t+11		48t+66+12m	12t+14
	48t+66+12m	12t+14		48t+67+12m	12t+15
	48t+67+12m	12t+15		48t+82+12m	12t+18
	48t+82+12m	12t+18		48t+83+12m	12t+19

Table 4. Proof of Lemma 4.1

Proposition 4.2. *If* $(v,6) \in B_3$, *then* $(v,6) \in A_3$.

Proof: For $v = 19, 22$ and 23 see Lemma 3.4. When $v \equiv 10 \pmod{12}$ and $v \geq 34$, we have $v - 6 \equiv 4 \pmod{12}$, so the conclusion follows from Lemma 3.9. When $v \equiv 3$ or $6 \pmod{12}$ and $v \geq 27$, see Table 1. So we need only consider the case $v \equiv 2, 7$ or $11 \pmod{12}$ and $v \geq 26$. For $v = 26, 35$ and 55 see Appendix C. Delete two points of one group of a TD$(6,5)$ and give weight 2 to each of the remaining three points of this group. As a GDD$[\{4\},3]$ of type $1^5 2^1$ and 1^5 exist (type $1^5 2^1$ is given in Appendix A and 1^5 is trivial), we obtain a GDD$[\{4\},3]$ of type $5^5 6^1$. So we can use Construction 2.4 to contruct a $(31,6;\{4\},3)$-IPBD. For $v = 43$, use Construction 2.1 by starting from an IGDD$[\{4\},3]$ of type $(9,1)^4(6,1)^1$ which comes from the existence of TD$(5,9)$. As $(10;2,1;1;\{4\},3)$-◇-IPBD and $(7,2;\{4\},3)$-IPBD exist, we may use Construction 2.1 by letting $a = b = 1$ to show that $(43,6) \in A_3$. For $v = 47$, use Construction 2.3 with $a = 2$ by starting from an IGDD$[\{4\},3]$ of type $(9,1)^4(9,0)^1$ (see [7]). As $(11,3;\{4\},3)$-IPBD and $(11,2;\{4\},3)$-IPBD exist, it shows that $(47,6) \in A_3$. For $v = 38$ and 50, see Table 3. For $67 \leq v \leq 119$ and $v \equiv 7 \pmod{12}$, let $w = (v-1)/3$, then $(v,w) \in A_3$ by Lemma 3.4. Since $22 \leq w \leq 38$, $(w,6) \in A_3$, and so $(v,6) \in A_3$ by Lemma 3.8. For $59 \leq v \leq 119$ and $v \equiv 11$ or $2 \pmod{12}$, let $w = (v-2)/3$ or $w = (v-5)/3$, then it is easy to see that $(v,6) \in A_3$ in a similar way. For $v \geq 119$, the conclusion comes from Lemma 4.1.

Lemma 4.3. *If* $v \in \{42,50,54,62,63,66\}$, *then* $(v,7) \in A_3$.

Proof: Delete one block from a TD$(6,m)$ to obtain an IGDD$[\{6\},1]$ of type $(m,1)^6$. Denote its groups as (G_i,H_i), $1 \leq i \leq 6$. Define weight $s(x)$ as follows:

$$s(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{i=1}^{5} G_i \\ 2 & \text{if } x \in H_6 \\ 0 \text{ or } 2 & \text{otherwise.} \end{cases}$$

Then an IGDD$[\{4\},3]$ of type $(m,1)^5(2r,2)^1$, where $1 \leq r \leq m-1$, is obtained since there exist GDD$[\{4\},3]$ of type 1^5 and $1^5 2^1$. Let $m = 8$, $r = 1, 5$ or 7. Use Construction 2.3 by letting $a = 0$, this shows that $\{(42,7),(50,7),(54,7)\} \subset A_3$. For $v = 63$, let $m = 9$, $r = 9$ and use Construction 2.3 by letting

$a = 0$. For $v = 62$ and 66, let $m = 11$, $r = 3$ or 5 and use Construction 2.1 by letting $a = 0$ and $b = 1$. This completes the proof.

Proposition 4.4. *If* $(v,7) \in B_3$, *then* $(v,7) \in A_3$.

Proof: For $v = 22, 23$ and 26 see Lemma 3.4. When $v \equiv 7$ or $10 \pmod{12}$ the conclusion comes from Theorem 1.1. When $v \equiv 11 \pmod{12}$ and $v \geq 35$, the conclusion follows from Lemma 3.9. Now we consider the case $v \equiv 2, 3$ or $6 \pmod{12}$ and $v \geq 27$. For $v = 30$ and 38, see Appendix C. For $v = 27, 39, 51$ and 75, see Table 3. For $v = 42$, 50, 54, 62, 63, and 66, see Lemma 4.3. From Lemma 3.4 we know that $(74, 23) \in A_3$, so $(74, 7) \in A_3$ by Lemma 3.5. For $78 \leq v \leq 114$, $(v, 22) \in A_3$ by Table 2, so $(v, 7) \in A_3$ by Lemma 3.5. For $v \geq 122$, the conclusion follows from Lemma 4.1.

Proposition 4.5. *If* $(v, 10) \in B_3$, *then* $(v, 10) \in A_3$.

