Hereditary Clique-Helly Graphs

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Abstract. We define the class of *hereditary clique-Helly graphs* or *HCH graphs*. It consists of those graphs, where the cliques of every induced subgraph obey the so-called 'Helly-property', namely, the total intersection of every family of pairwise intersecting cliques is nonempty. Several characterizations and an $O(|V|^2|E|)$ recognition algorithm for HCH graphs $G = (V, E)$ are given. It is shown that the clique graph of every HCH graph is a HCH graph, and that conversely every HCH graph is the clique graph of some HCH graph. Finally, it is shown that HCH graphs $G = (V, E)$ have at most $|E|$ cliques, whence a maximum cardinality clique can be found in time $O(|V||E|^2)$ in such a HCH graph.

1. Introduction.

All graphs in this note are finite. By *cliques* we always mean maximal complete subgraphs of a graph. A *clique-Helly graph* is a graph whose cliques obey the so-called 'Helly-property': For any set of pairwise intersecting cliques, the total intersection of these cliques is nonempty [2]. They had been introduced in connection with clique graphs in [5] and [3]. It turns out that this graph class is not closed under induced subgraphs. For instance, the graph $K_1 \star H$, constructed from the graph H by adding some new vertex and joining it to all 'old' vertices by edges, is a clique-Helly graph for every graph H. For me it seems that this is the reason why there is no polynomial recognition algorithm known for clique-Helly graphs up to now. Thus, it seems promising to consider the biggest hereditary subclass: A *hereditary clique-Helly graph* or *HCH graph* is defined by the property that every induced subgraph (including the graph itself) is a clique-Helly graph.

2. Characterizations.

In this section, we are going to give some characterizations of HCH graphs.

Theorem 2.1. *A graph is a HCH graph if and only if it contains none of the four graphs of Figure 1 as induced subgraph.*

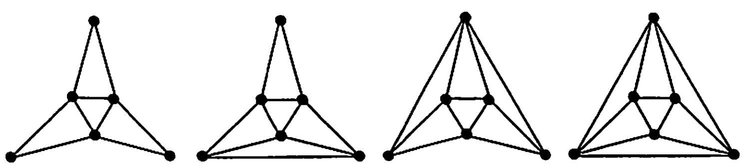

Figure 1

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Proof: None of the four graphs in Figure 1 is a clique-Helly graph, whence the necessity of the condition. Let conversely G be a graph without the graphs of Figure 1 as induced subgraphs. Assume G is no HCH graph, then some induced subgraph F of G is no clique-Helly graph. Let $C_i, i \in \{1,2,\ldots,k\}$, be pairwise intersecting cliques of F with empty total intersection, and let k be minimal with this property for F. Then each one of the intersections $C_2 \cap C_3 \cap \ldots \cap C_k, C_1 \cap C_3 \cap \ldots \cap C_k, C_1 \cap C_2 \cap C_4 \cap \ldots \cap C_k$ is nonempty. We choose vertices x_1, x_2, x_3 in these three sets, respectively. The three sets are disjoint, since $\cap_{j=1}^{k} C_i = \emptyset$, whence x_1, x_2, x_3 are distinct. By the same reason, C_1 and $C_2 \cap \ldots \cap C_k$ are disjoint. There is some vertex y_1 in C_1 not adjacent to x_1, because of the maximality of the clique C_1. In the same way we can find vertices y_2, y_3 in C_2, C_3, respectively, not adjacent to x_2 and x_3, respectively. We have $y_1 \notin V(C_2 \cup C_3), y_2 \notin V(C_1 \cup C_2)$, $y_3 \notin V(C_1 \cup C_2)$, so all the six vertices are distinct. They induce one of the graphs in Figure 1 in F, and, thus, in G also, a contradiction. ∎

The set of cliques of a graph obeys the *strong Helly-property*, if for all cliques $C_i, i \in I$, of the graph there holds:

$$\left|\bigcap_{i \in I} V(C_i)\right| = \min\{|V(C_i) \cap V(C_j)| / i \neq j \in I\}.$$

Corollary 2.2. *HCH graphs are exactly the graphs whose cliques obey the strong-Helly property.*

Proof: In [7] it was shown that the clique set of a graph obeys the strong-Helly property if and only if the graph contains none of the four graphs in Figure 1 as induced subgraph. ∎

Let us reformulate Theorem 2.1 slightly:

Corollary 2.3. *A graph is a HCH graph if and only for every triangle there is some edge such that any common neighbor of the vertices of the edge is also adjacent to the third vertex of the triangle.*

Such edges are called 'good' for the triangle. The corollary directly yields an $O(n^2 m)$-time recognition algorithm for HCH graphs with n vertices and m edges.

Algorithm 2.4. *Input: A graph $G = (V, E)$*

1. Compute all triangles of G; (* in time $O(nm)$ *)
2. FOR every triangle T of G DO
   - $M := E(T)$; (* the candidates for 'good' edges *)
   - FOR $v \in V\{x, y, z\}$ DO FOR $e \in E(T)$ DO IF $N_G(v) \cap V(T) = V(e)$ THEN $M := M \setminus \{e\}$ (* e is not 'good' *); OD;
   - IF $M = \emptyset$ THEN (PRINT('no HCH graph') ; STOP) ;
   - OD;

3. PRINT('G is a HCH graph'); STOP; END.

Now I give one more characterization, using hypergraph terminology. A *partial hypergraph* of a hypergraph $H = (V, (S_1, S_2, \ldots, S_t))$ is any hypergraph we can obtain by deleting hyperedges and vertices, that is, any hypergraph $P = (W, (W \cap S_j / j \in J))$, where $W \subseteq V$ and $J \subseteq \{1, 2, \ldots, t\}$. The *underlying graph* $U(H)$ of the hypergraph has the same vertex set as H, and two distinct vertices are adjacent in $U(H)$ if they lie in some common hyperedge. A hypergraph is *conformal*, if the hyperedges of the dual obey the Helly property.

