An algorithm for finding smallest defining sets of t-designs

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ABSTRACT

A set of blocks which is a subset of a unique t-(v, k, λ_t) design is called a *defining set* of that design. Using known results, an algorithm for finding smallest defining sets of any t-(v, k, λ_t) design is described. Then the results of this algorithm as applied to the two 2-(13,3,1) designs are given.

1. Introduction

A **block design** is a set of b k-subsets (blocks) chosen from a set of v elements, such that every element occurs in exactly r blocks. If every subset of t elements belongs to exactly λ_t blocks, the design is called a t**-design** and its parameters are indicated as t-(v, k, λ_t). An example is the 2-(7,3,1) design F, with blocks

$$F = \{124, 235, 346, 457, 561, 672, 713\}.$$

A 2-$(v, 3, 1)$ design is called a **Steiner triple system** of order v. These exist for all $v \equiv 1$ or 3 (modulo 6); see for example Mathon, Phelps and Rosa [8]. The design F given above is a Steiner triple system of order seven.

The following definitions were introduced by Gray [3].

Definition 1.1 *A set of blocks which is a subset of a unique t-(v, k, λ_t) design is said to be a* **defining set** *of the design, and will be denoted by $d(t$-$(v, k, \lambda_t))$. A* **minimal defining set**, *denoted by $d_m(t$-$(v, k, \lambda_t))$, is a defining set, no proper subset of which is a defining set. A* **smallest defining set**, *denoted by $d_s(t$-$(v, k, \lambda_t))$, is a defining set such that no other defining set has smaller cardinality.*

Every t-design has itself as a defining set and hence it must have a smallest defining set. A $d(t$-$(v, k, \lambda_t))$ defining set consisting of blocks of a particular t-(v, k, λ_t) design D is abbreviated to dD.

The set of blocks $R = \{457, 713, 672\}$ is a subset of two 2-(7,3,1) designs. These designs are $F = R \cup T_1$ and $R \cup T_2$, where $T_1 = \{124, 235, 346, 156\}$ and $T_2 = \{125, 234, 356, 146\}$. Hence R is not a defining set of either design. But the set of

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blocks $S = \{124, 235, 346\}$ can be completed to a 2-(7,3,1) design only by adjoining the blocks $\{457, 561, 672, 713\}$. Hence S is a defining set of the design F.

Definition 1.2 *A permutation of the elements of a design D which sends the set of blocks of D to itself is called an* **automorphism** *of the design.*

The set of all automorphisms of a design D is easily seen to be a group under composition which is denoted by $Aut(D)$.

Definition 1.3 *A* **single-transposition-free** *(STF) design is a t-(v, k, λ_t) design whose automorphism group contains no single transposition $(i\,j)$ of its elements.*

The definition of an STF design was introduced by Gray [5]. It can be shown that any t-$(v, k, 1)$ design with $k > t$ is STF. A **simple** t-(v, k, λ_t) design is one with no repeated blocks. Clearly any design with $\lambda_t = 1$ is simple.

In the remainder of this paper the known results on defining sets will be given, followed by a description of a new algorithm which arranges the results of [2], [3], [4], [5] into an implementable method which can be used to find the smallest defining sets of any t-(v, k, λ_t) design. The results of applying this algorithm to the 2-(13,3,1) designs will then be given. These designs are chosen to illustrate the algorithm as they are highly unsymmetric and the number of isomorphism classes involved in the search for their defining sets is extremely large. The problem of finding smallest defining sets of these designs is one which would be virtually impossible to solve using anything other than an implementable algorithm.

2. Background

In this section the background results are given which are needed to construct the algorithm. The concept of a trade is very important in the search for defining sets.

Definition 2.1 *A set of m blocks T_1 taken from a t-(v, k, λ_t) design is called a* **trade** *if another set of m k-sets T_2 can be found, containing exactly the same t-subsets. Often the sets T_1 and T_2 together are said to form a trade. A* **minimal trade** *is a trade, no proper subset of which is a trade.*

The set of blocks T_1, T_2 given above form a trade. Trades are also sometimes referred to as 'mutually balanced sets'; see Rodger [10]. Suppose the set of blocks T_1 of the design D can be traded for the set of k-sets T_2. Then the set $T_1\rho$ is a trade in D, as it can be traded with the set $T_2\rho$, for any $\rho \in Aut(D)$. The following results are taken from [3], [4], [5].

Lemma 2.2 *Every $d(t$-$(v, k, \lambda_t))$ defining set S of a design D contains a block of every possible trade $T_1 \subseteq D$.* □

Lemma 2.3 *Suppose S is a particular defining set of a t-(v, k, λ_t) design D. Then:*

(i) $S\rho$ *is a defining set of* D *for all* $\rho \in Aut(D)$;
(ii) $Aut(S)$ *is a subgroup of* $Aut(D)$. □

Lemma 2.4 *Any* $d(t\text{-}(v,k,\lambda_t))$ *defining set* S *of an STF design has at least* $v-1$ *elements occurring in its blocks.* □

Lemma 2.5 *Suppose that in a* $d(t\text{-}(v,k,\lambda_t))$ *defining set* S *of an STF design, elements* i, j *each occur only once. Then* i *and* j *must occur in different blocks of* S.□

Theorem 2.6 *For every STF* $t\text{-}(v,k,\lambda_t)$ *design* D,

$$|dD| \geq \frac{2(v-1)}{k^*+1}, \qquad \text{where} \qquad k^* = min(k, v-k).$$ □

Corollary 2.7 *If a* $d(t\text{-}(v,k,\lambda_t))$ *defining set* S *of an STF design consists of* s *blocks, then*

$$\frac{s(k-2)}{s-2} + 1 \leq v \leq \frac{s(k+1)}{2} + 1.$$ □

Theorem 2.8 *Suppose* S *is a* $d(t\text{-}(v,k,\lambda_t))$ *defining set of an STF design* D, *where* $t < k < v-t$ *and* $|S| = s$. *Then:*

(i) $k \leq 2^{s-1}$ *and* $(v-k) \leq 2^{s-1}$;
(ii) $v \leq 2^{s-1} + k^* \leq 2^s$, *where* $k^* = min(k, v-k)$. □

Theorem 2.9 *Suppose* S *is a* $d(2\text{-}(v,k,1))$ *defining set of an STF design* D, *where* $|S| = s$. *Then*

$$v \leq \binom{s}{2} + s + 1.$$ □

Theorem 2.10 *Suppose* S *is a* $d(t\text{-}(v,k,\lambda_t))$ *defining set of a simple STF design* D. *Let* $n(S:D)$ *denote the number of configurations of blocks in* D *isomorphic to* S. *Then*

$$n(S:D) = \frac{|Aut(D)|}{|Aut(S)|}.$$ □

3. The Algorithm

The algorithm developed to find smallest defining sets of a given $t\text{-}(v,k,\lambda_t)$ design D is now described. It arranges the results of [2], [3], [4], [5] as described above into an implementable method which can be used to find the smallest defining sets of any $t\text{-}(v,k,\lambda_t)$ design.