Proof: For $v = 31, 34$ and 35, see Lemma 3.4. For $v = 38, 39, 42$ and 54, see Table 3. For $v = 47$, use Construction 2.3 by starting from an IGDD[$\{4\}$, 3] of type $(10,2)^4(7,2)^1$ which is obtained by deleting 3 points of an $IA_3(10,2)$ (see [9]). Let $a = 0$, as the $(10, 2; \{4\}, 3)$-IPBD and $(7, 2; \{4\}, 3)$-IPBD exist by Lemma 3.4, this shows $(47, 10) \in A_3$. For $v = 51$, use a similar method by starting from an IGDD[$\{4\}$, 3] of type $(11,2)^4(7,2)^1$ which comes from an $IA_3(11, 2)$. For $v = 59$, add 3 points to every groups of a TD$(8, 7)$ and filling in groups of $(10, 3; \{4\}, 3)$-IPBD. For $v = 63$, see Appendix C. When $v \equiv 7$ or $10 \pmod{12}$, the conclusion comes from Theorem 1.1. When $v \geq 50$ and $v \equiv 2 \pmod{12}$, the conclusion comes from Lemma 3.9. Now consider the case $v \geq 66$ and $v \equiv 3, 6$ or $11 \pmod{12}$. For $66 \leq v \leq 90$, see Table 2. For $95 \leq v \leq 107$, let $m = 7$, $n = 3$, $a = 0$, $b = 1$ and $10 \leq r \leq 22$ in Lemma 3.1. This shows that $(v, 31) \in A_3$, so $(v, 10) \in A_3$ by Lemma 3.5. For $111 \leq v \leq 167$, $(v, 31) \in A_3$ by Table 2, so $(v, 10) \in A_3$ by Lemma 3.5. For $v \geq 169$, the conlusion follows from Lemma 4.1.

Proposition 4.6. *If* $(v, 2) \in B_3$, *then* $(v, 2) \in A_3$.

Proof: For $v \leq 23$, see Lemma 3.6. For $v \geq 26$, $(v, 7) \in A_3$ by Proposition 4.4, so $(v, 2) \in A_3$ by Lemma 3.5.

Proposition 4.7. *If* $(v, 3) \in B_3$, *then* $(v, 3) \in A_3$.

Proof: For $v \leq 19$, see Lemma 3.7. For $v = 22$ and 26 see Appendix C. For $v = 23$ see [8]. For $v = 27$ and 30, see Table 1. For $v \geq 31$, $(v, 10) \in A_3$ by Proposition 4.5, so $(v, 3) \in A_3$ by Lemma 3.5.

5. Recursive Constructions

In this section we use recursive constructions to obtain the main result except some small u which will be proved in the next section.

Lemma 5.1. *Let* $(v,u) \in B_3$. *Then* $(v,u) \in A_3$ *if* (v,u) *satisfies one of the following conditions.*

1) $u \geq 18, u \equiv 2,6$ *or* $14 \pmod{16}$ *and* $v \leq 3u + 3(u-2)/4 - 3$*;*
2) $u \geq 19, u \equiv 3,7$ *or* $15 \pmod{16}$ *and* $v \leq 3u + 3(u-3)/4 - 5$*;*
3) $u \geq 42, u \equiv 10 \pmod{16}$ *and* $v \leq 3u + 3(u-6)/4 - 11$*; or*
4) $u \geq 43, u \equiv 11 \pmod{16}$ *and* $v \leq 3u + 3(u-7)/4 - 13$.

Proof: For case 1), let $m = (u-2)/4$, $n = 2$, $a = 0$ and $b = 1$ in Lemma 3.1. The needed $(3m+1; m, 1; 0; \{4\}, 3)$-◇-IPBD is a $(3m+1, m; \{4\}, 3)$-IPBD and the $(r+1, 2; \{4\}, 3)$-IPBD comes from Proposition 4.6. Similarly, for the other cases, let $a = 0$, $b = 1$ and let $m = (u-3)/4, (u-6)/4$ or $(u-7)/4$, $n = 3, 6$ or 7 in Lemma 3.1 respectively, the needed IPBDs are given in section 4. The proof is complete.

Lemma 5.2. *Let* $(v,u) \in B_3$*, then* $(v,u) \in A_3$ *if* (v,u) *satisfies one of the following conditions.*

1) $u \geq 18, u \equiv 2$ *or* $6 \pmod{16}$ *and* $3u + (u-2)/4 \leq v \leq 4u - 6$*;*
2) $u \geq 19, u \equiv 3,7$ *or* $15 \pmod{16}$ *and* $3u + (u-3)/4 \leq v \leq 4u - 9$*;*
3) $u \geq 42, u \equiv 10 \pmod{16}$ *and* $3u + (u-6)/4 \leq v \leq 4u - 18$*;*
4) $u \geq 43, u \equiv 11 \pmod{16}$ *and* $3u + (u-7)/4 \leq v \leq 4u - 21$*; or*
5) $u \geq 62, u \equiv 14 \pmod{16}$ *and* $3u + (u-10)/4 \leq v \leq 4u - 30$.

Proof: If $m \equiv 0$ or $1 \pmod 4$ and $m \geq 4$, there is a TD$(4, m)$, so there is a $(4m; m, m; 0; \{4\}, 3)$-◇-IPBD. If $m \equiv 3 \pmod 4$ and $m \geq 7$, add one new point to every group of a TD$(4, m)$, then a $(4m+1; m+1, m+1; 1; \{4\}, 3)$-◇-IPBD is obtained easily. Now we can use Lemma 3.1 to prove this Lemma. For case 1), 3), 4) or 5), let $m = (u-2)/4, (u-6)/4, (u-7)/4$ or $(u-10)/4$, $n = 2, 6, 7$ or 10, $a = 0$ and $b = m$ in Lemma 3.1. For $u \equiv 3$ or $7 \pmod{16}$, let $m = (u-3)/4$, $n = 3$, $a = 0$ and $b = m$ in Lemma 3.1. For $u \equiv 15 \pmod{16}$, let $m = (u-3)/4$, $n = 2$, $a = 1$ and $b = m+1$ in Lemma 3.1.

By Lemmas 5.1 and 5.2 we can obtain the following lemma.

