On the smallest (1, 2)-eulerian weight of graphs

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Abstract. A weight $w: E(G) \to \{1,2\}$ is called a $(1,2)$-eulerian weight of graph G if the total weight of each edge-cut is even. A $(1,2)$-eulerian weight w of G is called smallest if the total weight w of G is minimum. In this note, we prove that if graph G is 2-connected and simple, and w_0 is a smallest $(1,2)$-eulerian weight, then either $|E_{w_0=\text{even}}| \leq |V(G)| - 3$ or $G = K_4$.

Let $G = (V, E)$ be a graph with vertex set V and edge set E. All graphs considered here are 2-connected and simple. An even subgraph of G is a subgraph of G such that the degree of each vertex is even in this subgraph. It is clear that an even subgraph is a union of edge-disjoint cycles. The set of all neighbors of a vertex v is denoted by $N(v)$. An edge-cut $[S, V - S]$ is the set of all edges with one end in S and another end in $V - S$.

A closed walk passing through all edges of a graph is called a postman tour of the graph. The Chinese Postman Problem (abreviated to CPP) is to find a shortest postman tour of the graph (the optimum solution of CPP). A weight $w: E(G) \to \{1,2\}$ is called a $(1,2)$-eulerian weight of the graph G if the total weight of each edge-cut is even. If a graph G with a $(1,2)$-eulerian weight w has a family of even subgraphs such that each edge e of G is contained in exactly $w(e)$ even subgraphs of the family, then this graph is said to be cycle w-covered by the family of even subgraphs. A graph is said to have the cycle cover property if G is cycle w-coverable with respect to every $(1,2)$-eulerian weight.

Denote

$$E_{w=\text{even}} = \{e \in E(G): w(e) \text{ is even}\}$$
$$E_{w=\text{odd}} = \{e \in E(G): w(e) \text{ is odd}\}$$

where w is a $(1,2)$-eulerian weight of G. It is trivial that $E_{w=\text{odd}}$ is an even subgraph of G. It is proved in this paper that if G is cycle w-coverable with respect to every $(1,2)$-eulerian weight such that $E_{w=\text{even}}$ is acyclic, then G has cycle covering property.

The shortest cycle covering problem (abbreviated to SCC) is to find a family of cycles F (or even subgraphs) in a graph G such that each edge of G is contained in some cycle(s) (or even subgraph(s)) of F and the total length of cycles in F is minimum.

Theorem 1. *If w is a $(1,2)$-eulerian weight of G, and C is a cycle of G, let w_1 be a $(1,2)$-weight of G such that $w_1(e) = w(e)$ if $e \notin C$, and $w_1(e) = 3 - w(e)$ if $e \in C$, then w_1 is a $(1,2)$-eulerian weight of G.*

Proof: In order to prove that w_1 is a $(1,2)$-eulerian weight of G, take any edge-cut $[S, V-S]$ of G. Because C is a cycle of G, the number of edges of C which are contained in the edge-cut $[S, V-S]$ must be even. Therefore, $|A| + |B|$ is even, where $A = \{e \in [S, V-S] : w(e) = 1, e \in C\}$, and $B = \{e \in [S, V-S] : w(e) = 2, e \in C\}$. Therefore, $|A| \equiv |B| \pmod 2$. It follows that

$$\sum_{e \in B} 1 + \sum_{e \in A} 2 \equiv \sum_{e \in B} 2 + \sum_{e \in A} 1 \cdots \pmod 2. \tag{1}$$

Also, we know that

$$\begin{aligned} \sum_{e \in [S,V-S]} w(e) &= \sum_{e \in [S,V-S] \setminus A \cup B} w(e) + \sum_{e \in A} 1 + \sum_{e \in B} 2 \\ &\equiv \sum_{e \in [S,V-S] \setminus A \cup B} w_1(e) + \sum_{e \in A} 2 + \sum_{e \in B} 1 \cdots \pmod 2 \\ &= \sum_{e \in [S,V-S]} w_1(e) \end{aligned}$$

Then, w_1 is a $(1,2)$-eulerian weight of G. This completes the proof. ∎

As a consequence of Theorem 1, we have

Theorem 2. *If a graph G is cycle w-coverable with respect to every $(1,2)$-eulerian weight w_T such that $G[E_{w=\text{even}}]$ is a forest, the graph G is cycle w-coverable with respect to any $(1,2)$-eulerian weight w.*

Proof: Suppose that G is not cycle w-coverable with respect to every $(1,2)$-eulerian weight w. Choose a $(1,2)$-eulerian weight w of G such that

(i) G is not cycle w-coverable,
(ii) $|E_{w=\text{even}}|$ is as small as possible.