It is assumed that the design D is both STF and simple, so the results of the previous section hold. First use Theorems 2.6, 2.8, 2.9 (if applicable) and Corollary 2.7 to calculate the lower bound n_0 on the size of a defining set. We start by determining, for each $\binom{b}{n_0}$ set of n_0 blocks of D, whether or not the set is a defining set. If there are no defining sets of n_0 blocks, consider sets with $n_0 + 1$ blocks, and so on. The first defining set found will be a smallest defining set: suppose it has n blocks. Work through every $\binom{b}{n}$ set of n blocks until a complete list of all smallest defining sets is found, then stop. Note that this process will terminate, as every design possesses smallest defining sets.

Suppose we have shown that no defining set of $n_0 + i$ blocks can exist, for $i = 0, 1, \ldots, j-1$, and we are now about to consider sets of $n_0 + j = n$. Arrange all the $\binom{b}{n}$ sets of n blocks of D into isomorphism classes: suppose there are m classes, $C_1, \ldots, C_m$. By Lemma 2.3(i), for each isomorphism class, either every set is a defining set or no set is a defining set. Hence it is enough to take a **transversal** of the isomorphism classes, that is, a set $\{ S_i | S_i \in C_i, \ i = 1, \ldots, m \}$, and decide, for each of these sets, whether or not it is a defining set. The results of the previous section give tests which can determine that a given set is not a defining set. These are applied to each set in the transversal, applying the easiest tests first.

Let j_i be $|C_i|$, the number of members of the i^{th} class. Then $j_i = n(S_i : D)$ in the notation of Theorem 2.10. Let $o_i = |Aut(S_i)|$ be the order of the automorphism group of S_i. Theorem 2.10 reworded states that if S_i is a defining set then $j_i \times o_i = |Aut(D)|$. Hence the reworded contrapositive states that if $j_i \times o_i \neq |Aut(D)|$, then S_i cannot be a defining set. Even more simply, if j_i is not a factor of $|Aut(D)|$ then, since o_i must be an integer, Theorem 2.10 implies S_i is not a defining set. Next, if S_i contains less than $v - 1$ elements within its blocks, then by Lemma 2.4 the set S_i cannot be a defining set. The next step requires the construction of the group $Aut(S_i)$: if the group $Aut(S_i)$ is not a subgroup of $Aut(D)$ then by Lemma 2.3(ii) S_i is not a defining set of D. This test can be carried out by the software package Cayley [1]. If the set S_i passes all these tests, then the class C_i is called **feasible**, if not it is called **infeasible**.

If any feasible classes still remain, a list of trades must be generated. In some cases it is easy to find small trades by hand. The structure of small trades is described in Hwang [7]. A trade in a t-design must contain at least 2^t blocks. Once any trade is found, several other trades can be generated from it simply by applying automorphisms of the design to it. By Lemma 2.2, if any trade found is disjoint from S_i, then S_i is not a defining set. If the small trades found do not rule out all classes, then the last step is to find all the t-(v, k, λ_t) designs containing S_i for every feasible class C_i, a process known as **completion** of S_i. If S_i can only be completed to one design, then by definition S_i is a defining set, and $d_s(D) = n$. However, suppose S_i can be completed to two or more designs. The set of blocks $D \setminus S_i$ is a trade disjoint from S_i, which contains a minimal trade T. Then T can generate more trades, using the automorphisms. Continue in this manner, accumulating trades.

When all the m sets of the transversal have been considered, we will be faced with one of two possibilities: either $n = d_s(D)$ and all the smallest defining sets of D have been found, or $d_s(D) > n$. If $n = d_s(D)$ then we also know how many isomorphism classes the smallest defining sets fall into, the size of each class and the automorphism groups of elements of the class. If $d_s(D) > n$ then start the process again for $n = n + 1$. The entire algorithm is given in Figure .1 in pseudocode. Note that new trades can be added to the list of trades at any stage.

The information produced by the algorithm is presented in table form, with a standard format. The table headings, left to right, are 'class number i', 'blocks of S_i', 'size j_i', 'group order o_i' and the final column headed 'trade number' if a list of trades was used, and 'feasible ?' if no trades were used. The symbols S_i, j_i, o_i are as defined above. All blocks of the design will have been numbered $1, \ldots, b$ and the block numbers, not the blocks, appear in the column labelled 'blocks of S_i'. The entries are ordered first by increasing size j_i, then by increasing group order o_i, and finally by lexicographical ordering on the block numbers of S_i. The entries in the 'class number i' column merely associate the numbers $1, \ldots m$ to the ordered entries. If the size j_i does not divide $|Aut(D)|$, then the entry in the 'group order o_i' column (and in the 'trade number' column, if present) will be a '*'. If j_i divides $|Aut(D)|$ but $j_i \times o_i \neq |Aut(D)|$ then the entry in the 'trade number' column (if present) will be '**'. Number all the trades in the list of known trades. If the set S_i is disjoint from trade number p then the number p will be placed in the 'trade number' column. Those sets which contain a block from every trade in the list will have an entry '***' in the 'trade number' column, indicating that they are possible defining sets. If there is no 'trade number' column then the 'feasible ?' column will have entry 'yes' in the i^{th} row if $j_i \times o_i = |Aut(D)|$, and entry 'no' otherwise. Hence classes with a '***' entry in the 'trade number' column, or a 'yes' entry in the 'feasible ?' column are feasible, all other classes are not. The representatives of the feasible classes need to be completed by hand to determine whether they are defining sets.

4. The 2-(13,3,1) designs

It is well known (see Mathon, Phelps and Rosa [8]) that there are only two non-isomorphic 2-(13,3,1) designs, or Steiner triple systems of order 13. The details of the search for smallest defining sets for the two 2-(13,3,1) designs is now given. Some designs are quite symmetric, and their defining sets could perhaps be found using some other method. However, consider the 2-(13,3,1) design D_1. The number of different isomorphism classes that need to be investigated to find defining sets of D_1 is extremely large. For instance, the $\binom{26}{9} = 3124550$ different sets of nine blocks of D_1 fall into 30376 isomorphism classes. Clearly, the problem of determining whether a representative of each class is a defining set is one which would be virtually impossible to solve using anything other than an implementable algorithm.

Two non-isomorphic 2-(13,3,1) designs are given in Tables 1, 2. For notational convenience the numbers 10, 11, 12 and 13 are represented in the design by the letters a, b, c and d respectively. Design D_1 in Table 1 is cyclic, with starter blocks 125 and 139. The design D_2 in Table 2 is obtained from D_1 by trading a set of four

Algorithm .1

```
input
D : design with b blocks;

begin
n := lower bound;        (* given by Theorems 2.6, 2.8, 2.9, Corollary 2.7 *)
found := false;
find initial trades;
use Aut(D) to generate more trades;
while not found do
    begin
    arrange all (b choose n) sets of n blocks of D into m isomorphism classes;
    for i := 1 to  m do
        begin
        j_i := size of isomorphism class i;
        if j_i divides |Aut(D)| then
            begin
            S_i := a representative of isomorphism class i;
            Aut(S_i) := the automorphism group of S_i;
            o_i := |Aut(S_i)|;
            if o_i × j_i = |Aut(D)| then
                if S_i contains at least v - 1 elements then
                    if Aut(S_i) is a subgroup of Aut(D) then
                        if S_i contains a block of every trade then
                            if S_i completes uniquely to design D then
                                begin
                                class i is a class of defining sets;
                                found := true;
                                end
                            else
                                begin
                                class i is not a class of defining sets;     (* by definition *)
                                find a new minimum trade;
                                use Aut(D) to generate more trades;
                                end
                        else
                            class i is not a class of defining sets;         (* by Lemma 2.2 *)
                    else
                        class i is not a class of defining sets;             (* by Lemma 2.3 *)
                else
                    class i is not a class of defining sets;                 (* by Lemma 2.4 *)
            else
                class i is not a class of defining sets;                     (* by Theorem 2.10 *)
            end
        else
            class i is not a class of defining sets;                         (* by Theorem 2.10 *)
        end
    n := n + 1;
    end
end.
```