Lemma 5.3. *Let* $(v,u) \in B_3$*, then* $(v,u) \in A_3$ *if* (v,u) *satisfies one of the following conditions.*

1) $u \geq 18, u \equiv 2$ *or* $6 \pmod{16}$ *and* $3u + 1 \leq v \leq 4u - 6$*;*
2) $u \geq 19, u \equiv 3,7$ *or* $15 \pmod{16}$ *and* $3u + 1 \leq v \leq 4u - 9$*;*
3) $u \geq 42, u \equiv 10 \pmod{16}$ *and* $3u + 1 \leq v \leq 4u - 18$*;*
4) $u \geq 43, u \equiv 11 \pmod{16}$ *and* $3u + 1 \leq v \leq 4u - 21$*; or*
5) $u \geq 62, u \equiv 14 \pmod{16}$ *and* $3u + 1 \leq v \leq 4u - 30$.

Lemma 5.4. *Suppose that there exists a resolvable* GDD$[\{4\}, 1]$ *of type* t^5 *then there exists an* IGDD$[\{4\}, 3]$ *of type* $(3t, t)^5 (r, n)^1$, *where* $2 \leq n \leq t(s-1)/3$ *and* $3n \leq r \leq t(s-1) + n$.

Proof: Add $t(s-1)/3$ new points to the resolvable GDD of type t^5 to obtain a GDD[$\{5\}$, 1] of type $t^5(t(s-1)/3)^1$. Denote the group of size $t(s-1)/3$ by G. Let $G' \subset G$ such that $|G'| = n$. Define weight $((s(x), t(x))$ as follows:

$$(s(x),t(x)) = \begin{cases} (3,1) & x \notin G, \\ (3,1) \text{ or } (4,1) & x \in G', \\ (0,0) \text{ or } (3,0) & x \in G \backslash G'. \end{cases}$$

Applying Construction 2.2 in a manner similar to what was done in the proof of Lemma 3.1, completes the proof.

Lemma 5.5. *If* $v \in \{151\} \cup \{v \geq 163 : v \equiv 2 \text{ or } 3 \pmod 4\}$, *then* $(v, 42) \in A_3$.

Proof: From a resolvable GDD[$\{4\}$, 1] of type 4^{10} (in fact it is a $(40,4,1)$-RBIBD, see [4]), we have an IGDD[$\{4\}$,3] of type $(12,4)^{10}(30,2)^1$ by Lemma 5.4. Using Construction 2.1 by letting $a = 0$ and $b = 1$, then a $(151, 42; \{4\}, 3)$-IPBD is obtained. By deleting some points from an $IA_3(4k, 9)$, where $k \geq 9$ (see [9]) we can easily construct an IGDD[$\{4,5\}$, 1] of type $(4k, 9)^4(r, 6)^1$, where $19 \leq r \leq 4k-3$ and $r \equiv 2$ or $3 \pmod 4$. As $\{(4k, 9), (r, 6)\} \subset A_3$, we can use Construction 2.3 by letting $a = 0$ to show that when $(v, 42) \in B_3$ and $16k + 19 \leq v \leq 20k - 5$ i.e. $v \geq 163, v \neq 178$ and $v \equiv 2$ or $3 \pmod 4$, $(v, 42) \in A_3$. But from Lemma 3.9 we know that $(178, 42) \in A_3$.

In Table 2 the case $u \equiv 3 \pmod 4$, $v - u = 171$ and 172 are not covered, so we need the following lemma.

Lemma 5.6. *If* $u \equiv 3 \pmod 4$ *and* $31 \leq u \leq 59$, *then* $\{(171 + u, u), (172 + u, u)\} \subset A_3$.

Proof: Since $172 \equiv 4 \pmod{12}$, $(172 + u, u) \in A_3$ when $31 \leq u < 59$ by Lemma 3.9. Let $m = 43$, $a = 19$ and $r = 40$ in Lemma 3.10, as $(62, 19) \in A_3$ by Lemma 5.2, then $(172 + 59, 59) \in A_3$. For $u = 31, 43$ and 55 $(171 + u, u) \in A_3$ by Theorem 1.1. From $IA_3(49, 8)$, $IA_3(45, 8)$ (see [9]), we may obtain IGDD[$\{4\}$,3] of type $(49, 8)^4(10, 3)^1$ and $(45, 8)^4(30, 7)^1$. So $(171 + u, u) \in A_3$ for $u = 35$ and 39 by Construction 2.3. From a TD$(5, 44)$ we obtain a GDD[$\{4\}$,3] of type $44^4 39^1$. Use Construction 2.4 by letting $a = 3, 7$ or 15. This shows that $(171 + u, u) \in A_3$ for $u \in \{47, 51, 59\}$.

Now we are in a position to prove the main result of this section.

Proposition 5.7. *If* $(v, u) \in B_3$, $u = 22$ *or* $u \geq 31$ *and* $v \leq 9u + 4$, *then* $(v, u) \in A_3$.

Proof: For $(v, u) \in B_3$, $u = 22$ or $31 \leq u \leq 111$ and $v \leq 9u + 4$, we use Lemmas 5.1–5.6 and Table 2 to prove the conclusion. The details are listed in Table 5. When $u \geq 114$, Lemma 5.3 shows that for $3u + 1 \leq v \leq 4u - 30$,

$(v,u) \in A_3$. Since the maximum distance between two successive values of m in Table 2, when $m \geq 43$, is at most 8, we can choose m in Table 2 such that $\max\{43, u/3\} \leq m \leq (3u-30)/7$, i.e., $m+2 \leq u < 3m$ and $7m \leq 3u-30$. On the other hand, we can choose another m in Table 2 such that $(8u+4)/10 \leq m \leq u-2$, i.e., $m+2 \leq u \leq 3m$, and $8u+4 \leq 10m$. Note that in Table 2, when $m+2 \leq u \leq 3m$, where $m \geq 43$, then it shows $(v,u) \in A_3$ for $7m \leq v-u \leq 10m$. Also note that in Table 2, the values of $v-u$ fill all the positive integers greater than 174. So Table 2 shows that when $u \geq 114$ and $4u-3 \leq v \leq 9u+4$, $(v,u) \in A_3$. For the same reason, when $v \geq 70$ we have not listed the upper bound of v from Table 2. The conclusion follows.

