On the Second Order Chromatic Number and Maximal Criticality of a Graph

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Abstract. Given positive integers p and q, a (p,q)-colouring of a graph G is a mapping $\theta : V(G) \to \{1,2,\ldots,q\}$ such that $\theta(u) \neq \theta(v)$ for all distinct vertices u, v in G whose distance $d(u,v) \leq p$. The pth order chromatic number $\chi^{(p)}(G)$ of G is the minimum value of q such that G admits a (p,q)-colouring. G is said to be (p,q)-maximally critical if $\chi^{(p)}(G) = q$ and $\chi^{(p)}(G+e) > q$ for each edge e not in G. In this paper we study the structure of $(2,q)$-maximally critical graphs. Some necessary or sufficient conditions for a graph to be $(2,q)$-maximally critical are obtained. Let G be a $(2,q)$-maximally critical graph with colour classes $V_1, V_2, \ldots, V_q$. We show that if $|V_1| = |V_2| = \cdots = |V_k| = 1$ and $|v_{k+1}| = \cdots = |V_q| = h \geq 1$ for some k, where $1 \leq k \leq q-1$, then $h \leq h^*$, where

$$h^* = \max\{k, \min\{q-1, 2(q-1-k)\}\}.$$

Furthermore, for each h with $1 \leq h \leq h^*$, we are able to construct a $(2,q)$-maximally critical connected graph with colour classes $V_1, V_2, \ldots, V_q$ such that $|V_1| = |V_2| = \cdots = |V_k| = 1$ and $|V_{k+1}| = \cdots = |V_q| = h$.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Given positive integers p and q, a **pth order q-colouring** of G is a mapping

$$\theta : V(G) \to \{1,2,\ldots,q\}$$

satisfying the condition that $\theta(u) \neq \theta(v)$ for all distinct vertices u, v in G whose distance $d(u,v) \leq p$. For simplicity, a pth order q-colouring of G is also called a **(p,q)-colouring** of G. The **pth order chromatic number** $\chi^{(p)}(G)$ of G is defined as the minimum value of q such that G admits a (p,q)-colouring. In particular, we have $\chi^{(1)}(G) = \chi(G)$, which is the usual chromatic number of G. The notions of a (p,q)-colouring and the generalized chromatic number $\chi^{(p)}(G)$ of G were introduced and studied by F. Kramer and H. Kramer around 1970 (see [3], [4] and [5]). Recently, motivated by a problem in cellular telecommunication technology, the above notions have been investigated again by Baldi [1].

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The p-power G^p of G is the graph defined by

$$V(G^p) = V(G) \quad \text{and} \quad E(G^p) = \{uv \mid u, v \in V(G) \quad \text{and} \quad d(u,v) \leq p\}.$$

It follows by definition that $\chi^{(p)}(G) = \chi(G^p)$. Also, if H is a subgraph of G, then $\chi^{(p)}(H) \leq \chi^{(p)}(G)$. In particular, $\chi^{(p)}(G) \leq \chi^{(p)}(G+e)$ for each $e \in E(\overline{G})$, where $\overline{G}$ is the complement of G.

A graph G is said to be **(p,q)-maximally critical** if $\chi^{(p)}(G) = q$ and $\chi^{(p)}(G+e) > q$ for each $e \in E(\overline{G})$. Let θ be a (p,q)-colouring of G. The graph G is said to be **maximally critical wrt θ** if θ is no longer a (p,q)-colouring of $G+e$ for each $e \in E(\overline{G})$. Thus, G is (p,q)-maximally critical iff G is maximally critical wrt. θ for all (p,q)-colourings θ of G. Also, it is clear that every graph G with $\chi^{(p)}(G) = q$ is a spanning subgraph of a (p,q)-maximally critical graph. It should be pointed out that a different notion of criticality of G (i.e., $\chi^{(p)}(G-v) < \chi^{(p)}(G)$ for each $v \in V(G)$) was introduced in [4].

Despite the fact that $\chi^{(p)}(G) = \chi(G^{(p)})$, there are graphs G which are (p,q)-maximally critical but their G^P are not $(1,q)$-maximally critical (for instance, the path P_n of order n where $n \geq 5$ and $n \not\equiv 0 \pmod 3$ is $(2,3)$-maximally critical but P_n^2 is not $(1,3)$-maximally critical); and on the other hand, there are graphs G which are not (p,q)-maximally critical but their G^p are $(1,q)$-maximally critical (for instance, take G to be the graph of Figure 1).