Figure 1: The general algorithm - pseudocode

blocks. The trade consists of blocks

$$T_1 = \{ 125, 458, d14, d28 \} \tag{1}$$

from design D_1 and blocks

$$T_2 = \{ 12d, 4d8, 514, 528 \}$$

from design D_2.

block number	block	block number	block
1	125	14	139
2	236	15	24*a*
3	347	16	35*b*
4	458	17	46*c*
5	569	18	57*d*
6	67*a*	19	681
7	78*b*	20	792
8	89*c*	21	8*a*3
9	9*ad*	22	9*b*4
10	*ab*1	23	*ac*5
11	*bc*2	24	*bd*6
12	*cd*3	25	*c*17
13	*d*14	26	*d*28

Table 1: The cyclic 2-(13,3,1) design D_1

The automorphism groups of the two 2-(13,3,1) designs are well known, see for example [8]. For these particular designs they were conveniently found using *nauty* [9]. The order of the automorphism group of D_1 is 39. The group is generated by the permutations

$$\begin{aligned} \psi &= (1\,2\,3\,4\,5\,6\,7\,8\,9\,a\,b\,c\,d), \\ \phi &= (2\,4\,a)(3\,7\,6)(5\,d\,b)(8\,9\,c). \end{aligned}$$

The order of the automorphism group of D_2 is 6. This group is generated by the permutations

$$\begin{aligned} \omega &= (2\,9)(3\,d)(4\,5)(6\,a)(8\,b), \\ \chi &= (1\,4)(2\,b)(3\,8)(6\,7)(9\,d). \end{aligned}$$

By an earlier remark both of these designs are STF, so the results of Section 2 give bounds on the size of possible defining sets. The result of Theorem 2.6 shows that

$$|d(2\text{-}(13,3,1))| \geq \frac{2(13-1)}{3+1} = 6.$$

block number	block	block number	block
1	$12d$	14	139
2	236	15	$24a$
3	347	16	$35b$
4	$4d8$	17	$46c$
5	569	18	$57d$
6	$67a$	19	681
7	$78b$	20	792
8	$89c$	21	$8a3$
9	$9ad$	22	$9b4$
10	$ab1$	23	$ac5$
11	$bc2$	24	$bd6$
12	$cd3$	25	$c17$
13	514	26	528

Table 2: The non-cyclic 2-(13,3,1) design D_2

So a defining set must contain at least six blocks. Using Corollary 2.7 a design with a defining set of 6 blocks must satisfy

$$\frac{5}{2} = \frac{6(3-2)}{6-2} + 1 \leq v \leq \frac{6(3+1)}{2} + 1 = 13.$$

Since here $v = 13$ satisfies both inequalities, this result does not rule out the possibility of a defining set of six blocks. Similarly the results of Theorems 2.8 and 2.9 do not disallow a defining set of six blocks, as the equations $k \leq 32$, $v - k \leq 32$, $v \leq 32 + k^* \leq 64$ and $v \leq \binom{6}{2} + 6 + 1$ all hold when $v = 13$ and $k = k^* = 3$.

However the following argument shows that in fact no defining set of six blocks can exist.

Lemma 4.1 *There is no* 2-(13, 3, 1) *defining set consisting of six blocks.*

Proof. Suppose S is a 2-(13,3,1) defining set which consists of only six blocks. By Lemma 2.4 there are at least 12 elements appearing in the blocks of S. By Lemma 2.5, at most six elements can appear precisely once each. The only possibility is for six elements to occur once each and six elements to occur twice each. Take elements $1, \ldots, 6$ in one block each and let elements $7, 8, 9, a, b, c$ occur twice. Without loss of generality the blocks may be completed to

$$178,\ 289,\ 39-,\ 4--,\ 5--,\ 6--.$$

Now the third block may be completed in two ways, by adding the element 7 or a new element. If the element 7 is added then the completed set of blocks is

$$178,\ 289,\ 379, 4ab,\ 5bc,\ 6ca.$$

On the other hand, if a new element is added (the element a for instance) then the completed set of blocks is

$$178,\ 289,\ 39a,\ 4ab,\ 5bc,\ 6c7.$$

But the orders of the automorphism groups of these sets are 72 and 12 respectively. By Lemma 2.3(ii), the order of the automorphism group of a defining set must divide the order of the automorphism group of the design. Since neither 72 nor 12 is a factor of either 39 or 6, there is no 2-(13,3,1) defining set consisting of six blocks. □

5. The cyclic 2-(13,3,1) design D_1

Before starting the search for defining sets, some trades were found. There is the trade T_1 of volume four defined in (1), which was used to go from the cyclic design D_1 to the non-cyclic design D_2. Some automorphisms of D_1 were applied to T_1 to give the first ten trades in Table 3. The next eight trades in Table 3 were obtained by applying automorphisms to the set of blocks

$$U_1 = \{\,347,\ 3cd,\ 38a,\ 458,\ 57d,\ 5ac\,\}$$

which can be traded with the set of blocks

$$U_2 = \{\,348,\ 37d,\ 3ac,\ 457,\ 58a,\ 5cd\,\}.$$

Each trade T in the table is also displayed as either $T_1\rho$ or $U_1\rho$, where ρ is a product of the generators ψ, ϕ of the group $Aut(D_1)$.

These 18 trades were enough to rule out all sets of seven or eight blocks, but in order to handle sets of nine blocks, many trades were needed. Table 4 shows 24 non-isomorphic trades. Using the automorphisms of D_1, each trade in this table yields a set of 39 trades. The $24 \times 39 = 936$ trades formed in this way are enough to rule out all infeasible sets of nine blocks (in fact less than 100 of them are actually needed). The trades in Table 4 are numbered using Roman numerals to emphasise the fact that they are non-isomorphic generating trades, and that the entire list of trades used is obtained by applying the automorphisms of the design to these trades.

The lower bound on the size of defining sets of 2-(13,3,1) designs is seven, as shown by Lemma 4.1. The $\binom{26}{7} = 657800$ sets of seven blocks of the design D_1 fall into 1186 isomorphism classes. There are 31 classes satisfying j_i divides 39, five have size $j_i = 13$ and the rest have size $j_i = 39$. The information about these classes is given in Table 5. Clearly, none of the sets in the omitted classes can be a defining set, and similarly none of the sets shown in Table 5 is a defining set.

Next consider all the $\binom{26}{8} = 1562275$ sets of eight blocks from the design D_1. These sets fall into 6776 isomorphism classes. Of these classes, 899 have a size j_i which is a factor of 39. There are 11 classes with size $j_i = 13$, and the information about these classes is given in Table 6. The remaining 888 classes have size $j_i = 39$, and of these classes, a total of 169 classes have automorphism group order $o_i = 1$. The information about these classes is given in Tables 7 - 11. None of these sets is a defining set of D_1.