By the assumption in the theorem, the subgraph induced by $E_{w=\text{even}}$ is not a forest. Thus, let C be a cycle in $E_{w=\text{even}}$. We construct a new $(1,2)$-eulerian weight as follows: $w_1(e) = w(e)$ if $e \notin E(C)$, and $w_1(e) = 3 - w(e)$ if $e \in E(C)$. By Theorem 1, we know that w_1 is a $(1,2)$-eulerian weight of G. By the inductive hypothesis, since $|E_{w=\text{even}}| \geq |E_{w_1=\text{even}}| + 1$ it follows that G is w_1-cycle coverable by a family of cycles F. But $F + \{C\}$ is a cycle w-cover of G. This contradicts our assumption and completes the proof. ∎

Let Q be the family of all the $(1,2)$-eulerian weights of G. An element w of Q is called smallest if the total weight w of G is minimum. Obviously, if w_0 is a smallest $(1,2)$-weight of G, the subgraph induced by $E_{w_0=\text{even}}$ is a forest.

Theorem 2 can be considered as a partial result towards the following conjecture:

Conjecture (Zhang [5]). *Let $w_0 \in Q$ be such that $|E_{w_0=\text{even}}|$ is minimum. If G is cycle w_0-coverable, then G is w-cycle coverable with respect to every $(1,2)$-eulerian weight w.*

If w_0 is a smallest $(1,2)$-eulerian weight of G, we have the following property.

Proposition. *Let w_0 be a smallest $(1,2)$-eulerian weight of G. Then $|E_{w_0=\text{even}}| + |E|$ = the length of the optimum solution of CPP.*

Proof: Let T be an optimum solution of CPP of G. Define a weight w_T by $w_T(e) = h$ if T passes through the edge e of G h times. Obviously, $1 \le h \le 2$, and it follows that w_T is a $(1,2)$-eulerian weight of G. Then, $|E_{w_T=\text{even}}| \geq |E_{w_0=\text{even}}|$.

Since $E_{w_0=\text{odd}}$ is an even subgraph, and $E_{w_0=\text{even}}$ is a forest, we form a new graph G^* from G by doubling each edge of $E_{w_0=\text{even}}$. The new graph G^* is a eulerian graph (even graph). Then,

$$|E_{w_0=\text{even}}| \geq |E_{w_T=\text{even}}|.$$

Therefore,

$$|E_{w_T=\text{even}}| = |E_{w_0=\text{even}}|.$$

This completes the proof. ∎

By $H(w)$ we denote the subgraph $H(w) = (V(G), E_{w=\text{even}})$, where w is a $(1,2)$-eulerian weight of G. As another consequence of Theorem 1, we have

Theorem 3. *Let G be 2-connected and w_0 be a smallest $(1,2)$-eulerian weight of G. Then*

(1) *$H(w_0)$ is a forest with at least two components; and*
(2) *$|E_{w_0=\text{even}}| \leq |V| - 2$; and*
(3) *every edge of $E(G) - E_{w_0=\text{even}}$ must be in the edge-cut between some components of $H(w_0)$.*

Proof: Using contradiction, we may assume that $H(w_0)$ is a spanning tree of G. It follows that $E_{w_0=\text{odd}}$ is the cotree of G. Then we know that there exists a cycle C which is contained in $H(w_0) + e$, where $e \in E_{w_0=\text{odd}}$. Let $w_1(e) = w(e)$ if $e \notin C$, and $w_1(e) = 3 - w(e)$ if $e \in C$. By Theorem 1, w_1 is a $(1,2)$-eulerian weight, it is easy to see that

$$|E_{w_0=\text{even}}| \geq |E_{w_1=\text{even}}| + 1.$$

This contradiction shows that the subgraph induced by $E_{w_0=\text{even}}$ is not a spanning tree of G. Note that $H(w_0)$ is a forest with at least two components. Then $|E_{w_0=\text{even}}| \leq |V| - 2$. It is easy to see that the third conclusion is true. This completes the proof of Theorem 3. ∎

Now we are in the position to prove our main result.