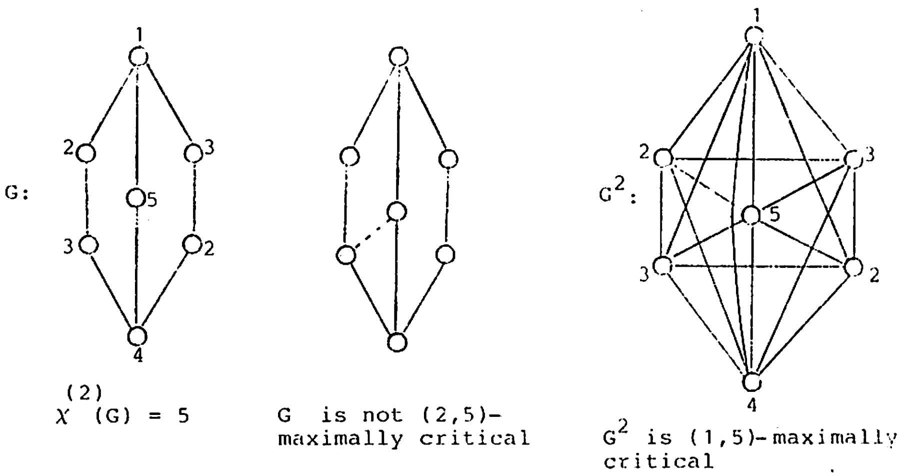


Figure 1

A general problem is: Given positive integers p and q, characterize (p,q)-maximally critical graphs. When $p = 1$, this problem is trivial. Indeed, a graph G is $(1,q)$-maximally critical iff G is a complete q-partite graph. For $p \geq 2$, the problem is, however, difficult, and has not yet been settled. In this paper, we make the first attempt to study this family of graphs when $p = 2$. In section 2 below, we give various families of $(2,q)$-maximally critical graphs, and then proceed to section 3 to study the structure of such a graph. Some necessary or sufficient conditions for a graph to be $(2,q)$-maximally critical are obtained. Let G be a graph with $\chi^{(2)}(G) = q$, and let $V_1, V_2, \ldots, V_q$ be the colour classes of G such that $|V_1| \leq |V_2| \leq \cdots \leq |V_q|$. Suppose k is the least integer such that $|V_{k+1}| = |V_q|$. It is believed that if G is $(2,q)$-maximally critical, then the largest possible value of $|v_{k+1}|$ depends on $k, q, |V_1|, |V_2|, \ldots$ and $|V_k|$. In section 4, we examine this problem for the case when $|V_1| = |V_2| = \cdots = |V_k| = 1$, and prove that in this case $|V_{k+1}| \leq h^*$ where $h^* = \max\{k, \min\{q-1, 2(q-1-k)\}\}$. Finally, for each h with $1 \leq h \leq h^*$, we show that there is a $(2,q)$-maximally critical connected graph G such that $|V_1| = |V_2| = \ldots |V_k| = 1$ and $|V_{k+1}| = \cdots = |V_q| = h$.

Throughout this paper, for simplicity, we shall call a $(2,q)$-colouring of G a **q-colouring** of G, and a $(2,q)$-maximally critical graph a **q-critical graph**, or simply a **critical graph** if 'q' is clear from the context or immaterial. We shall denote by $v(G)$, $e(G)$ and diam G the order, size and diameter of G. For $v \in V(G)$, we write $N(v) = \{u \in V(G) | uv \in E(G)\}$, and write deg v for the degree of v. For $A \subseteq V(G)$, we write $N(A) = \cup(N(a) | a \in A)$, and write $\langle A \rangle$ for the subgraph of G induced by A. We refer to [2] for other notation or terms not defined here.

2. Families of Critical Graphs

We provide in this section a number of families of critical graphs. But first of all, we take note of the following observation (*): for any graph $G, \chi^{(2)}(G) \geq \Delta(G) + 1$, where $\Delta(G) = \max\{\deg v | v \in V(G)\}$.

(1) A graph G of order n is 1-critical iff $G \cong O_n$, an empty graph of order n; and 2-critical iff G is a union of independent edges when n is even or G is a union of independent edges and an isolated vertex when n is odd.

(2) A path P_n of order n is 3-critical iff $n \not\equiv 0 \pmod 3$. A cycle C_n of order n is 3-critical iff $n \equiv 0 \pmod 3$. Every complete graph K_n of order n is, by definition, n-critical; and a graph G with diam $G \leq 2$ is critical iff G is a complete graph.

(3) For a connected graph G, G is 3-critical iff $G \cong P_n$ when $n \geq 4$ and $n \not\equiv 0 \pmod 3$ or $G \cong C_n$ when $n \geq 3$ and $n \equiv 0 \pmod 3$.

(4) The cartesian product $C_n \times P_2 (n \geq 3)$ is critical iff $n \equiv 0 \pmod 4$. For $n \equiv 0 \pmod 4$, we note that the graph $C_n \times P_2$ is uniquely 4-colourable (see Figure 2).