Finally, the $\binom{26}{9} = 3124550$ sets of nine blocks of D_1 fall into 30376 isomorphism classes. A total of 11418 of these classes satisfy j_i divides 39; 14 classes have size $j_i = 13$ and the information about these classes is given in Table 12. These sets are not feasible. The remaining 11404 classes have size $j_i = 39$, and of these classes a total of 6834 classes also have $o_i = 1$. All but 17 of these classes are disjoint from a trade generated by the trades in Table 4. The information about these 17 classes and a few other sample classes is given in Table 13, where the trade numbers referred to are those given in Table 3. The representatives S_i of the feasible classes are listed below. They all have only the trivial automorphism, and there are 39 isomorphic copies of each S_i in the design D_1.

$$S_1 = \{125, 236, 347, 458, 67a, 89c, 9ad, bc2, 35b\},$$

$$S_2 = \{125, 236, 347, 458, 67a, 89c, 9ad, 35b, bd6\},$$
$$S_3 = \{125, 236, 347, 458, 67a, 89c, cd3, 9b4, bd6\},$$
$$S_4 = \{125, 236, 347, 458, 89c, 9ad, ab1, 35b, 46c\},$$
$$S_5 = \{125, 236, 347, 67a, 89c, 57d, 9b4, ac5, bd6\},$$
$$S_6 = \{125, 236, 347, 78b, 9ad, ab1, 46c, 681, 9b4\},$$
$$S_7 = \{125, 236, 347, 78b, 9ad, 46c, 9b4, ac5, d28\},$$
$$S_8 = \{125, 236, 347, 89c, 9ad, bc2, 46c, 57d, 681\},$$
$$S_9 = \{125, 236, 347, 89c, 9ad, 35b, 46c, 57d, 681\},$$
$$S_{10} = \{125, 236, 347, 89c, 9ad, 46c, 57d, 681, 9b4\},$$
$$S_{11} = \{125, 236, 347, 89c, ab1, 35b, 57d, 9b4, d28\},$$
$$S_{12} = \{125, 236, 347, 89c, ab1, 46c, 57d, 9b4, bd6\},$$
$$S_{13} = \{125, 236, 347, 89c, ab1, 57d, 792, ac5, bd6\},$$
$$S_{14} = \{125, 236, 78b, 9ad, 24a, 46c, 57d, 681, c17\},$$
$$S_{15} = \{125, 236, 78b, 9ad, 46c, 57d, 681, 9b4, c17\},$$
$$S_{16} = \{125, 236, 78b, 9ad, 46c, 681, 8a3, 9b4, c17\},$$
$$S_{17} = \{125, 236, 78b, 9ad, 46c, 8a3, 9b4, c17, d28\}.$$

All of these sets are defining sets. The proof of this assertion is only given for the set S_{10}. For a full account with all proofs, see [6].

Lemma 5.1 *The set of blocks*

$$S_{10} = \{125, 236, 347, 89c, 9ad, 46c, 57d, 681, 9b4\}$$

is a 2-(13, 3, 1) *defining set.*

Proof. Element 9 must be in three more blocks, with the elements 1, 2, 3, 5, 6,

7. Blocks 125, 236 and the pair 16 have all occurred, forcing blocks 139, 279, 569. Now 6 must appear twice more, with the elements 7, a, b, d. Pairs ad, $7d$ force blocks $67a$, $6bd$. Now 7 must occur in two more blocks, with the elements 1, 8, b, c. Pairs 18, $8c$ force blocks $78b$, $17c$. Element 1 must be in two more blocks, with elements 4, a, b, d. Pairs ad, bd force blocks $14d$, $1ab$. Next, b is in two more blocks, with elements 2, 3, 5, c. Pairs 23, 25 force blocks $2bc$, $35b$. Element 2 must appear twice more, with elements 4, 8, a, d. Pairs $4d$, ad force blocks $28d$, $24a$. Now element 4 must be in block 458, 5 in block $5ac$, 8 in the block $38a$ and the last block must be $3cd$. Hence the set S_{10} completes uniquely to the 2-(13,3,1) design D_1 and so it is a defining set. □

Theorem 5.2 *There are, up to isomorphism, exactly 17 smallest defining sets for the cyclic 2-$(13,3,1)$ design D_1. Each consists of nine blocks and has no non-trivial automorphisms, and the number of isomorphic copies of each is $\frac{|Aut(D_1)|}{1} = 39$. Hence there are exactly 663 distinct smallest defining sets of D_1.*

Proof. Lemma 4.1 shows that there are no defining sets of D_1 of six blocks. Tables 14 - 11 show that there are no defining sets of D_1 with seven or eight blocks. Tables 12 and 13 show that there are exactly 17 feasible classes of sets of nine blocks. Lemmas similar to Lemma 5.1 show that representatives of these classes are defining sets of D_1. Hence they are smallest defining sets of D_1, and $|d_s(D_1)| = 9$. □

6. The non-cyclic 2-(13,3,1) design D_2

The lower bound on defining sets of 2-(13,3,1) designs was shown to be seven blocks, by Lemma 4.1. The $\binom{26}{7} = 657800$ sets of seven blocks of D_2 form exactly 1259 isomorphism classes. Of these classes, all but four are infeasible as the size of the class j_i is not a factor of 6, the order of the automorphism group of D_2. The remaining four classes are shown in Table 14, and clearly none of these sets is a defining set. Finally, consider all $\binom{26}{8} = 1562275$ sets of eight blocks of D_2. The sets fall into 7843 isomorphism classes. A total of 54 of these classes satisfy j_i divides 6. The information about these classes is given in Tables 15, 16. Exactly two of these classes are feasible, classes 13 and 14.

The set of blocks

$$S_1 = \{12d,\ 347,\ 78b,\ 9ad,\ 24a,\ 35b,\ 5ac,\ 6bd\}$$

belongs to isomorphism class number 13. The order of its automorphism group is 1 and there are six isomorphic copies of S_1 in the design D_2.

The set of blocks

$$S_2 = \{12d,\ 569,\ 89c,\ 1ab,\ 3cd,\ 145,\ 24a,\ 46c\}$$

belongs to isomorphism class number 14. The order of its automorphism group is 1 and there are six isomorphic copies of S_2 in the design D_2.

trade number	blocks	automorphism
1	2, 3, 6, 15	$T_1\psi^2$
2	3, 4, 7, 16	$T_1\psi^3$
3	4, 5, 8, 17	$T_1\psi^4$
4	5, 6, 9, 18	$T_1\psi^5$
5	6, 7, 10, 19	$T_1\psi^6$
6	7, 8, 11, 20	$T_1\psi^7$
7	8, 9, 12, 21	$T_1\psi^8$
8	9, 10, 13, 22	$T_1\psi^9$
9	2, 11, 12, 24	$T_1\psi^{11}$
10	3, 12, 13, 25	$T_1\psi^{12}$
11	3, 4, 12, 18, 21, 23	U_1
12	4, 5, 13, 19, 22, 24	$U_1\psi$
13	4, 8, 9, 15, 23, 26	$U_1\psi^5$
14	5, 9, 10, 14, 16, 24	$U_1\psi^6$
15	6, 10, 11, 15, 17, 25	$U_1\psi^7$
16	7, 11, 12, 16, 18, 26	$U_1\psi^8$
17	5, 6, 8, 19, 23, 25	$U_1\psi^4\phi$
18	6, 7, 9, 20, 24, 26	$U_1\psi^4\phi\psi$

Table 3: The 18 trades which rule out $n = 7$, $n = 8$ for the design D_1

The next lemma shows that S_1 is a defining set of the design D_2. For the proof that S_2 is also a defining set of D_2, see [6].

