The Spectrum of Support Sizes of BIB Designs with $v = 10$ and $k = 5$

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Abstract. The support of a t-design is the set of all distinct blocks of the design. The support size of a design is denoted by b^*. In this paper, except for $b^* = 23$, we completely determine the spectrum of support sizes of the case $v = 10$, $k = 5$ and $t = 2$.

1. Introduction

A t-(v, k, λ) design is a pair $(V, \mathcal{B})$ in which $|V| = v$ and $\mathcal{B}$ is a collection of k-subsets (called blocks) such that any t-subset appears exactly λ times in $\mathcal{B}$. We call the set of all distinct blocks of a t-design, the *support* of the design and its cardinality, which is denoted by b^*, is the support size of the design. The total number of blocks of $\mathcal{B}$ is denoted by b.

Recently the problem of the spectrum of support sizes of triple systems ($k = 3, t = 2$, and arbitrary v), except for few exceptions, was completely determined [1,5]. For $k = 4$, some progress has been made [4,6], but the solution is far from complete. The construction procedure for these cases are essentially recursive. But as in the case of triple systems, the spectrums for small v's have many irregularities and exceptions to the rule. Therefore, treating these cases separately is justifiable.

In this paper we intend to treat the spectrum problem for the case $v = 10$ and $k = 5$, and except for $b^* = 23$, all possible (b, b^*) are given. In Section 2, it is shown that the minimum possible value of b^*'s is 18, and there is no $2-(10,5)$ design with $b^* = 19$ or 20, and if $b^* = 21$, then λ is a multiple of 12. In Section 3, the method of construction of desgins with various b^*'s is described.

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2. Nonexistence of Some Support Sizes

For BIB designs with $v = 10$ and $k = 5$, we have $b = 18m, r = 9m$ and $\lambda = 4m$, for $m = 1, 2, \ldots$.

Let $(V, \mathcal{B})$ be a $2-(v, k, \lambda)$ design, and for

$$\overline{\mathcal{B}} := \{V \setminus B | B \in \mathcal{B}\},$$

let $\overline{D} := (V, \overline{\mathcal{B}})$. Then it is easy to check that $\overline{D}$ is a $2-(v, v-k, \lambda')$ design with $\lambda' = b + \lambda - 2r$. (Note that D and $\overline{D}$ have equal support sizes.)

Suppose that $D = (V, \mathcal{B})$ is a t-(v, k) design and $x \in V$. Let

$$\mathcal{B}_x := \{B | B \cup \{x\} \in \mathcal{B}\},$$
$$\mathcal{B}^x := \{B \in \mathcal{B} | x \notin B\}.$$

Then it is well-known that $D_x = (V \setminus \{x\}, \mathcal{B}_x)$ and $D^x = (V \setminus \{x\}, \mathcal{B}^x)$ are $(t-1)$-$(v-1, k-1)$ and $(t-1)$-$(v-1, k)$ designs, respectively.

Alltop [3] has shown that for t even, a self-complementray t-$(2k, k, \lambda)$ design (i.e., a design $(V, \mathcal{B})$ for which $\mathcal{B} = \overline{\mathcal{B}}$) is already a $(t+1)$-design.

Using these results one can easily prove that if $(V, \mathcal{B})$ is a BIB design with $v = 2k$, then $(V, \mathcal{B} \cup \overline{\mathcal{B}})$ is a 3-design.

Theorem 1. *For every BIB design with $v = 10$ and $k = 5$ with support size b^* we have: $b^* \geq 18$, $b^* \neq 19, 20$ and if $b^* = 21$, then λ is a multiple of 12.*

Proof: Let $D = (V, \mathcal{B})$ be a BIB design with $v = 10$, $k = 5$ and support size b^*. Suppose that $V = \{0, 1, \ldots, 9\}$. Then $\Delta = D \cup \overline{D} = (V, \mathcal{B} \cup \overline{\mathcal{B}})$ is a 3-design, and its support size $\overline{b^*}$ is at most $2b^*$. Let b_1^* and b_2^* be the support sizes of 2-$(9, 4, 3m)$ and 2-$(9, 5, 5m)$ designs Δ_0 and Δ^0, respectively. Note that $\overline{b^*} = b_1^* + b_2^*$. Since is self-complementary, we have $\Delta^0 = \overline{\Delta_0}$, and hence $b_1^*; = b_2^*$. Consequently, $b_1^* \leq b^*$.