(5) For $n \geq 6$ and $n \equiv 2 \pmod 4$, the cartesian product $P_n \times P_2$ is not critical. Let a and b be the two end vertices of P_n and let $V(P_2) = \{c, d\}$. If G_n is the graph obtained from $P_n \times P_2$ by adding two new edges $e_1 = (a,c)(b,d)$ and $e_2 = (a,d)(b,c)$, then G_n is 4-critical (see Figure 3).

(6) Given positive integers r, s, t, c_1 and c_2 with $r, s, t \geq 2$, $t \leq s$ and $c_1 +$

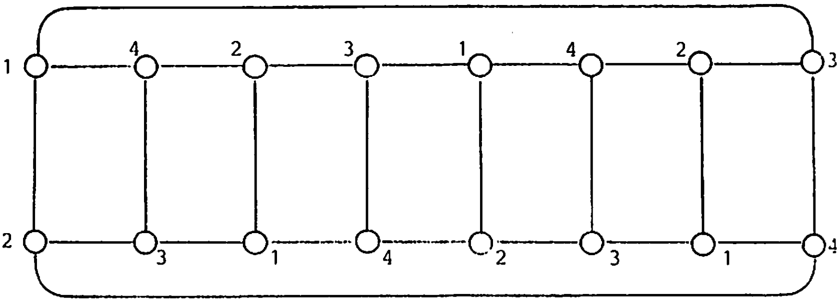


Figure 2. 4-colouring of $C_8 \times P_2$

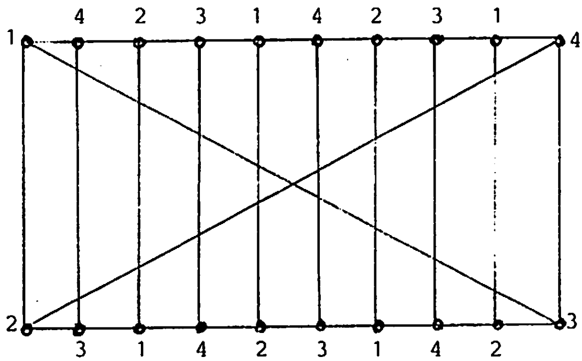


Figure 3. 4-colouring of G_{10}.

$c_2 = s - 1$, let G be the graph whose $V(G)$ is partitioned into subsets $V_1, V_2, \ldots, V_{2r}$ where $V_1 = V_3 = \cdots = V_{2r-1} = \{1, 2, \ldots, s\}$ and $V_2 = V_4 = \cdots = V_{2r} = \{1, 2, \ldots, t\}$ such that

(i) $\langle V_i \rangle \cong \begin{cases} K_s & \text{if } i \text{ is odd} \\ K_t & \text{if } i \text{ is even} \end{cases}$

(ii) For $i = 1, 3, \ldots, 2r-1$, the vertex 'j' in V_i is adjacent to the vertices '$j+1$', '$j+2$', …, '$j+c_1$' (mod s) in V_{i+1} (and no more in V_{i+1}) if they are available, and

(iii) For $i = 2, 4, \ldots, 2r$, the vertex 'j' in V_i is adjacent to the vertices 'j', '$j+1$', …, '$j+c_2$' (mod s) in V_{i+1} (and no more in V_{i+1}) if they are available (here $V_{2r+1} = V_1$).

It can be shown that G is a $(s+t)$-critical graph. Figure 4 shows the graph when $r = 3$, $s = 4$, $t = 3$, $c_1 = 1$ and $c_2 = 2$.

It should be noted that for any graph G, $\Delta(G) \leq \chi^{(2)}(G) - 1$ by the observation (*). Thus, if G is $(\chi^{(2)}(G) - 1)$-regular, then G must be critical.

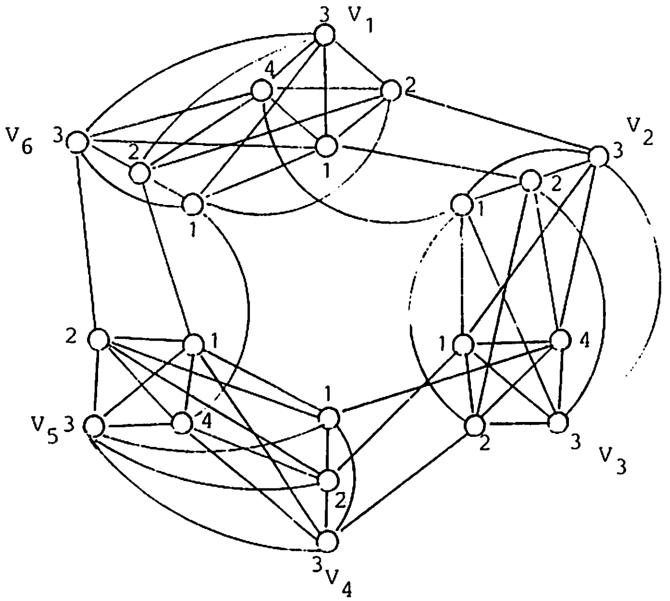


Figure 4. A 7-critical graph

The cycles C_n when $n \equiv 0 \pmod 3$ and those shown in (4), (5) and (6) are examples of this type. Some families of critical graphs which need not be regular are given below.