Lemma 6.1 *The set of blocks*

$$S_1 = \{\, 12d,\ 347,\ 78b,\ 9ad,\ 24a,\ 35b,\ 5ac,\ 6bd \,\}$$

is a 2-$(13,3,1)$ *defining set.*

Proof. Now element a must occur in three more blocks, with elements 1, 3, 6, 7, 8, b. The block $78b$ and pairs 37, $3b$, $6b$ have all occurred, forcing completion of the three blocks to $67a, 38a, 1ab$. The element d must occur in three more blocks, with elements 3, 4, 5, 7, 8, c. The block 347 and pairs 78, 38, 35 have all appeared, forcing blocks $3cd, 48d, 57d$. The element 3 occurs twice more, with elements 1, 2, 6, 9. The pair 12 is present, so there are two cases:

Case α: The blocks containing 3 are $136, 239$.
Now element 7 must occur in two more blocks, with elements 1, 2, 9, c. Pairs 12, 29 force blocks $179, 27c$. Now element b must occur twice more, with elements 2, 4, 9, c. But pairs 24, 29, $2c$ are all already present, so Case α is impossible.

Case β: The blocks containing 3 are $139, 236$.

trade number	blocks
i	2, 3, 6, 15
ii	3, 4, 12, 18, 21, 23
iii	4, 5, 8, 12, 13, 14, 18, 19, 20, 26
iv	4, 6, 10, 11, 15, 18, 23, 24, 25, 26
v	4, 10, 11, 14, 15, 16, 20, 21, 22, 25
vi	5, 7, 9, 14, 18, 19, 20, 21, 24, 26
vii	5, 10, 11, 14, 16, 17, 20, 22, 23, 25
viii	8, 10, 11, 12, 13, 14, 15, 19, 24, 26
ix	8, 10, 11, 12, 13, 14, 15, 20, 25, 26
x	4, 5, 8, 12, 13, 14, 15, 19, 21, 23, 26
xi	4, 8, 10, 12, 13, 14, 15, 16, 21, 22, 26
xii	4, 5, 6, 7, 10, 11, 16, 20, 21, 22, 23, 25
xiii	4, 5, 6, 8, 10, 11, 14, 16, 20, 22, 23, 25
xiv	4, 5, 10, 12, 13, 14, 16, 17, 19, 21, 22, 23
xv	4, 5, 9, 12, 14, 15, 18, 19, 20, 21, 23, 26
xvi	4, 5, 10, 11, 12, 14, 15, 16, 19, 21, 23, 26
xvii	4, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 25
xviii	6, 7, 9, 11, 14, 15, 17, 19, 20, 21, 25, 26
xix	4, 5, 7, 11, 12, 13, 17, 18, 20, 21, 22, 23, 25, 26
xx	4, 5, 9, 10, 11, 13, 14, 15, 18, 19, 20, 21, 25, 26
xxi	4, 7, 10, 11, 12, 13, 14, 15, 16, 20, 22, 23, 25, 26
xxii	5, 6, 7, 9, 11, 12, 14, 15, 16, 17, 20, 21, 24, 25
xxiii	3, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 19, 20, 21, 24
xxiv	4, 5, 6, 7, 10, 11, 12, 13, 14, 16, 17, 20, 21, 22, 23, 24, 26

Table 4: The 24 trades which generate all trades needed for $n = 9$

Now element 7 must occur twice more, with elements 1, 2, 9, c. Pairs 12, 19 force blocks $17c, 279$. Now element b must be in two more blocks, with elements 2, 4, 9, c. Pairs 24, 29 have occurred, forcing blocks $2bc, 49b$. Element 4 occurs twice more, with elements 1, 5, 6, c. Pairs $1c$, $5c$ force blocks $145, 46c$. Element c must occur in block $89c$, element 9 must occur in block 569, element 1 must occur in block 168 and the last block must be block 258. Hence the set S_1 completes uniquely to the 2-(13,3,1) design D_2 and so it is a defining set. □

Theorem 6.2 *There are, up to isomorphism, exactly two smallest defining sets of the* 2-(13, 3, 1) *design* D_2, *both of these sets consist of eight blocks. Each has no non-trivial automorphisms, and the number of isomorphic copies of each is* $\frac{|Aut(D_2)|}{1} = 6$. *Thus the total number of distinct smallest defining sets of* D_2 *is* 12.

Proof. Lemma 4.1 shows that a defining set of D_2 must have at least seven blocks. But only four isomorphism classes of sets of seven blocks of D_2 have a size which is

class number i	blocks of S_i	size j_i	group order o_i	feasible ?
1	1, 2, 5, 7, 15, 17, 23	13	36	no
2	1, 3, 8, 14, 20, 24, 25	13	36	no
3	1, 2, 5, 14, 15, 17, 23	13	144	no
4	1, 2, 9, 11, 17, 19, 25	13	144	no
5	1, 2, 5, 14, 16, 19, 20	13	576	no
6	1, 2, 6, 19, 21, 22, 26	39	6	no
7	1, 3, 14, 15, 20, 24, 25	39	6	no
8	1, 2, 9, 17, 18, 19, 21	39	8	no
9	1, 2, 19, 21, 22, 24, 26	39	8	no
10	1, 3, 14, 16, 17, 22, 26	39	8	no
11	1, 3, 15, 16, 18, 19, 24	39	8	no
12	1, 3, 15, 16, 19, 20, 22	39	8	no
13	1, 3, 16, 17, 19, 20, 22	39	8	no
14	1, 2, 7, 16, 24, 25, 26	39	12	no
15	1, 2, 19, 21, 23, 24, 26	39	12	no
16	1, 2, 5, 7, 14, 18, 21	39	16	no
17	1, 2, 6, 14, 15, 16, 17	39	16	no
18	1, 2, 7, 15, 17, 22, 24	39	16	no
19	1, 3, 8, 16, 17, 18, 24	39	16	no
20	1, 3, 16, 19, 22, 23, 26	39	16	no
21	1, 3, 17, 19, 20, 21, 23	39	16	no
22	1, 17, 18, 20, 21, 22, 24	39	24	no
23	1, 2, 3, 4, 5, 7, 14	39	48	no
24	1, 2, 3, 6, 8, 15, 18	39	48	no
25	1, 2, 3, 6, 8, 15, 24	39	48	no
26	1, 2, 3, 8, 10, 18, 24	39	48	no
27	1, 2, 6, 18, 19, 22, 23	39	48	no
28	1, 2, 4, 8, 14, 17, 19	39	96	no
29	1, 2, 5, 7, 14, 16, 19	39	192	no
30	1, 2, 3, 5, 6, 14, 15	39	288	no
31	1, 2, 4, 15, 19, 21, 26	39	288	no

