The Edge Reconstruction of 3-Connected Projective Graphs With Minimum Valency 5

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Abstract. In this paper, we prove that 3-connected projective graphs with minimum valency 5 are edge reconstructible.

In this paper, all graphs $G = (V(G), E(G))$ considered will be finite and simple. Let $n = |V(G)|$, $m = |E(G)|$. For a vertex $v \in V(G)$, let $d(v)$ be the degree of v. If $d(v) = k$, we call v a *k-vertex* and we use $n_k(G)$ to denote the number of k-vertices in G. The maximum degree in G is denoted by Δ.

A graph G is said to have *connectivity* $k = k(G)$ if G is a complete graph on $k+1$ vertices or the deletion of some set of k vertices disconnects G, and k is the least integer with this property. For any $t \leq k$, G is said to be *t-connected.* A graph is *planar* (or *projective*) if it can be embedded in the plane (or projective plane). Such an embedding is called a *plane* (or *projective plane*) *graph.*

Let P be the projective plane. We identify a graph G embedded in P with its image in P. We use S to denote a sphere. Since the sphere S is a double cover of the projective plane as a surface covering, it induces a double cover G' embedded in the sphere of any graph G embedded in the projective plane.

Let G be embedded in a surface Σ, where $\Sigma = S$ or P. The connected components of $\Sigma - G$ are called the *faces* of the embedding of G in Σ. If F is a face and v is a vertex on the boundary of F, then we say that v is incident to F. If the boundary of F is a k-circuit, we say that F is a *k-face.* If all the faces of an embedding of G in Σ are homeomorphic to the open unit disc, then the embedding is said to be a 2-cell *embedding.* All embeddings considered in this paper are 2-cell embeddings. If a graph G with n vertices and m edges has an embedding in a surface Σ of characteristic $\chi(\Sigma)$, and if f is the number of faces of the embedding, then Euler's formula states that

$$n - m + f = \chi(\Sigma).$$

Since G is finite and simple, each face has degree ≥ 3. This implies that $m \leq 3n - 3\chi(\Sigma)$, with equality holding if and only if G triangulates Σ. The theory of surface embeddings of graphs is dealt with in detail in [4], to which the reader is referred.

A graph G is *edge reconstructible* if its isomorphism class is uniquely determined by the collection $\mathcal{D}(G) = \{G - e : e \in E(G)\}$ of single-edge-deleted subgraphs of G. The edge form of the reconstruction conjecture states that every graph with at least four edges is edge reconstructible. In [3], it is proved that 3-connected planar graphs with minimum valency 5 are edge reconstructible. In this paper, we focus our attention on 3-connected projective graphs and give a simple proof of the following theorem.

Theorem. *The 3-connected projective graphs with minimum valency 5 are edge reconstructible.*

Before we prove our theorem, we need several lemmas. The first lemma is due to Hoffman [See 1].

Lemma 1. *Let G be a graph of minimum valency δ. Suppose that, for some $k \geq 0$, there is a vertex in G of degree $\delta + k$ adjacent to $k + 1$ or more vertices of degree δ. Then G is edge reconstructible.*

We define $E_{\delta,i}(G) = \{e \in E(G)$: one end of e is incident to a δ-vertex and the other end of e is incident to an i-vertex$\}$ and $t_{\delta,i}(G) = |E_{\delta,i}(G)|$, for $i \geq \delta$. By Lemma 1, if a graph G of minimum valency δ is not edge reconstructible, then $t_{\delta,i}(G) \leq (i - \delta) n_i(G)$.

The second lemma is a special case of Lemma 1.4 of [2].

Lemma 2. *If a graph G of minimum valency δ contains a triangle which is incident to one δ-vertex and two $(\delta + 1)$-vertices, then G is edge reconstructible.*

Lemma 3. *Let G be a planar (or projective) graph with minimum valency 5. If G satisfies $t_{5,i} \leq (i - 5) n_i$, for all $i \in \{5, 6, \ldots, \Delta\}$, then there exists at least one 5-vertex which is adjacent to at least four 6-vertices.*