Theorem 2.5. *Let Θ denote the class of all conformal hypergraphs without C_3 as partial hypergraph. Then the underlying graphs of the members of Θ are exactly the HCH graphs.*

Proof: If $Q_1, Q_2, \ldots, Q_t$ are all cliques of a graph $G = (V, E)$, then its *clique hypergraph* is defined by $\kappa(G) = (V, \{V(Q_1), V(Q_2), \ldots, V(Q_t)\})$. It is well-known that a hypergraph H is conformal if and only if the partial hypergraph $R(H)$ generated by the inclusion-maximal hyperedges is the clique-hypergraph of some graph, namely, of its underlying graph $U(H)$. So if C_3 is a partial hypergraph of H, then there are 3 vertices x_1, x_2, x_3 and three cliques Q_1, Q_2, Q_3 of $U(H)$ such that each Q_j does not contain x_j, but the other two vertices, for $j \in \{1, 2, 3\}$. Then the three cliques hurt the strong Helly property, whence $U(H)$ is no HCH graph by Corollary 2.2. Conversely, if $U(H)$ is no HCH graph, it must contain some triangle $\{x_1, x_2, x_3\}$ with no 'good' edge, see 2.3. Then there must be cliques Q_1, Q_2, Q_3 in $U(H)$ with $x_j \notin V(Q_j)$, but $x_k \in V(Q_j)$, for $k \neq j \in \{1, 2, 3\}$. $V(Q_1)$, $V(Q_2)$, $V(Q_3)$ are hyperedges in H, since H is conformal. Then these three vertices induce in these three hyperedges a partial hypergraph isomorphic to C_3. ∎

3. Subclasses and superclasses.

By definition every HCH graph is a clique-Helly graph. But there is another superclass, the class of *irreducible* graphs. It contains those graphs where every clique contains some edge that lies in no other clique. In [7] it has been shown that every graph without induced subgraphs as in Figure 1 must be irreducible. From 2.1 there follows, since none of the graphs in Figure 1 is irreducible, when we define *hereditary irreducible graphs* analogously as those graphs where every induced subgraph is irreducible:

Corollary 3.1. *The class of HCH graphs is exactly the class of hereditary irreducible graphs.*

The next result is immediate (it follows from 2.1, for example).

Remark 3.2: $(K_4\,e)$-free graphs (and in particular triangle-free graphs) are HCH graphs.

Corollary 3.3. *Every strongly chordal graph (and in particular every interval graph and every block graph) is a HCH graph.*

Proof: Strongly chordal graphs are exactly the underlying graphs of totally balanced hypergraphs, see [1], which are defined as hypergraphs without graph cycles as partial hypergraphs. But C_3 is a graph cycle, and we use Theorem 2.5. ∎

Finally, we can state a common neighborhood condition for HCH graphs. *Trivially perfect graphs* [4] do not contain any induced subgraph isomorphic to the path P_4 or cycle C_4 with 4 vertices.

Proposition 3.4. *The common neighborhood of any two distance 2 vertices in an HCH graph induces a trivially perfect graph.*

Proof: Let x and y be two vertices of distance 2 in the HCH graph G. If their common neighborhood contains some induced C_4, then x, y, and this C_4 induces the octahedron $\overline{3K_2}$, the last graph of Figure 1. If their common neighborhood contains some induced 4-vertex path P_4, then x, y, induce together with this P_4 the third graph of Figure 1. But both these graphs are forbidden induced subgraphs of G by Theorem 2.1, a contradiction. ∎

4. Cliques.

The *clique graph* $C(G)$ of a graph G is the intersection graph of the set of all cliques of G. That is, $C(G)$ has all the cliques of G as vertices, and two distinct vertices are adjacent whenever they have nonempty intersection. In [2] there has been shown that certain classes Γ of graphs (including the class of clique-Helly graphs) are *fixed* under C. This means, that the clique graph of every graph in Γ also lies in Γ, and every graph of Γ is the clique graph of another graph of Γ.

Corollary 4.1. *The class of HCH graphs is fixed under C.*

Proof: The proof is straightforward by Theorem 2.5 and Lemma 3.1 in [2]. ∎

Clique-Helly graphs (though they are in a sense very restricted) may have an exponential number of cliques, measured in the vertex number. Choose any graph H with $\Omega(2^n)$ cliques, for instance $\overline{tK_3}$. Then the clique-Helly graph $K_1 \star H$ has as many cliques as H, but only one vertex more. Contrary to this, we get by 3.1 for HCH graphs:

Corollary 4.2. *No connected HCH graph has more cliques than edges.*

Corollary 4.3. *The cardinality of a maximum clique can be computed in time $O(nm^2)$ for every HCH graph with n vertices and m edges.*

Proof: The algorithm of [6] generates all the cliques of a graph in time $O(nmc)$, where c denotes the number of cliques. ∎

Surely the last two results even hold for irreducible graphs.

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