Table 5: The classes of sets of seven blocks of D_1 for which j_i divides 39

class number i	blocks of S_i	size j_i	group order o_i	feasible ?
1	1, 2, 3, 7, 9, 10, 12, 17	13	6	no
2	1, 14, 17, 21, 22, 23, 24, 26	13	12	no
3	1, 2, 6, 12, 19, 21, 22, 26	13	18	no
4	1, 2, 3, 4, 7, 9, 11, 16	13	36	no
5	1, 2, 5, 7, 14, 15, 17, 23	13	36	no
6	1, 2, 5, 7, 14, 18, 24, 26	13	36	no
7	1, 2, 6, 12, 18, 22, 23, 25	13	36	no
8	1, 2, 9, 11, 16, 17, 19, 25	13	36	no
9	1, 17, 18, 20, 21, 22, 24, 25	13	36	no
10	1, 2, 4, 10, 15, 16, 19, 21	13	288	no
11	1, 2, 5, 7, 14, 16, 19, 20	13	576	no

Table 6: The classes of sets of eight blocks of D_1 for which $j_i = 13$

a factor of $|Aut(D_2)| = 6$. These four are shown in Table 14 and they are infeasible. Hence there are no defining sets of seven blocks. The information given in Tables 15, 16 shows that there are only two feasible classes of sets of eight blocks. Lemma 6.1 and a similar lemma for S_2 show that representatives of these classes are defining sets of D_2. Hence they are smallest defining sets of D_2, and $|d_s(D_2)| = 8$. □

class number i	blocks of S_i	trade number	class number i	blocks of S_i	trade number
1	1, 2, 3, 4, 7, 9,22,26	15	40	1, 2, 3, 10, 16, 18, 22, 26	3
2	1, 2, 3, 5, 7, 8, 9,23	15	41	1, 2, 3, 10, 18, 20, 22, 26	3
3	1, 2, 3, 5, 7,21,24,25	8	42	1, 2, 3, 10, 19, 20, 22, 26	3
4	1, 2, 3, 6, 8,19,22,26	14	43	1, 2, 3, 11, 16, 19, 22, 26	3
5	1, 2, 3, 6,16,19,22,26	3	44	1, 2, 3, 16, 19, 21, 22, 26	3
6	1, 2, 3, 7, 8, 9,20,23	12	45	1, 2, 3, 16, 19, 22, 25, 26	3
7	1, 2, 3, 7, 8, 9,22,23	15	46	1, 2, 3, 18, 22, 23, 24, 25	3
8	1, 2, 3, 7, 9,10,17,22	17	47	1, 2, 3, 19, 21, 22, 24, 26	3
9	1, 2, 3, 7, 9,16,22,26	3	48	1, 2, 4, 6, 12, 21, 22, 24	6
10	1, 2, 3, 7, 9,19,22,26	3	49	1, 2, 4, 6, 12, 22, 23, 24	6
11	1, 2, 3, 7, 9, 22, 23, 26	3	50	1, 2, 4, 6, 16, 19, 22, 25	6
12	1, 2, 3, 7, 10, 12, 17, 21	4	51	1, 2, 4, 6, 16, 22, 24, 25	6
13	1, 2, 3, 7, 10, 12, 14, 17	4	52	1, 2, 4, 8, 10, 21, 23, 24	4
14	1, 2, 3, 7, 10, 12, 17, 19	4	53	1, 2, 4, 9, 19, 23, 24, 25	6
15	1, 2, 3, 7, 10, 18, 22, 26	3	54	1, 2, 4, 9, 22, 23, 24, 25	5
16	1, 2, 3, 7, 10, 19, 22, 26	3	55	1, 2, 4, 12, 14, 15, 23, 24	4
17	1, 2, 3, 7, 17, 19, 22, 23	4	56	1, 2, 4, 12, 14, 17, 20, 23	4
18	1, 2, 3, 8, 9, 15, 16, 25	5	57	1, 2, 4, 12, 15, 21, 22, 24	4
19	1, 2, 3, 8, 9, 15, 19, 25	16	58	1, 2, 4, 14, 17, 20, 23, 26	4
20	1, 2, 3, 8, 9, 19, 22, 23	15	59	1, 2, 4, 18, 22, 23, 24, 25	5
21	1, 2, 3, 8, 10, 14, 20, 23	4	60	1, 2, 4, 21, 22, 23, 24, 25	4
22	1, 2, 3, 8, 10, 17, 18, 22	18	61	1, 2, 5, 7, 15, 17, 18, 22	7
23	1, 2, 3, 8, 16, 17, 18, 19	8	62	1, 2, 6, 8, 15, 16, 19, 22	10
24	1, 2, 3, 8, 16, 19, 22, 26	4	63	1, 2, 6, 8, 17, 18, 22, 26	2
25	1, 2, 3, 8, 17, 18, 22, 26	5	64	1, 2, 7, 8, 15, 17, 18, 22	10
26	1, 2, 3, 9, 10, 16, 17, 26	6	65	1, 2, 7, 8, 15, 19, 23, 24	4
27	1, 2, 3, 9, 10, 16, 22, 26	3	66	1, 2, 7, 8, 17, 18, 21, 22	10
28	1, 2, 3, 9, 11, 18, 19, 22	3	67	1, 2, 7, 8, 17, 18, 22, 26	10
29	1, 2, 3, 9, 14, 18, 19, 23	3	68	1, 2, 7, 9, 14, 24, 25, 26	3
30	1, 2, 3, 9, 16, 19, 22, 26	3	69	1, 2, 7, 9, 15, 17, 18, 19	10
31	1, 2, 3, 9, 17, 18, 19, 21	6	70	1, 2, 7, 9, 17, 18, 22, 26	10
32	1, 2, 3, 9, 17, 19, 22, 23	6	71	1, 2, 7, 9, 17, 22, 24, 25	11
33	1, 2, 3, 9, 19, 20, 22, 23	3	72	1, 2, 7, 9, 18, 19, 22, 26	3
34	1, 2, 3, 9, 19, 21, 22, 26	3	73	1, 2, 7, 9, 19, 22, 23, 26	3
35	1, 2, 3, 9, 19, 22, 23, 26	3	74	1, 2, 7, 9, 19, 22, 25, 26	3
36	1, 2, 3, 10, 11, 18, 22, 26	3	75	1, 2, 7, 9, 19, 23, 24, 26	3
37	1, 2, 3, 10, 11, 19, 22, 26	3	76	1, 2, 7, 14, 15, 17, 18, 22	7
38	1, 2, 3, 10, 12, 19, 20, 23	3	77	1, 2, 7, 14, 17, 18, 21, 25	8
39	1, 2, 3, 10, 12, 21, 22, 26	3	78	1, 2, 7, 14, 17, 18, 22, 26	7