u	v		u	v	
	L. 5.1,5.2	Table 2		L. 5.1,5.2	Table 2
22	67–82	78–212	71	214–275	246–
31	94–115	103–201,206–321	74	223–278	263–
34	103–130	111–354	75	226–279	250–
35	106–131	126–205,210–365	78	235–282	267–
38	115–146	126–358	79	238–307	282–
39	118–147	130–209,214–409	82	247–322	271–
42	127–150	154–362	83	250–323	286–
43	130–151	143–213,218–453	86	259–338	303–
46	139–168	158–476	87	262–339	290–
47	142–179	166–217,222–457	90	271–342	307–
50	151–194	183–520	91	274–343	315–
51	154–195	170–221,226–541	94	283–346	311–
54	163–210	187–524	95	286–371	319–
55	166–211	183–225,230–585	98	295–386	339–
58	175–214	191–618	99	298–387	330–
59	178–215	195–229,234–619	102	307–402	343–
62	187–218	206–622	103	310–403	362–
63	190–243	238–623	106	319–406	407–
66	199–258	227–706	107	322–407	366–
67	202–259	242–706	110	331–410	411–
70	211–274	231–	111	334–435	370–

Table 5. Proof of Proposition 5.7

6. Proof of Theorem 1.5 and Theorem 1.6

In this section, we first prove that when $u \in \{11, 14, 15, 18, 19, 23, 26, 27, 30\}$ and $(v,u) \in B_3$, then $(v,u) \in A_3$. From which we complete the proof of Theorem 1.5. Then we give the proof of Theorem 1.6.

Lemma 6.1. *If there exists a* TD$(6,m)$, *then there exists a* GDD$[\{4\},3]$ *of type* $m^5(2r)^1$, *where* $0 \leq r \leq m$.

Proof: Give r points of one group of a TD$(6,m)$ weight 2 and the remainders weight 0. Give the points of other 5 groups weight 1. As there exist GDD$[\{4\},3]$ of type 1^5 and $1^5 2^1$, the conclusion follows from Construction 2.2.

Lemma 6.2. *If there exists a* TD$(6,m)$, *then there exists a* GDD$[\{4\},3]$ *of type* $m^4(2m)^1(m+r)^1$, *where* $0 \leq r \leq m$.

Proof: Give r points of one group of a TD$(6,m)$ weight 2. Give another group of the TD weight 2 and the others of this TD weight 1. As there exist GDD$[\{4\},3]$ of type $1^5 2^1$ and $1^4 2^2$ (see Appendix B), the conclusion follows from Construction 2.2.

Proposition 6.3. *If* $(v,11) \in B_3$ *and* $34 \leq v \leq 102$, *then* $(v,11) \in A_3$.

Proof: For $34 \leq v \leq 38$, see Lemma 3.4. A $(39,11;\{4\},3)$-IPBD is constructed in [8]. For $v \in \{42,43,55,59\}$, see Table 3. For $v \in \{47,50,62\}$, see Table 1. Use Lemma 6.1 by letting $r = 4$ and $m = 7, 8$ or 11 to obtain a GDD$[\{4\},3]$ of type $m^5 8^1$. Make use of Construction 2.4 from these GDDs by letting $a = 3$. This shows $(v,11) \in A_3$ for $v \in \{46,51,66\}$. $(63,11) \in A_3$ comes from Lemma 3.9. For $v \in \{54,58\}$, see Appendix C. For $67 \leq v \leq 101$ see Table 2. Let $m = 9$, $a = 2$, $n = 2$ and $r = 24$ in Lemma 3.8. This shows $(102,11) \in A_3$.

Proposition 6.4. *If* $(v,14) \in B_3$ *and* $43 \leq v \leq 129$, *then* $(v,14) \in A_3$.

Proof: For $43 \leq v \leq 47$, see Lemma 3.4. For $50 \leq v \leq 62$ and $v \neq 51$, see Table 2. For $v \in \{51,67\}$, see Appendix C. Give 2 points of one group of a TD$(8,7)$ weight 3 and the others of this TD weight 1 and use the GDD$[\{4\},3]$ of type 1^8, $1^7 3^1$, to obtain a GDD$[\{4\},3]$ of type $7^7 11^1$. Now with $a = 3$ in Construction 2.4 we obtain $(63,14) \in A_3$. For $v \in \{66,126\}$, the conclusion comes from Lemma 3.9. Let $m = 11$, $a = 3$, $n = 2$ and $r = 32$ in Lemma 3.8, then $(127,14) \in A_3$. For $70 \leq v \leq 124$, see Table 2.

Proposition 6.5. *If* $(v,15) \in B_3$ *and* $46 \leq v \leq 138$, *then* $(v,15) \in A_3$.

Proof: For $46 \leq v \leq 50$, see Lemma 3.4. Let $m = 7$ and $r = 1$ in Lemma 6.2, then a GDD of type $7^4 14^1 8^1$ is obtained. Use Construction 2.4 by letting $a = 1$, then $(51,15) \in A_3$. For $54 \leq v \leq 67$, $v \neq 55$, see Table 3. Let $m = 8$ or 11, $r = 7$ in Lemma 6.1 to obtain GDDs of type $8^5 14^1$ or $11^5 14^1$. Use Construction 2.4 with these GDDs by letting $a = 1$. This shows $(v,15) \in A_3$ for $v \in \{55,70\}$. For $71 \leq v \leq 138$, see Table 2.

Proposition 6.6. *If* $(v,18) \in B_3$ *and* $55 \leq v \leq 165$, *then* $(v,18) \in A_3$.