Theorem 4. *Let G be a 2-connected graph and w_0 be a smallest element of Q. Then one of the following 2 statements holds:*

(1) *The subgraph $H(w_0) = (V(G), E_{w_0=\text{even}})$ has at least 3 components implying that $|E_{w_0=\text{even}}| \leq |V| - 3$; or*

(2) $G = K_4$.

Proof: Let the subgraph $H(w_0)$ of G be G^*. If G^* contains at least three components, the theorem is done because the subgraph is a forest. By Theorem 3, we may assume that G^* contains exactly two components T_1, T_2. Then $E(G) \setminus E(G^*)$ is exactly the edge cut between T_1 and T_2. Let P_1 be a longest path of T_1, and P_2 be a longest path of T_2. Without loss of generality, let $|P_1| \leq |P_2|$.

We claim that $|P_i|$ is at least 1, $i = 1$ or 2. Assume that $|P_1| = 0$ and $G \neq K_4$. Thus, T_1 is a single vertex and T_2 must have at least 2 vertices. That is, $|P_2|$ is at least 1. Let v_1, v_2 be the ends of P_2. Note that v_i ($i = 1,2$) is incident with precisely one edge of weight 2. Since w_0 is an eulerian-weight and G is 2-connected, v_i must be incident with at least 2 weight one edges in G. Thus, any end vertex v_i of T_i ($i = 1,2$) must be incident with at least two weight 1 edges in G, say, $v_1 u \in E(G)$ and $v_1 u' \in E(G)$. But u, u' are different vertices of T_1. This contradicts that $|P_1| = 0$.

We claim that $|P_i|$ is 1, $i = 1$ or 2. Otherwise, without loss of generality, we assume that $|P_2| \geq 2$. Let v_1, v_2 be the ends of P_2. As above, we can find two different vertices, $u_1 \in T_1$, $u_2 \in T_1$, such that $u_1 v_1 \in E(G)$, $u_2 v_2 \in E(G)$. Let P^* be a path between u_1 and u_2 in T_1. Obviously, $|P^*| \geq 1$. Then, $v_1 u_1 P^* u_2 v_2 P_2 v_1$ is a cycle of G with length at least 5. Let C be such a cycle. It is easy to see that

$$|\{e \in C: w_0 = 2\}| \geq |\{e \in C: w_0 = 1\}| + 1. \tag{2}$$

Let $w_1(e) = w_0(e)$ if $e \notin C$; $w_1(e) = 3 - w_0(e)$ if $e \in C$. By Theorem 1, w_1 is a $(1,2)$-eulerian weight of G, and

$$|E_{w_0=\text{even}}| \geq |E_{w_1=\text{even}}| + 1.$$

This contradicts the definition of w_0.

Now we know that $|P_i|$ is 1. Because the end vertices of P_i have degree at least 3, and G is 2-connected, it is easy to see that G is K_4.

This completes the proof. ∎

Corollary 5. *If G admits a nowhere zero 4-flow, other than K_4, or contains no subdivision of the Petersen graph, then the total length ℓ of a shortest cycle cover satisfies $\ell \leq |E| + |V| - 3$.*

Proof: In [5], it is shown that if G admits a nowhere zero 4 flow, then G has the circuit property. In [1], we know that if G contains no subdivision of Petersen graph, then G has the cycle cover property. In [5], it is also easy to prove that if G

has the circuit property, then the optimum solution of the CPP equals the solution of SCC. Then the corollary is true by our proposition and Theorem 4 above. This completes the proof. ∎

In fact, we get the following stronger conclusion:

Corollary 6. *If G has a cycle covering property, then either $G = K_4$ or G has a shortest cycle cover with total length at most $|E| + |V| - 3$.*

Proof: The same as Corollary 5. ∎

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