Proof. We only consider the case when G is a planar graph. For the other case, the proof is similar. Since $m \leq 3n - 6$, and since $2m = \sum_{i=5}^{\Delta} i n_i$ and $n = \sum_{i=5}^{\Delta} n_i$, we have

$$n_5 \geq n_7 + \cdots + (\Delta - 6) n_\Delta + 12.$$

By $5 n_5 = \sum_{i=6}^{\Delta} t_{5,i}$, we have

$$\begin{aligned} t_{5,6} &= 5 n_5 - t_{5,7} - \ldots - t_{5,\Delta} \\ &\geq 3 n_5 + (2 n_7 - t_{5,7}) + \ldots + \left[(\Delta - 5) n_\Delta - t_{5,\Delta}\right] + n_8 + \ldots + (\Delta - 7) n_\Delta + 24 \\ &\geq 3 n_5 + n_8 + \ldots + (\Delta - 7) n_\Delta + 24 . \end{aligned}$$

Hence $t_{5,6} > 3 n_5$. This implies that G has at least one 5-vertex which is adjacent to at least four 6-vertices. ∎

Before we state our next lemma, we give two more definitions. A simple closed curve Γ of P is *essential* if $P - \Gamma$ is connected. For any graph G embedded in P, the *representativity* $\rho(G)$ of G is the minimum of $|\Gamma \cap G|$ where Γ ranges over all essential curves in P. Representativity of an embedding is a measure of how "densely" the graph is embedded onto the surface. A major effect of high representativity is to make the embedding highly "locally planar".

Lemma 4. *Let G be a 3-connected projective graph with minimum valency 5 which is embedded in P with $\rho(G) \geq 3$. If G satisfies $t_{5,i} \leq (i-5)n_i$ for all $i \in \{5,6,\ldots,\Delta\}$, then there is a 3-face in the embedding of G which is incident to one 5-vertex and two 6-vertices.*

Proof. Assume that our lemma is not true. Then there does not exist any 3-face which is incident to one 5-vertex and two 6-vertices in the embedding of G. Since $\rho(G) \geq 3$, by [4], the embedding is a wheel-neighborhood embedding. Since $t_{5,5} = 0$, there do not exist any two adjacent 5-vertices in G. Hence from G, we can construct G' by adding edges to G such that each 5-vertex of G is a 5-vertex of G' and every 5-vertex of G' is incident to five 3-faces. By our construction of G', it is clear that the inequalities $t_{5,i}(G') \leq (i-5)n_i(G')$, for all $i \in \{5,6,\ldots,\Delta\}$, are held in G'. It is also clear that the embedding of G' contains no 3-face which is incident to one 5-vertex and two 6-vertices. But by Lemma 3, there exists at least one 5-vertex in G' which is adjacent to at least four 6-vertices. This implies that there is a 3-face in the embedding of G' which is incident to one 5-vertex and two 6-vertices. This is a contradiction. Hence, our lemma is true. ∎

Lemma 5. *Let G be a 3-connected plane graph with minimum valency 5. If G satisfies $t_{5,i} \leq (i-5)n_i$ for all $i \in \{5,6,\ldots,\Delta\}$, then there is a 3-face in G which is incident to one 5-vertex and two 6-vertices.*

The proof is similar to the one of Lemma 4.

Proof of Theorem. Let G be a 3-connected projective graph with minimum valency 5 which is embedded in P. We assume that G is not edge reconstructible. We consider the following cases according to $\rho(G)$.

Case 1. $\rho(G) = 1$.

By [4], G is a 3-connected planar graph. By [3], G is edge reconstructible.

Case 2. $\rho(G) \geq 3$.

Since G is not edge reconstructible, by Lemma 1, it satisfies $t_{5,i} \leq (i-5)n_i$ for all $i \in \{5,6,\ldots,\Delta\}$. By Lemma 4, there is a 3-face in the embedding of G which is incident to one 5-vertex and two 6-vertices. Hence by Lemma 2, G is edge reconstructible.

Case 3. $\rho(G) = 2$.

Since S is a double cover of P, we can lift G to S and obtain the graph G' in S which is a double cover of the graph G. Since G is a 3-connected projective graph with minimum valency 5, it is easy to see that G' is a 3-connected plane graph with minimum valency 5. Since G satisfies $t_{5,i}(G) \leq (i-5)n_i(G)$, it is clear that G' satisfies $t_{5,i}(G') \leq (i-5)n_i(G')$. Hence by Lemma 5, there is a 3-face in G' which is incident to one 5-vertex and two 6-vertices. Therefore there is a 3-circuit in G which is incident to one 5-vertex and two 6-vertices. By Lemma 2, G is edge reconstructible. ∎

References

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