(7) For $n \geq 5$, $n \equiv 1$ or $3 \pmod 4$, the cartesian product $P_n \times P_2$ is 4-critical (see Figure 5).

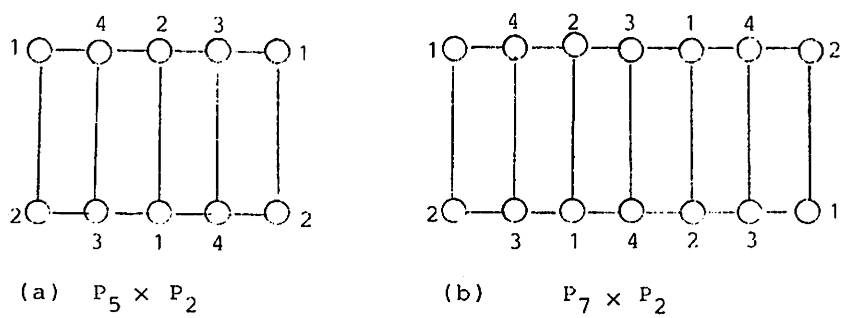


Figure 5

(8) Given any two integers $r, s \geq 2$, let G be the graph obtained from a K_r and rK_s's by gluing to each vertex in K_r a K_s (see Figure 6 for $r = 4$ and $s = 3$). It is easy to check that G is a $(r + s - 1)$-critical graph.

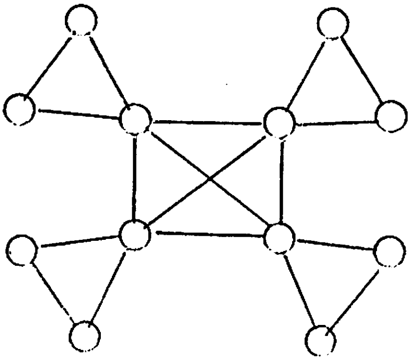


Figure 6. A non-regular 6-critical graph

(9) Given integers r, s and t with $r \geq 2$, $s \geq t \geq 1$, let G be the graph whose $V(G)$ is partitioned into subsets V_1, V_2, U_1 and U_2 such that
 (i) $\langle V_1 \rangle \cong K_s, \langle V_2 \rangle \cong K_t$,
 (ii) $\langle U_1 \rangle \cong \langle U_2 \rangle \cong O_r$,
 (iii) $\langle U_1 \cup U_2 \rangle \cong K_{r,r} - F$ where F is a 1-factor of the complete bipartite graph $K_{r,r}$,
 (iv) v and u are adjacent if ($v \in V_1$ and $u \in U_1$) or ($v \in V_2$ and $u \in U_2$).
 It can be proved that G is a $(r+s)$-critical graph. Figure 7 shows an example when $r = 3$, $s = 2$ and $t = 1$.

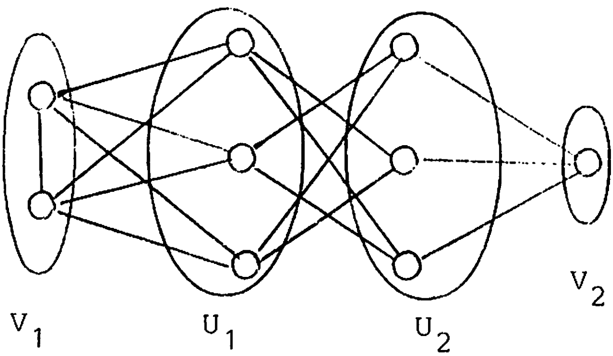
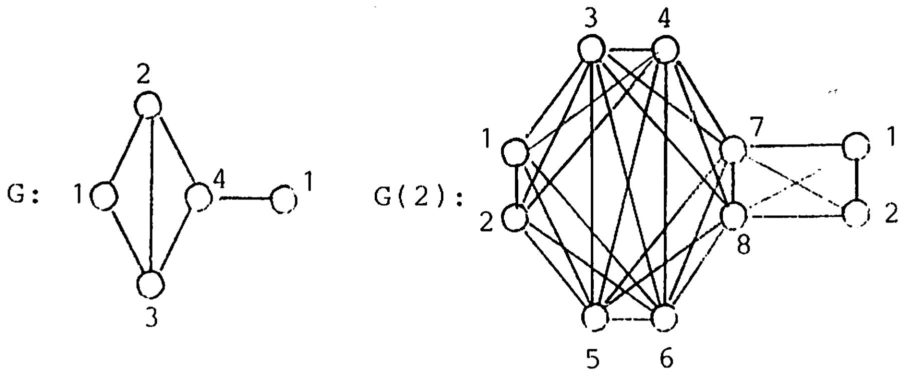


Figure 7. A non-regular 5-critical graph

The last example shows how a q-critical graph can give rise to a qr-critical graph for each positive integer r.