In [8], it is proved that for BIB designs with $v = 9$ and $k = 4$, the minimum possible support size is equal to 18. In [9], it is shown that there is no 2-$(9, 4, 3m)$ design with support size equal to 19 or 20, and if the support size is 21, then m is a multiple of 3.

Therefore, the relation $b_1^* \leq b^*$ implies that for BIB designs with $v = 10$ and $k = 5$, the minimum possible support size is 18.

To prove the other claims, it is enough to show that if $b_1^* = 18$, then $b^* \notin \{19, 20, 21\}$.

Assume that $b_1^* = 18$. In [8] it is proved that if in a 2-$(9, 4, 3m)$ design there exists a block with frequency $> m$, then the support size is at least 21. Now, since Δ_0 has $18m$ blocks, hence Δ_0 is m-uniform (i.e., the frequency of each block of Δ_0 is equal to m). Since $\Delta_0 = \overline{\Delta_0}$, the design Δ is also m-uniform; so in Δ the frequency of each block is at most m. For any $B \in \Delta$, let $n_1(B)$, $n_2(B)$

and $n(B)$ denote the frequencies of B in D, $\overline{D}$ and Δ, respectively. Then, the following relations are immediate consequences of definitions of $\overline{D}$ and Δ.

$$n_1(B) = n_2(V\backslash B),$$
$$n_1(B) + n_2(B) = n(B).$$

Since $n(B) = m$, so $n_1(B) < m$ if and only if B and $V\backslash B$ both belong to D. Let

$$\Gamma_1 := \{B \in \mathcal{B} | V\backslash \mathcal{B}\} = \{B \in \mathcal{B} | n_1(B) < m\},$$
$$\Gamma_2 := \mathcal{B}\backslash\Gamma_1 = \{B \in \mathcal{B} | n_1(B) = m\}.$$

We define r_1 and r_2 to be the number of distinct blocks in Γ_1 and Γ_2, respectively. It follows that

$$r_1 + r_2 = b^*,$$
$$r_1 + 2r_2 = \overline{b^*} = 2b_1^* = 36.$$

Hence r_1 is even, and

$$b^* = \frac{1}{2}(r_1 + r_2) + \frac{1}{2}r_1 = 18 + \frac{1}{2}r_1.$$

So, to complete the proof, we have to show that either $r_1 = 0$ or $r_1 \geq 8$; and since r_1 is even, it suffices to show that either $r_1 = 0$ or $r_1 > 6$.

For every pair $\{x, y\}$ of elements of V, the number of blocks in Γ_2 which contain $\{x, y\}$ is a multiple of m (probably equal to zero), and since $\lambda = 4m$, so the number of blocks in Γ_1 which contain $\{x, y\}$ is also a multiple of m. Hence, if a pair $\{x, y\}$ appears in a block B of Γ_1, then it must appear in another block $B' \in \Gamma_1$ such that $B' \neq B$ (because $n_1(B) < m$).

If $\Gamma_1 = \emptyset$, then $r_1 = 0$ and $b^* = 18$. Otherwise, with no loss of generality, we assume that $B_1 = \{1,2,3,4,5\} \in \Gamma_1$. Now, the pairs $\{1, i\}$, $2 \leq i \leq 5$, should appear in at least two distinct blocks of Γ_1. Let $B_2 = \{1,2,x,y,z\}$ be the other block of Γ_1 such that $B_2 \neq B_1$. With no loss of generality, we assume that $z \notin \{3,4,5\}$; so let $z = 6$. Suppose that B_3 is a block in Γ_1 which contains $\{1,6\}$ and $B_3 \neq B_2$. (Obviously $B_3 \neq B_1$, because $6 \notin B_1$.) Let $B_3 = \{1,6,x',y',z'\}$. Now B_i and $\overline{B_i}$, $i = 1,2,3$, are 6 distinct blocks of Γ_1. Hence $\Gamma_1 > 6$. If $\Gamma_1 = 6$, then

$$\Gamma_1 = \{B_1, B_2, B_3, \overline{B_1}, \overline{B_2}, \overline{B_3}\}.$$