Proof: For $55 \leq v \leq 66$, the conclusion comes from Lemmas 5.3. Let $m = 9$ and $r = 4$ in Lemma 6.2 to obtain a GDD of type $9^4 18^1 13^1$. Use of Construction 2.4 with this GDD and with $a = 0$ shows that $(67,18) \in A_3$. For $v \in \{70,71\}$, see Table 3. For $74 \leq v \leq 165$, see Table 2.

Proposition 6.7. *If* $(v,19) \in B_3$ *and* $58 \leq v \leq 174$, *then* $(v,19) \in A_3$.

Proof: For $58 \leq v \leq 67$, the conclusion comes from Lemma 5.3. For $v \in \{70,74\}$, see Table 3. For $v = 71$, use Lemma 3.5 by letting $m = r = 13$ and

$a = 6$. As $(19, 6; \{4\}, 3)$-IPBD exists, $(71, 19) \in A_3$. For $75 \leq v \leq 174$, see Table 2.

Proposition 6.8. *If* $(v, 23) \in B_3$ *and* $70 \leq v \leq 210$, *then* $(v, 23) \in A_3$.

Proof: For $70 \leq v \leq 83$, the conclusion comes from Lemma 5.3. For $83 \leq v \leq 191$, see Table 2. Let $m = 17$, $a = 6$, $n = 16$ and $20 \leq r \leq 36$ in Lemma 3.8 to show $(v, 23) \in A_3$ for $194 \leq v \leq 210$.

Proposition 6.9. *If* $(v, 26) \in B_3$ *and* $79 \leq v \leq 237$, *then* $(v, 26) \in A_3$.

Proof: For $79 \leq v \leq 83$, see Lemma 3.4. For $v = 86$ see Table 3. For $87 \leq v \leq 237$, see Table 2.

Proposition 6.10. *If* $(v, 27) \in B_3$ *and* $82 \leq v \leq 246$, *then* $(v, 27) \in A_3$.

Proof: For $82 \leq v \leq 86$, see Lemma 3.4. For $v = 87$, see Table 3. For $90 \leq v \leq 195$ and $202 \leq v \leq 246$, see Table 2. For $v = 198$ see Table 1. From Lemma 3.9, we know that $(199, 27) \in A_3$.

Proposition 6.11. *If* $(v, 30) \in B_3$ *and* $91 \leq v \leq 273$, *then* $(v, 30) \in A_3$.

Proof: If $91 \leq v \leq 108$, then $(v, 30) \in A_3$ from Lemma 5.1. For $107 \leq v \leq 273$, see Table 1.

Now we are in the position to prove the main results of this paper. We restate the theorems here for the readers convenience.

Theorem 1.5. *A* $(v, u; \{4\}, 3)$*-IPBD exists if* $v, u \equiv 2$ *or* $3 \pmod 4$ *and* $v \geq 3u + 1$.

Proof: In section 4, we have proved that for $u \in \{2, 3, 6, 7, 10\}$ the conclusion is true. From Propositions 6.3–6.11, we know that when $u \in \{11, 14, 15, 18, 19, 23, 26, 27, 30\}$ and $v < 9u + 4$, $(v, u) \in A_3$. Combining the results of section 5 we obtain that if $(v, u) \in B_3$ and $v < 9u + 4$, then $(v, u) \in A_3$. Now if $v \geq 9u + 4$, we can choose u', $u' \equiv 2$ or $3 \pmod 4$ with $u' \geq 3u + 1$, such that $3u' \leq v < 9u + 4$, then $(v, u') \in A_3$. If $u' < 9u + 4$, then $(v, u) \in A_3$ by Lemma 3.5. If $u' \geq 9u + 4$, we can choose u'' with $u'' \geq 3u + 1$ and such that $3u'' + 1 \leq u' < 9u'' + 4$, so $(u', u'') \in A_3$ and then $(v, u'') \in A_3$. Following this method we can prove that $(v, u) \in A_3$ after finite steps. This completes the proof.

Theorem 1.6. *A* $(v, u, \{4\}, \lambda)$*-IPBD exists if and only if the triple* (v, u, λ) *satisfies* $(*)$.

Proof: The condition $(*)$ can be simplified according to the congruences of modulo 6. In addition to $v \geq 3u + 1$ the conditions are:

1) If $\lambda \equiv 1$ or $5 \pmod 6$, then $v, u \equiv 1$ or $4 \pmod{12}$ or $v, u \equiv 7$ or $10 \pmod{12}$;

2) If $\lambda \equiv 2$ or $4 \pmod 6$, then $v, u \equiv 1 \pmod 3$;
3) If $\lambda \equiv 3 \pmod 6$, then $v, u \equiv 0$ or $1 \pmod 4$ or $v, u \equiv 2$ or $3 \pmod 4$;
4) If $\lambda \equiv 0 \pmod 6$, then $v \geq 4$.

In the above four classes, $\lambda = 1, 2, 3$ and 6 are the smallest cases and are already proved. For other cases we may prove the conclusion by repeating blocks. For example, when $\lambda \equiv 2$ or $4 \pmod 6$ and $\lambda > 2$, repeat each block of a $(v, u; \{4\}, 2)$-IPBD $\lambda/2$ times to obtain a $(v, u; \{4\}, \lambda)$-IPBD. The proof is completed.