(10) Given a graph G and a positive integer r, let $G(r)$ denote the graph obtained from G by replacing each vertex v of G by an r-complete graph $K_r(v)$ such

that $a \in V(K_r(v))$ and $b \in V(K_r(u))(u \neq v)$ are adjacent in $G(r)$ iff v and u are adjacent in G (see Figure 8). It is clear that if G is a q-critical graph, then $G(r)$ is a qr-critical graph.



a 4-critical graph   a 8-critical graph

Figure 8

3. Structures of Critical Graphs

In this section we study the structures of critical graphs. Some necessary or sufficient conditions for a graph to be critical are also given.

In what follows, we shall denote by $n(\Delta)$ the number of vertices v in G such that $\deg v = \Delta(G)$.

Theorem 1. *Suppose that G is a graph whose $V(G)$ is partitioned into subsets $A_1, A_2, \ldots A_q$ such that*

(i) $|A_1| \leq |A_2| \leq \cdots \leq |A_q|$,

(ii) *A_i is independent for each $i = 1, 2, \ldots, q$, and*

(iii) *for all $i, j = 1, \ldots, q, i < j$, $\langle A_i \cup A_j \rangle$ consists of $|A_i|$ independent edges.*

Then

(1) $\Delta(G) = q - 1$;

(2) *If k is the largest integer such that $|A_k| = |A_1|$, then*

$$n(\Delta) \geq \begin{cases} k|A_1| & \text{if } k \geq 2 \\ 2|A_1| & \text{if } k = 1; \end{cases}$$

in particular, $n(\Delta) \geq 2$;

(3) $n(\Delta) \leq q|A_1|$;

(4) $\chi^{(2)}(G) = q$;

(5) *If G is uniquely q-colourable, then G is q-critical;*

(6) *If $|A_q| - |A_1| \leq 1$, then G is q-critical.*

Proof (1) For each v in $A_i (i = 1, 2, \ldots, q)$, it follows from (ii) and (iii) that $N(v) \cap A_i = \emptyset$ and $|N(v) \cap A_j| \leq 1$ if $j \neq i$, with equality if $i = 1$. Thus, we have $\deg v \leq q - 1$ with equality if $i = 1$. This proves (1).

(2) By the above argument, we have $\deg v = q - 1$ for all $v \in A_i$ if $|A_i| = |A_1|$. Thus, $n(\Delta) \geq k|A_1|$ if $k \geq 2$. For $k = 1$, we note that every vertex in $N(A_1) \cap A_2$ is also of degree (G). Thus $n(\Delta) \geq |A_1| + |N(A_1) \cap A_2| \geq 2|A_1|$.

(3) It follows from (ii) and (iii) that if v is a vertex of maximum degree, then

$$v \in A_1 \cup \bigcup\{A_i \cap N(A_1) | 2 \leq i \leq q\},$$

and so

$$\begin{aligned} n(\Delta) &\leq |A_1| + \sum_{i=2}^{q} |A_i \cap N(A_1)| \\ &\leq |A_1| + (q-1)|A_1| = q|A_1|. \end{aligned}$$

(4) Define a colouring θ of G by colouring all vertices in A_i by "i" ($i = 1, 2, \ldots, q$). Since $d(u, v) \geq 3$ for all distinct vertices u, v in $A_i (i = 1, 2, \ldots, q)$ by (ii) and (iii), θ is a q-colouring of G and so $\chi^{(2)}(G) \leq q$. On the other hand, $\chi^{(2)}(G) \geq \Delta(G) + 1 = q$. Hence $\chi^{(2)}(G) = q$, proving (4).

(5) If G is uniquely q-colourable, then the colouring θ of G defined in (4) is the only q-colouring of G. By (ii) and (iii), it is clear that θ is no longer a colouring of $G + e$ for each $e \in E(\overline{G})$. Thus, G is critical.