Table 7: Sets of eight blocks of D_1 with size $j_i = 39$, group order $o_i = 1$: classes 40..78

class number i	blocks of S_i	trade number	class number i	blocks of S_i	trade number
79	1, 2, 7, 14, 17, 22, 24, 26	4	118	1, 2, 14, 18, 22, 23, 24, 25	2
80	1, 2, 7, 14, 22, 24, 25, 26	3	119	1, 2, 15, 18, 19, 21, 22, 24	2
81	1, 2, 7, 15, 16, 17, 18, 19	7	120	1, 3, 8, 14, 17, 18, 19, 21	8
82	1, 2, 7, 15, 16, 17, 18, 20	7	121	1, 2, 17, 18, 22, 23, 24, 25	2
83	1, 2, 7, 15, 16, 17, 18, 22	7	122	1, 3, 8, 14, 18, 19, 22, 23	9
84	1, 2, 7, 15, 16, 17, 18, 25	7	123	1, 3, 8, 14, 18, 22, 23, 24	5
85	1, 2, 7, 15, 17, 18, 21, 22	10	124	1, 3, 8, 15, 16, 19, 22, 24	4
86	1, 2, 7, 15, 17, 18, 22, 23	7	125	1, 3, 9, 14, 15, 17, 23, 24	5
87	1, 2, 7, 15, 17, 18, 22, 24	7	126	1, 3, 9, 14, 16, 19, 22, 26	3
88	1, 2, 7, 15, 18, 19, 22, 24	3	127	1, 3, 9, 16, 19, 22, 25, 26	3
89	1, 2, 7, 15, 18, 19, 23, 24	3	128	1, 3, 14, 15, 16, 19, 22, 26	3
90	1, 2, 7, 15, 19, 20, 23, 24	3	129	1, 3, 14, 15, 18, 22, 23, 24	3
91	1, 2, 7, 15, 19, 23, 24, 26	3	130	1, 3, 14, 16, 17, 19, 22, 26	4
92	1, 2, 7, 17, 18, 21, 22, 25	13	131	1, 3, 14, 17, 18, 19, 22, 23	6
93	1, 2, 8, 14, 17, 18, 22, 26	2	132	1, 3, 14, 17, 18, 22, 23, 26	5
94	1, 2, 8, 14, 18, 19, 21, 24	2	133	1, 3, 14, 17, 20, 24, 25, 26	4
95	1, 2, 8, 15, 16, 19, 22, 25	4	134	1, 3, 14, 17, 21, 22, 24, 26	4
96	1, 2, 8, 15, 17, 18, 22, 26	2	135	1, 3, 14, 17, 22, 23, 24, 25	4
97	1, 2, 8, 16, 17, 20, 21, 25	4	136	1, 3, 14, 18, 22, 23, 24, 25	3
98	1, 2, 8, 16, 18, 19, 22, 26	10	137	1, 3, 15, 16, 18, 19, 20, 24	3
99	1, 2, 8, 16, 18, 20, 21, 25	5	138	1, 3, 15, 16, 18, 19, 22, 24	3
100	1, 2, 8, 16, 18, 22, 25, 26	5	139	1, 3, 15, 16, 19, 20, 22, 24	3
101	1, 2, 8, 16, 20, 21, 24, 25	4	140	1, 3, 15, 16, 19, 20, 23, 24	3
102	1, 2, 8, 17, 18, 19, 22, 26	2	141	1, 3, 15, 16, 19, 20, 24, 25	3
103	1, 2, 8, 17, 18, 20, 22, 26	2	142	1, 3, 15, 16, 19, 22, 25, 26	3
104	1, 2, 8, 17, 18, 22, 25, 26	2	143	1, 3, 15, 18, 19, 21, 22, 24	3
105	1, 2, 8, 18, 22, 23, 24, 25	2	144	1, 3, 15, 19, 21, 22, 24, 26	3
106	1, 2, 9, 11, 14, 15, 17, 18	2	145	1, 3, 16, 17, 18, 19, 20, 21	8
107	1, 2, 9, 11, 17, 18, 19, 21	2	146	1, 3, 16, 17, 18, 19, 20, 22	7
108	1, 2, 9, 14, 15, 18, 21, 25	2	147	1, 3, 16, 17, 19, 20, 21, 22	4
109	1, 2, 9, 15, 17, 18, 21, 25	2	148	1, 3, 16, 17, 19, 22, 24, 25	4
110	1, 2, 9, 15, 17, 19, 22, 23	2	149	1, 3, 16, 17, 19, 22, 25, 26	4
111	1, 2, 9, 17, 19, 22, 23, 25	2	150	1, 3, 16, 18, 19, 22, 25, 26	3
112	1, 2, 9, 17, 19, 22, 23, 26	2	151	1, 3, 16, 19, 20, 22, 24, 25	3
113	1, 2, 9, 17, 21, 22, 25, 26	2	152	1, 3, 16, 19, 22, 24, 25, 26	3
114	1, 2, 9, 18, 22, 23, 24, 25	2	153	1, 3, 17, 18, 19, 20, 21, 24	8
115	1, 2, 9, 19, 22, 23, 24, 26	2	154	1, 3, 17, 18, 19, 20, 22, 23	7
116	1, 2, 9, 19, 23, 24, 25, 26	2	155	1, 3, 17, 18, 19, 22, 23, 24	6
117	1, 2, 9, 22, 23, 24, 25, 26	2	156	1, 3, 17, 19, 21, 22, 24, 26	4

Table 8: Sets of eight blocks of D_1 with size $j_i = 39$, group order $o_i = 1$: classes 79..156

class number i	blocks of S_i	trade number
157	1, 3, 17, 21, 22, 23, 24, 26	4
158	1, 3, 18, 19, 22, 23, 24, 25	3
159	1, 3, 18, 21, 22, 23, 24, 25	3
160	1, 14, 15, 17, 18, 22, 23, 26	2
161	1, 14, 17, 18, 20, 22, 23, 26	1
162	1, 14, 17, 18, 20, 24, 25, 26	1
163	1, 14, 17, 18, 21, 22, 23, 24	1

class number i	blocks of S_i	trade number
164	1, 14, 17, 18, 22, 23, 24, 26	1
165	1, 15, 16, 17, 18, 19, 20, 22	7
166	1, 15, 16, 17, 18, 19, 20, 24	7
167	1, 15, 16, 17, 18, 19, 22, 24	6
168	1, 15, 16, 17, 18, 21, 22, 25	5
169	1, 15, 16, 17, 20, 21, 24, 25	4

Table 9: Sets of eight blocks of D_1 with size $j_i = 39$, group order $o_i = 1$: classes 157..169

class number i	blocks of S_i	size j_i	group order o_i	trade number
1	1, 2, 5, 15, 17, 21, 22, 23, 25	13	3	2
2	1, 2, 3, 4, 7, 11, 18, 21, 22	13	6	**
3	1, 2, 3, 7, 9, 12, 18, 20, 26	13	6	**
4	1, 2, 3, 7, 9, 12, 19, 22, 23	13	6	**
5	1, 2, 5, 15, 16, 17, 19, 20, 23	13	6	**
6	1, 3, 8, 16, 17, 18, 19, 22, 26	13	6	**
7	1, 3, 9, 16, 17, 19, 21, 23, 24	13	6	**
8	1, 2, 3, 5, 8, 10, 21, 22, 25	13	12	**
9	1, 3, 9, 14, 15, 17, 18, 19, 24	13	12	**
10	1, 2, 5, 16, 18, 19, 20, 24, 26	13	18	**
11	1, 3, 8, 14, 15, 20, 21, 23, 25	13	18	**
12	1, 2, 3, 5, 8, 10, 18, 24, 26	13	24	**
13	1, 2, 4, 6, 8, 9, 14, 17, 18	13	36	**
14	1, 2, 3, 5, 6, 9, 14, 15, 18	13	72	**