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Appendix A: Some classes of IPBDs

$(12t+7, 4t+2; \{4\}, 3)$-IPBD
points: $Z_{8t+5} \cup \{\infty_i : 1 \leq i \leq 4t+2\}$
blocks: $\{\{\infty_i, 0, i, 2i+2\} : 1 \leq i \leq 4t\}$
$\{\infty_{4t+1}, 0, 1, 2\}$ $\{\infty_{4t+2}, 0, 2, 4t+3\}$ mod $8t+5$

$(12t+10, 4t+2; \{4\}, 3)$-IPBD
points: $Z_{8t+8} \cup \{\infty_i : 1 \leq i \leq 4t+2\}$
blocks: $\{\{0, 2t+2, 4t+4, 6t+6\}$: repeat 3 times$\}$
$\{\{\infty_i, 0, i, 2i+1\} : 1 \leq i \leq 2t\}$ $\{\{\infty_i, 0, i, 2i\} : 2t+3 \leq i \leq 3t+1\}$
$\{\{\infty_i, 0, i, 2i+2\} : 3t+3 \leq i \leq 4t+1\}$ $\{\infty_{2t+1}, 0, 2t+1, 4t+5\}$
$\{\infty_{2t+2}, 0, 3t+2, 3t+3\}$ $\{\infty_{3t+2}, 0, 3t+2, 3t+4\}$
$\{\infty_{4t+2}, 0, 4t+2, 4t+3\}$ mod $8t+8$

$(12t+11, 4t+2; \{4\}, 3)$-IPBD
points: $Z_{8t+9} \cup \{\infty_i : 1 \leq i \leq 4t+2\}$
blocks: $\{0, 2t+2, 4t+4, 6t+6\}$
$\{\{\infty_i, 0, i, 2i+1\} : 1 \leq i \leq 4t+2, i \neq t+2, 2t+1, 2t+2, 3t+3\}$
$\{\infty_{t+2}, 0, t+2, t+3\}$ $\{\infty_{2t+1}, 0, 2t+1, 6t+4\}$
$\{\infty_{2t+2}, 0, 4t+3, 8t+7\}$ $\{\infty_{3t+3}, 0, 3t+3, 3t+4\}$ mod $8t+9$

$(12t+10, 4t+3; \{4\}, 3)$-IPBD
points: $Z_{8t+7} \cup \{\infty_i : 1 \leq i \leq 4t+3\}$
blocks: $\{\{\infty_i, 0, i, 2i+2\} : 1 \leq i \leq 4t+1\}$ $\{\infty_{4t+2}, 0, 1, 2\}$
$\{\infty_{4t+3}, 0, 2, 4t+4\}$ mod $8t+7$

$(12t+11, 4t+3; \{4\}, 3)$-IPBD
points: $Z_{8t+8} \cup \{\infty_i : 1 \leq i \leq 4t+3\}$
blocks: $\{0, 2t+2, 4t+4, 6t+6\}$ $\{\{\infty_i, 0, i, 2i+1\} : 1 \leq i \leq 2t\}$
$\{\{\infty_i, 0, i, 2i\} : 2t+3 \leq i \leq 3t+1\}$
$\{\{\infty_i, 0, i, 2i+2\} : 3t+3 \leq i \leq 4t+1\}$ $\{\infty_{2t+1}, 0, 2t+1, 4t+5\}$
$\{\infty_{2t+2}, 0, 3t+2, 3t+3\}$ $\{\infty_{3t+2}, 0, 3t+2, 3t+4\}$
$\{\infty_{4t+2}, 0, 4t+2, 4t+3\}$ $\{\infty_{4t+3}, 0, 2t+2, 4t+4\}$ mod $8t+8$

$(12t+14, 4t+3; \{4\}, 3)$-IPBD
points: $Z_{8t+11} \cup \{\infty_i : 1 \leq i \leq 4t+3\}$
blocks: $\{0, 1, 2, 4t+6\}$ $\{\{\infty_i, 0, i, 2i+1\} : 1 \leq i \leq 4t+3\}$ mod $8t+11$

Appendix B: Some CDD[{4},3]

Type $1^7 3^1 2^1$

groups: $\{a_1, a_2\}$ $\{b_1, b_2, b_3\}$ $\{\{i\} : 0 \le i \le 6\}$

blocks: $\{0,1,3,6\}$ $\{0,2,4,5\}$ $\{a_1,1,4,5\}$ $\{a_2,2,3,6\}$ $\{b_1,a_1,0,2\}$
$\{b_1,a_1,4,6\}$ $\{b_1,a_2,3,4\}$ $\{b_1,1,2,4\}$ $\{b_1,a_1,3,5\}$ $\{b_1,a_2,1,2\}$
$\{b_1,a_2,0,5\}$ $\{b_1,0,1,6\}$ $\{b_1,3,5,6\}$ $\{b_2,a_1,0,6\}$ $\{b_2,a_2,0,4\}$
$\{b_2,0,2,5\}$ $\{b_2,a_1,1,4\}$ $\{b_2,a_2,2,6\}$ $\{b_2,1,5,6\}$ $\{b_2,a_1,1,3\}$
$\{b_2,a_2,3,5\}$ $\{b_2,2,3,4\}$ $\{b_3,a_1,0,3\}$ $\{b_3,a_2,0,1\}$ $\{b_3,0,3,4\}$
$\{b_3,a_1,2,6\}$ $\{b_3,a_2,4,6\}$ $\{b_3,1,2,3\}$ $\{b_3,a_1,2,5\}$ $\{b_3,a_2,1,5\}$
$\{b_3,4,5,6\}$