(6) Let φ be any q-colouring of G, and let $V_1, V_2, \ldots, V_q$ be the colour classes of G determined by φ (i.e., $V_i = \{v \in V(G) | \varphi(v) = i\}, i = 1, 2, \ldots, q$). We may assume that $|V_1| \leq |V_2| \leq \cdots \leq |V_q|$. Since $\chi^{(2)}(G) = q, V_i \neq \emptyset$ for each $i = 1, 2, \ldots, q$. Since φ is a colouring of G, for each $i = 1, 2, \ldots, q$, V_i is independent, and for all i, j with $i < j$, $\langle V_i \cup V_j \rangle$ consists of at most $|V_i|$ independent edges. Accordingly, we have $e(G) \leq \alpha$, where

$$\alpha = (q-1)|V_1| + (q-2)|V_2| + \cdots + 2|V_{q-2}| + |V_{q-1}|.$$

It can be shown that α attains its maximum value $\alpha_{\max}$ iff $|V_q| - |V_1| \leq 1$. Thus if the assumption $|A_q| - |A_1| \leq 1$ holds in G, then by (ii) and (iii), we have $e(G) = \alpha_{\max}$, which implies that G must be critical. ∎

Corollary. *Let G be a q-critical graph, and let $\{V_i | i = 1, 2, \ldots, q\}$ be any family of colour classes of G with $|V_1| \leq |V_2| \leq \cdots \leq |V_q|$. Then*

(1) *V_i is independent for each $i = 1, 2, \ldots, q$;*

(2) *$\langle V_i \cup V_j \rangle$ consists of $|V_i|$ independent edges for all i, j with $i < j$;*

(3) *$\Delta(G) = q - 1$;*

(4) *If k is the largest integer such that* $|V_k| = |V_1|$, *then*

$$n(\Delta) \geq \begin{cases} k|V_1| & \text{if } k \geq 2 \\ 2|V_1| & \text{if } k = 1 \end{cases}$$

(5) $n(\Delta) \leq q|V_1|$;
(6) $e(G) = qv(G) - \sum_{j=1}^{q} j|V_j|$. ∎

It was pointed out in section 2 that if G is a graph of order n with $\chi^{(2)}(G) = q$, $\Delta(G) = q-1$ and $n(\Delta) = n$, then G must be critical. We call such a graph G a regular q-critical graph. Our next result considers the situation when $1 \leq n - n(\Delta) \leq 2$.

Theorem 2. *Let G be a graph of order $n \geq 4$ with $\chi^{(2)}(G) = q \geq 3$, diam $G \geq 3$ and $\Delta(G) = q-1$.*

(1) *If $n(\Delta) = n-1$, then G is critical, and in this case, $G = G' \cup O_1$ where G' is a regular q-critical graph.*
(2) *If $n(\Delta) = n-2$, then*
   (a) *G is critical or*
   (b) *$G \cong G^* \cup O_2$ or $G \cong G^\# - e$ where G^* and $G^\#$ are regular q-critical graphs, and $e \in E(G^\#)$.*

Remark We exclude the trivial case when diam $G \leq 2$, which was mentioned in family (2). Also, if G is a regular q-critical graph of order n, then $q|n$ and each colour class has the same number of vertices.

Proof. Let $\{V_1, V_2, \ldots, V_q\}$ be a family of colour classes of G with $|V_1| \leq |V_2| \leq \cdots \leq |V_q|$. Note that $n(\Delta) \leq q|V_1|$ holds also in G.
(1) If $n(\Delta) = n-1$, then as $n(\Delta) \leq q|V_1|$, either

$$|V_1| = |V_2| = \cdots = |V_q|$$

or

$$|V_1| = |V_2| = \cdots = |V_{q-1}| = |V_q| - 1.$$

In the former case, the fact that $n(\Delta) \geq n-1$ implies that $n(\Delta) = n$, a contradiction. In the latter case, the fact that $n(\Delta) = n-1$ implies that V_q contains an isolated vertex w and $G - w$ is a regular q-critical graph. Thus G is critical by Theorem 1(6), proving (1).
(2) If $n(\Delta) = n-2$, then as $n(\Delta) \leq q|V_1|$, we have

$$|V_1| = |V_2| = \cdots = |V_{q-1}| = |V_q| - 2$$
$$|V_1| = \cdots = |V_{q-2}| = |V_{q-1}| - 1 = |V_q| - 1,$$

or

$$|V_1| = |V_2| = \cdots = |V_q|.$$

In the first case, $G \cong G^* \cup O_2$ as stated in the theorem. In the second case, either $G \cong G^* \cup O_2$ again or G is critical by Theorem 1(6). In the third case, $G \cong G^\# - e$ as stated in the theorem. ∎

4. A Sharp Bound and Constructions

We first establish the following result as stated in the introduction.

Theorem 3. *Let G be a connected q-critical graph, where $q \geq 3$ and let $\{V_1, V_2, \ldots, V_q\}$ be a family of colour classes of G. If $|V_1| = |V_2| = \cdots = |V_k| = 1$ and $|V_{k+1}| = |V_{k+2}| = \cdots = |V_q| = h \geq 1$ for some k, where $1 \leq k \leq q-1$, then*

$$h \leq \max\{k, \min\{q-1, 2(q-1-k)\}\}.$$

Proof. Since the number on the right hand side of the above inequality is always exceeding 1, we may assume that $h \geq 3$.