Since the pair $\{2,6\}$ should appear in at least two blocks of Γ_1, and since it does not appear in B_1, $\overline{B_1}$, $\overline{B_2}$ and $\overline{B_3}$, hence $\{2,6\} \subseteq B_3$. So $B_3 = \{1,2,6,x',y'\}$. Also, each of the pairs

$$\{1,3\}, \{1,4\}, \{1,5\}, \{1,x\}, \{1,y\}, \{1,x'\}, \{1,y'\}$$

must appear in at least two distinct blocks of Γ_1. But none of them appear in $\overline{B_1}$, $\overline{B_2}$ or $\overline{B_3}$, hence, with no loss of generality we can assume that

$$x = 3, y = 4, x' = 3, y' = 5.$$

Therefore, up to isomorphism, the structures of the blocks of Γ_1 are as follows

$$B_1 = \{1,2,3,4,5\}, \overline{B_1} = \{6,7,8,9,0\},$$
$$B_2 = \{1,2,3,4,6\}, \overline{B_2} = \{5,7,8,9,0\},$$
$$B_3 = \{1,2,3,5,6\}, \overline{B_3} = \{4,7,8,9,0\}.$$

But then the pair $\{4,5\}$ appears in exactly one block of Γ_1, which is a contradiction. Hence $r_1 > 6$; and this completes the proof. ∎

3. Construction Methods

In [11], it is shown that there are exactly 21 nonisomorphic BIB designs with $b = b^* = 18$. These 21 designs are explicitly listed in Table 1 of [2]; we will refer to the ith design of this table as "BDi".

Recently, in [10] a large set for BIB designs with $v = 10$ and $k = 5$, based on BD5 was given. In Table 2 of [2] altogether 232 designs with different b^*'s are given. All simple designs in this table are constructed from the large set. The design with $b^* = 21$ and $b = 54$ was constructed directly:

$$4\{12345\} + 3\{12389, 12470, 13560, 24569, 34578, 14790, 15680, 23890, 25679, 34678,\}$$
$$+ 2\{12678, 13679, 14689, 15789, 23670, 24680, 25780, 34690, 35790, 45890\}$$

and the other designs appearing in this table are constructed by "random permutation method". In this method we start with a given design D and by applying different random permutations (on 10 points) on BD5, we find the design with desired b^* as $D + \sigma$ BD5, where σ is the appropriate permutation. Note that when there is a design of the form $D + \sigma BD_i$ with b blocks and support size equal to b^*, then by adding copies of BD_i to this design we obtain designs with $b + 18m$ blocks ($m \geq 1$) and the same support size.

Designs with $b^* = 247, 279, 250$ and 251 do not exist with $b = 252$, since otherwise trades of volumes $5, 3, 2$ and 1 must exist, respectively. But there are no trades of these volumes [7]. Therefore the designs with the above mentioned b^*'s were obtained with $b = 270$.

The following theorem summarizes what we have found in connection with the problem of support sizes of BIB designs with $v = 10$ and $k = 5$.

Theorem 2. *For $b^* \neq 23$, there is a $2-(10,5)$ design with b blocks and support size equal to b^* if and only if the following conditions hold*

(i) $b = 18m$ *for* $m \geq 1, b > b^*, 18 \leq b^* \leq 252$ *and* $b^* \neq 19, 20$;
(ii) *if* $b^* = 21$, *then* $b = 54t$ *for* $t = 1, 2, \ldots$;
(iii) *if* $b^* \in \{247, 249, 250, 251\}$, *then* $b \geq 270$.

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