Type $1^7 3^2$

groups: $\{a_1, a_2, a_3\}$ $\{b_1, b_2, b_3\}$ $\{\{i\} : 0 \le i \le 6\}$

blocks: $\{1,3,5,6\}$ $\{2,3,4,5\}$ $\{1,2,4,6\}$ $\{a_1,0,1,4\}$ $\{a_2,0,2,5\}$
$\{a_3,0,3,6\}$ $\{b_1,0,1,2\}$ $\{b_2,0,3,4\}$ $\{b_3,0,5,6\}$ $\{b_1,a_1,0,5\}$
$\{b_1,a_2,0,6\}$ $\{b_1,a_3,1,5\}$ $\{b_2,a_1,1,2\}$ $\{b_2,a_2,0,1\}$ $\{b_2,a_3,0,2\}$
$\{b_1,a_1,2,3\}$ $\{b_1,a_2,1,3\}$ $\{b_1,a_3,2,4\}$ $\{b_2,a_1,4,5\}$ $\{b_2,a_2,2,3\}$
$\{b_2,a_3,1,6\}$ $\{b_1,a_1,4,6\}$ $\{b_1,a_2,4,5\}$ $\{b_1,a_3,3,6\}$ $\{b_2,a_1,5,6\}$
$\{b_2,a_2,4,6\}$ $\{b_2,a_3,3,5\}$ $\{b_3,a_1,0,3\}$ $\{b_3,a_2,3,4\}$ $\{b_3,a_3,0,4\}$
$\{b_3,a_1,1,3\}$ $\{b_3,a_2,2,6\}$ $\{b_3,a_3,2,5\}$ $\{b_3,a_1,2,6\}$ $\{b_3,a_2,1,5\}$
$\{b_3,a_3,1,4\}$

Type $1^4 2^2$

groups: $\{4,a\}$ $\{5,b\}$ $\{\{i\} : 0 \le i \le 3\}$

blocks: $\{0,1,2,3\}$ $\{b,4,0,1\}$ $\{b,a,2,3\}$ $\{b,4,1,2\}$ $\{b,a,3,0\}$ $\{b,4,2,3\}$
$\{b,a,0,1\}$ $\{5,4,0,2\}$ $\{5,a,1,2\}$ $\{5,4,0,3\}$ $\{5,a,1,3\}$ $\{5,4,1,3\}$
$\{5,a,0,2\}$

Appendix C: Some IPBDs

$(10,2;\{4\},3)$-IPBD

points: $Z_8 \cup \{\infty_i : 1 \le i \le 2\}$

blocks: $\{0,1,4,5\}$ $\{0,2,4,6\}$ $\{0,1,3,\infty_1\}$ $\{0,1,3,\infty_2\}$ mod 8

$(15,2;\{4\},3)$-IPBD

points: $Z_{13} \cup \{\infty_i : 1 \le i \le 2\}$

blocks: $\{0,1,3,6\}$ $\{0,1,4,6\}$ $\{0,1,5,\infty_1\}$ $\{0,2,6,\infty_2\}$ mod 13

$(19,2;\{4\},3)$-IPBD

points: $Z_{17} \cup \{\infty_i : 1 \le i \le 2\}$

blocks: $\{0,1,5,8\}$ $\{0,1,4,6\}$ $\{0,2,5,9\}$ $\{0,1,7,\infty_1\}$ $\{0,2,8,\infty_2\}$ mod 17

$(15,3;\{4\},3)$-IPBD

points: $Z_{12} \cup \{\infty_i : 1 \le i \le 3\}$

blocks: $\{0,3,6,9\}$ $\{0,1,3,6\}$ $\{\{0,1,5,\infty_i\} : i = 1,2\}$ $\{0,2,4,\infty_3\}$ mod 12

(22, 3; {4}, 3)-IPBD
points: $Z_{19} \cup \{\infty_i : 1 \le i \le 3\}$
blocks: $\{0,1,5,8\}$ $\{0,2,6,11\}$ $\{0,2,5,9\}$ $\{0,6,13,\infty_1\}$ $\{0,1,3,\infty_2\}$
$\{0,1,9,\infty_3\}$ mod 19

(26, 3; {4}, 3)-IPBD
points: $Z_{23} \cup \{\infty_i : 1 \le i \le 3\}$
blocks: $\{0,2,5,9\}$ $\{0,4,11,21\}$ $\{0,6,14,18\}$ $\{0,7,10,15\}$ $\{0,1,2,\infty_1\}$
$\{0,1,11,\infty_2\}$ $\{0,3,9\infty_3\}$ mod 23

(23, 6; {4}, 3)-IPBD
points: $Z_{17} \cup \{\infty_i : 1 \le i \le 6\}$
blocks: $\{0,1,3,7\}$ $\{0,1,2,\infty_1\}$ $\{0,2,7,\infty_2\}$ $\{0,3,8,\infty_3\}$ $\{0,4,9,\infty_4\}$
$\{0,3,9,\infty_5\}$ $\{0,4,10,\infty_6\}$ mod 17

(26, 6; {4}, 3)-IPBD
points: $Z_{20} \cup \{\infty_i : 1 \le i \le 6\}$
blocks: $\{0,5,15,10\}$ repeat 3 times
$\{0,2,3,9\}$ $\{0,4,12,\infty_1\}$ $\{0,2,11,\infty_2\}$ $\{0,6,14,\infty_3\}$ $\{\{0,3,7,\infty_i\} :$
$i = 4,5\}$ $\{0,1,2,\infty_6\}$ mod 20

(35, 6; {4}, 3)-IPBD
points: $Z_{29} \cup \{\infty_i : 1 \le i \le 6\}$
blocks: $\{\{0,1,6,14\}$: repeat 2 times$\}$ $\{\{0,2,9,12\}$: repeat 2 times$\}$
$\{0,1,6,\infty_1\}$ $\{0,2,13,\infty_2\}$ $\{0,3,12,\infty_3\}$ $\{0,4,14,\infty_4\}$ $\{0,7,18,\infty_5\}$
$\{0,4,8,\infty_6\}$ mod 29

(55, 6; {4}, 3)-IPBD
points: $Z_{49} \cup \{\infty_i : 1 \le i \le 6\}$
blocks: $\{0,1,10,21\}$ $\{0,2,6,19\}$ $\{0,5,8,12\}$ $\{0,2,16,34\}$ $\{0,1,23,47\}$
$\{0,5,11,35\}$ $\{0,7,19,40\}$ $\{0,8,18,41\}$ $\{0,9,22,32\}$ $\{0,1,13,\infty_1\}$
$\{0,4,11,\infty_2\}$ $\{0,3,18,\infty_3\}$ $\{0,5,29,\infty_4\}$ $\{0,6,28,\infty_5\}$
$\{0,14,34,\infty_6\}$ mod 49