Let $A = V_1 \cup V_2 \cdots \cup V_k$. We claim that for each $u \in A$ and $v \in V(G) \setminus A$, $d(u,v) \leq 2$. Otherwise, suppose $d(u,v) \geq 3$ for some $u \in V_i$ and $v \in V_j$, where $1 \leq i \leq k$ and $k+1 \leq j \leq q$. Now, if we keep the colours of all vertices of G except v, which is re-coloured by colour "i", we obtain a new q-colouring θ of G. Since $h \geq 3$, there exists a $w \in V_j \setminus \{v\}$ such that $w \notin N(u)$. Clearly, θ is a q-colouring of $G + vw$, which contradicts the fact that G is q-critical. Fix an arbitrary set V_t, where $k+1 \leq t \leq q$. Let

$$X = V_t \cap N(A).$$

Observe that $|X| \leq k$, and for each $u \in A$,

$$|N(u) \setminus (A \cup X)| \leq q-k-1.$$

From this and the earlier claim, it follows that

$$|V_t \setminus X| \leq q-k-1. \qquad (1)$$

Thus

$$h = |V_t| = |X| + |V_t \setminus X| \leq k + (q-k-1) = q-1.$$

Hence, if $2k \leq q-1$, then

$$\begin{aligned} &\max\{k, \min\{q-1, 2(q-1-k)\}\} \\ =&\max\{k, (q-k-1) + \min\{k, q-k-1\}\} \\ =&\max\{k, q-1\} \\ =&q-1 \geq h. \end{aligned}$$

We now confine ourselves to the case where $2k > q-1$, and note that in this case, it suffices to show that

$$h \leq \max\{k, 2(q-k-1)\}$$

and to consider the case when $h > k$.

For each $t > k$, let $b_t = |V_t \setminus N(A)|$. From (1), we have $b_t \leq q-k-1$. Let $b = \max\{b_t | t > k\}$. Without loss of generality, we may assume that $b_{k+1} = b$. Let

$$B = V_{k+1} \setminus N(A).$$

Since $h > k$, B is nonempty. Fom the earlier claim, for each $u \in A$ and $v \in B$, $N(u) \cap N(v) \neq \emptyset$. Thus, if we denote by $e(S,T)$ the number of edges joining a vertex in S to a vertex in T, where $S, T \subseteq V(G)$, then

$$e(N(B), A) \geq bk.$$

Since $e(V(G) \setminus (A \cup V_{k+1}), A) \leq k(q-k-1)$, we have

$$\begin{aligned} e(V(G) \setminus (A \cup V_{k+1} \cup N(B)), A) &\leq k(q-k-1) - bk \\ &= k((q-k-1) - b). \end{aligned}$$

Hence

$$|(V(G) \setminus (A \cup V_{k+1} \cup N(B))) \cap N(A)| \leq k((q-k-1) - b).$$

Since

$$\begin{aligned} &|V(G) \setminus (A \cup V_{k+1})| \\ =&|(V(G) \setminus (A \cup V_{k+1})) \cap N(A)| + |(V(G) \setminus (A \cup V_{k+1})) \cap \overline{N(A)}|, \end{aligned}$$

we have

$$\begin{aligned} (q-k-1)h &\leq k((q-k-1) - b) + |N(B)| + (|\overline{N(A)}| - |B|) \\ &\leq k((q-k-1) - b) + b(q-k-1) + b(q-k-1) \\ &= k(q-k-1) + b(2(q-k-1) - k). \end{aligned}$$

Case 1. $2(q-k-1) - k \leq 0$.
Then

$$(q-k-1)h \leq k(q-k-1)$$

or

$$h \leq k = \max\{k, 2(q-k-1)\}.$$

Case 2. $2(q-k-1)-k > 0$.
In this case,

$$\begin{aligned}(q-k-1)h &\leq k(q-k-1) + (q-k-1)(2(q-k-1)-k) \\ &= 2(q-k-1)^2,\end{aligned}$$

or

$$h \leq 2(q-k-1) = \max\{k, 2(q-k-1)\}.$$

The proof is thus complete. ∎

We shall now provide methods of construction to show that for all k, q with $1 \leq k \leq q-1$ and $q \geq 3$, and for each h with $1 \leq h \leq h^*$, where

$$h^* = \max\{k, \min\{q-1, 2(q-1-k)\}\},$$

there is a connected q-critical graph with colour classes $\{V_1, V_2, \ldots, V_q\}$ such that $|V_1| = |V_2| = \cdots = |V_k| = 1$ and $|V_{k+1}| = \cdots = |V_q| = h$.