Table 10: The classes of sets of nine blocks of D_1 for which $j_i = 13$

class number i	blocks of S_i	size j_i	group order o_i	trade number
1	1, 2, 3, 4, 6, 8, 9, 11, 16	39	1	***
2	1, 2, 3, 4, 6, 8, 9, 16, 24	39	1	***
3	1, 2, 3, 4, 6, 8, 12, 22, 24	39	1	***
4	1, 2, 3, 4, 8, 9, 10, 16, 17	39	1	***
5	1, 2, 3, 4, 9, 10, 15, 16, 24	39	1	6
6	1, 2, 3, 5, 8, 16, 20, 24, 25	39	1	5
7	1, 2, 3, 6, 8, 18, 22, 23, 24	39	1	***
8	1, 2, 3, 7, 8, 18, 22, 23, 24	39	1	15
9	1, 2, 3, 7, 9, 10, 17, 19, 22	39	1	***
10	1, 2, 3, 7, 9, 17, 22, 23, 26	39	1	***
11	1, 2, 3, 8, 9, 11, 17, 18, 19	39	1	***
12	1, 2, 3, 8, 9, 16, 17, 18, 19	39	1	***
13	1, 2, 3, 8, 9, 16, 22, 23, 25	39	1	5
14	1, 2, 3, 8, 9, 17, 18, 19, 22	39	1	***
15	1, 2, 3, 8, 10, 16, 18, 22, 26	39	1	***
16	1, 2, 3, 8, 10, 17, 18, 22, 24	39	1	***
17	1, 2, 3, 8, 10, 18, 20, 23, 24	39	1	***
18	1, 2, 3, 9, 14, 15, 18, 22, 23	39	1	3
19	1, 2, 3, 12, 17, 18, 22, 23, 24	39	1	5
20	1, 2, 4, 8, 10, 15, 21, 23, 24	39	1	4
21	1, 2, 4, 17, 20, 21, 22, 24, 26	39	1	4
22	1, 2, 7, 8, 15, 19, 22, 23, 24	39	1	4
23	1, 2, 7, 9, 15, 17, 18, 19, 25	39	1	***
24	1, 2, 7, 9, 17, 18, 19, 22, 25	39	1	***
25	1, 2, 7, 9, 17, 19, 21, 22, 25	39	1	***
26	1, 2, 7, 9, 17, 21, 22, 25, 26	39	1	***
27	1, 2, 7, 15, 17, 18, 19, 23, 24	39	1	7
28	1, 2, 9, 14, 16, 17, 20, 23, 26	39	1	5
29	1, 2, 16, 19, 20, 21, 22, 24, 25	39	1	3
30	1, 3, 15, 16, 19, 20, 21, 22, 25	39	1	3

Table 11: Some classes of sets of nine blocks of D_1 for which $j_i = 39$

class number i	blocks of S_i	size j_i	group order o_i	feasible ?
1	1, 4, 6, 8, 11, 14, 16	6	48	no
2	1, 2, 6, 8, 10, 15, 24	6	36	no
3	1, 3, 9, 10, 11, 19, 24	6	36	no
4	2, 3, 6, 8, 13, 15, 24	6	192	no

Table 12: The classes of sets of seven blocks of D_2 for which j_i divides 6

class number i	blocks of S_i	size j_i	group order o_i	feasible ?
1	3, 5, 8, 11, 12, 13, 18, 19	3	4	no
2	1, 4, 6, 8, 11, 12, 14, 16	3	12	no
3	1, 4, 6, 8, 11, 13, 14, 16	3	12	no
4	1, 2, 7, 9, 17, 21, 22, 25	3	16	no
5	1, 5, 6, 8, 12, 18, 21, 22	3	16	no
6	1, 6, 12, 13, 14, 17, 20, 23	3	16	no
7	2, 3, 7, 9, 11, 13, 19, 23	3	16	no
8	3, 6, 8, 11, 12, 13, 15, 24	3	16	no
9	1, 3, 8, 10, 15, 16, 18, 24	3	24	no
10	2, 6, 7, 8, 12, 13, 20, 24	3	24	no
11	1, 6, 9, 11, 13, 21, 22, 23	3	64	no
12	1, 9, 10, 11, 13, 15, 22, 23	3	192	no
13	1, 3, 7, 9, 15, 16, 23, 24	6	1	yes
14	1, 5, 8, 10, 12, 13, 15, 17	6	1	yes
15	1, 3, 5, 8, 10, 11, 13, 15	6	2	no
16	1, 3, 6, 11, 14, 21, 24, 26	6	2	no
17	1, 3, 8, 9, 10, 13, 17, 24	6	2	no
18	1, 5, 7, 8, 12, 13, 15, 21	6	2	no
19	1, 5, 8, 10, 11, 13, 15, 17	6	2	no
20	1, 5, 8, 10, 12, 15, 22, 25	6	2	no
21	1, 2, 3, 6, 7, 9, 15, 16	6	4	no
22	1, 2, 5, 6, 8, 10, 13, 15	6	4	no
23	1, 3, 4, 6, 8, 11, 14, 16	6	4	no
24	1, 3, 5, 9, 10, 11, 13, 17	6	4	no
25	1, 3, 8, 10, 16, 17, 18, 24	6	4	no
26	1, 5, 6, 9, 11, 13, 21, 25	6	4	no
27	1, 5, 8, 10, 13, 15, 17, 24	6	4	no

Table 13: Sets of eight blocks of D_2 for which j_i divides 6: classes 1..27

class number i	blocks of S_i	size j_i	group order o_i	feasible ?
28	1, 5, 10, 11, 15, 18, 21, 22	6	4	no
29	1, 5, 7, 8, 10, 12, 13, 15	6	6	no
30	1, 2, 6, 7, 9, 12, 13, 14	6	8	no
31	1, 3, 5, 8, 10, 13, 15, 24	6	8	no
32	1, 3, 5, 11, 14, 19, 21, 26	6	8	no
33	1, 3, 8, 10, 12, 13, 17, 18	6	8	no
34	1, 5, 7, 8, 11, 12, 14, 15	6	8	no
35	1, 5, 8, 10, 12, 15, 18, 22	6	8	no
36	1, 2, 5, 7, 9, 12, 14, 15	6	12	no
37	1, 2, 6, 8, 10, 13, 15, 24	6	12	no
38	1, 2, 4, 8, 10, 13, 18, 26	6	16	no
39	1, 8, 10, 13, 16, 17, 18, 24	6	16	no
40	1, 5, 7, 10, 11, 12, 15, 25	6	18	no
41	1, 2, 7, 9, 13, 14, 17, 23	6	24	no
42	1, 2, 6, 8, 11, 12, 16, 24	6	32	no
43	2, 3, 5, 7, 9, 15, 18, 25	6	32	no
44	1, 2, 3, 6, 8, 10, 13, 15	6	36	no
45	1, 2, 3, 6, 8, 10, 15, 24	6	36	no
46	1, 3, 4, 5, 10, 11, 21, 25	6	36	no
47	1, 3, 8, 9, 10, 11, 19, 24	6	36	no
48	1, 3, 8, 10, 13, 16, 18, 24	6	36	no
49	1, 5, 6, 9, 10, 11, 18, 25	6	36	no
50	2, 3, 6, 8, 13, 15, 17, 24	6	64	no
51	1, 2, 9, 10, 14, 19, 21, 24	6	96	no
52	1, 3, 12, 13, 14, 18, 20, 25	6	96	no
53	2, 3, 5, 6, 9, 15, 18, 20	6	96	no
54	1, 2, 4, 12, 15, 17, 19, 21	6	288	no

Table 14: Sets of eight blocks of D_2 for which j_i divides 6: classes 28..54

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