(30, 7; {4}, 3)-IPBD
points: $Z_{23} \cup \{\infty_i : 1 \le i \le 7\}$
blocks: $\{0,1,3,11\}$ $\{0,4,9,15\}$ $\{0,7,26,\infty_1\}$ $\{0,2,5,\infty_2\}$ $\{0,4,10,\infty_3\}$
$\{0,1,2,\infty_4\}$ $\{0,3,11,\infty_5\}$ $\{0,4,9,\infty_6\}$ $\{0,6,16,\infty_7\}$ mod 23

(38, 7; {4}, 3)-IPBD
points: $Z_{31} \cup \{\infty_i : 1 \le i \le 7\}$
blocks: $\{0,1,7,15\}$ $\{0,2,15,13\}$ $\{0,4,13,25\}$ $\{0,1,3,15\}$ $\{0,1,10,\infty_1\}$
$\{0,4,9,\infty_2\}$ $\{0,2,14,\infty_3\}$ $\{0,3,13,\infty_4\}$ $\{0,5,11,\infty_5\}$ $\{0,7,15,\infty_6\}$
$\{0,4,11,\infty_7\}$ mod 31

$(63,10;\{4\},3)$-IPBD

points: $Z_{53} \cup \{\infty_i : 1 \le i \le 10\}$

blocks: $\{0,1,11,23\}$ $\{0,2,15,29\}$ $\{0,3,19,20\}$ $\{0,4,22,25\}$ $\{0,6,13,18\}$ $\{0,7,16,21\}$ $\{0,8,23,25\}$ $\{0,9,17,28\}$ $\{0,13,29,\infty_1\}$ $\{0,4,22,\infty_2\}$ $\{0,20,23,\infty_3\}$ $\{0,7,26,\infty_4\}$ $\{0,1,12,\infty_5\}$ $\{0,2,10,\infty_6\}$ $\{0,4,9,\infty_7\}$ $\{0,6,21,\infty_8\}$ $\{0,10,24,\infty_9\}$ $\{0,6,26,\infty_{10}\}$ mod 53

$(54,11;\{4\},3)$-IPBD

points: $Z_{43} \cup \{\infty_i : 1 \le i \le 11\}$

blocks: $\{0,1,10,21\}$ $\{0,2,6,19\}$ $\{0,5,8,12\}$ $\{0,3,15,33\}$ $\{0,7,21,23\}$ $\{0,11,27,\infty_1\}$ $\{0,2,23,\infty_2\}$ $\{0,9,19,\infty_3\}$ $\{0,1,13,\infty_4\}$ $\{0,1,15,\infty_5\}$ $\{0,3,7,\infty_6\}$ $\{0,5,11,\infty_7\}$ $\{0,5,19,\infty_8\}$ $\{0,6,15,\infty_9\}$ $\{\{0,8,26,\infty_i\} : i = 10,11\}$ mod 43

$(58,11;\{4\},3)$-IPBD

points: $Z_{47} \cup \{\infty_i : 1 \le i \le 11\}$

blocks: $\{0,1,12,22\}$ $\{0,2,15,23\}$ $\{0,3,9,27\}$ $\{0,4,7,13\}$ $\{0,7,11,25\}$ $\{0,14,19,31\}$ $\{0,1,15,\infty_1\}$ $\{0,1,17,\infty_2\}$ $\{0,7,28,\infty_3\}$ $\{0,11,31,\infty_4\}$ $\{0,3,8,\infty_5\}$ $\{0,2,10,\infty_6\}$ $\{0,6,18,\infty_7\}$ $\{0,9,22,\infty_8\}$ $\{0,4,27,\infty_9\}$ $\{0,2,19,\infty_{10}\}$ $\{0,5,15,\infty_{11}\}$ mod 47

$(51,14;\{4\},3)$-IPBD

points: $Z_{37} \cup \{\infty_i : 1 \le i \le 14\}$

blocks: $\{0,1,7,18\}$ repeat 2 times $\{0,1,7,\infty_1\}$ $\{0,2,17,\infty_2\}$ $\{0,3,13,\infty_3\}$ $\{0,4,8,\infty_4\}$ $\{0,4,9,\infty_5\}$ $\{0,8,16,\infty_6\}$ $\{0,9,18,\infty_7\}$ $\{0,11,25,\infty_8\}$ $\{\{0,2,14,\infty_i\} : i = 9,10\}$ $\{\{0,3,16,\infty_i\} : i = 11,12\}$ $\{\{0,5,15,\infty_i\} : i = 13,14\}$ mod 37

$(67,14;\{4\},3)$-IPBD

points: $Z_{53} \cup \{\infty_i : 1 \le i \le 14\}$

blocks: $\{0,1,11,23\}$ $\{0,2,15,29\}$ $\{0,3,19,20\}$ $\{0,4,22,25\}$ $\{0,5,9,17\}$ $\{0,6,19,21\}$ $\{0,5,11,\infty_1\}$ $\{0,7,27,\infty_2\}$ $\{0,8,24,\infty_3\}$ $\{0,9,23,\infty_4\}$ $\{0,7,25,\infty_5\}$ $\{0,10,20,\infty_6\}$ $\{0,1,7,\infty_7\}$ $\{0,2,19,\infty_8\}$ $\{0,3,11,\infty_9\}$ $\{0,4,29,\infty_{10}\}$ $\{0,5,23,\infty_{11}\}$ $\{0,9,21,\infty_{12}\}$ $\{0,13,27,\infty_{13}\}$ $\{0,15,31,\infty_{14}\}$ mod 53