Let V be a set of $k + h(q-k)$ vertices which is partitioned into q subsets $V_1, V_2, \ldots, V_q$ such that $|V_1| = |V_2| = \cdots = |V_k| = 1$ and $|V_{k+1}| = \cdots = |V_q| = h$. For convenience, we label the vertices of V_i, as v_{ij}, where $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, |V_i|$, and denote

$$W_0 = \{v_{11}, v_{21}, \ldots, v_{k1}\}$$

and

$$W_i = \{v_{(k+1)i}, v_{(k+2)i}, \ldots, V_{qi}\}, \quad i = 1, 2, \ldots, h.$$

We consider two cases.

Construction I. $h \leq k$.

Let G be a graph with $V(G) = V$. The adjacency of vertices in G is defined as follows (see Figure 9): The vertices $u \in W_s \cap V_i$ and $v \in W_{s'} \cap V_{i'}$, where $s \leq s'$, $i \neq i'$, are adjacent iff

(i) $s = s'$ (thus each W_i forms a clique in G)
(ii) $s = 0$ and $s' = \min\{i, h\}, i = 1, 2, \ldots, k$.

It is not hard to see that the graph G so constructed is connected and q-critical. Note that if $k > \frac{2}{3}(q-1)$, then $q-k-1 \leq \frac{1}{3}(q-1)$ and hence the value of h runs through the interval $[1, h^*]$.

Construction II. $k < h \leq h^*$.

Write $r = \min\{k, q-k-1\}$, $r_1 = \min\{r, h-r\}$ and $r_2 = \max\{r, h-r\}$. Observe that $r_2 \leq q-k-1$.

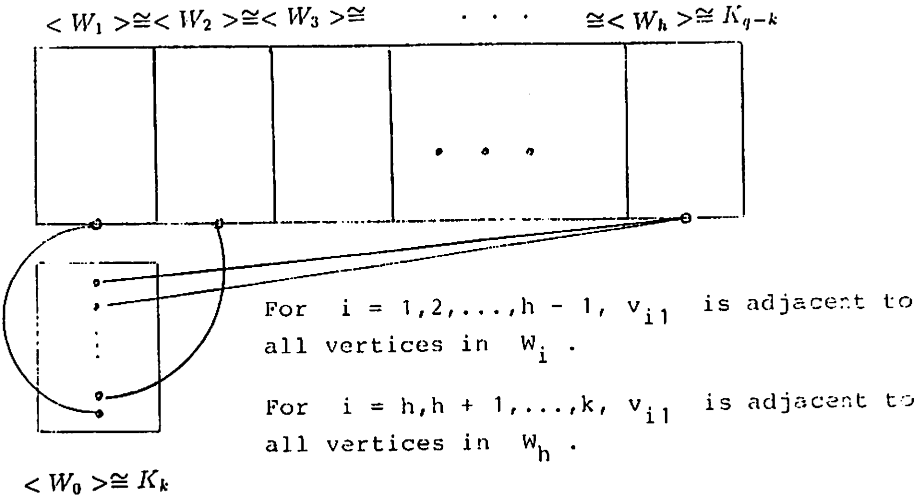


Figure 9

Let G be a graph with $V(G) = V$. The adjacency of vertices in G is defined as follows: two vertices $u \in W_s \cap V_i$ and $v \in W_{s'} \cap V_{i'}$, where $s \le s'$ and $i \ne i'$, are adjacent iff one of the following 4 conditions is satisfied:

(i) $s = s' = 0$ (thus W_0 forms a clique in G);
(ii) $s = 0$ and $s' = \min\{i, r\}$;
(iii) $1 \le s \le r, r + 1 \le s' \le h, i, i' \in [k+1, q]$ and $(s + i) - (s' + i') \equiv 0 \pmod{r_2}$;
(iv) If ($r_2 = r$ and $s, s' \le r$) or ($r_2 > r$ and $s, s' > r$), then $s + s' \equiv h + 2r + 1 \pmod{r_2}$ and $i - (s' + i') \equiv t \pmod{r_2}$ where $r_2 \ge t \ge r_1 + 1$.

The adjacencies defined in (iii) and (iv) are illustrated respectively as follows. Suppose $k = 3$, $q = 8$ and $h = 5$. In this case $r = 3$, $r_1 = h - r = 2$ and $r_2 = r = 3$. Take, for instance, (s, i) to be one of the following in $W_1 \cup W_2 \cup W_3$:

$$a_1 = (1,8), a_2 = (2,7), a_3 = (3,6), a_4 = (1,5), a_5 = (2,4).$$

Then the vertices adjacent to these a_i's by (iii) are those b_j's in $W_4 \cup W_5$ as shown in Figure 10, where

$$b_1 = (4,5), b_2 = (5,4), b_3 = (4,8), b_4 = (5,7).$$

Observe that the subgraph induced by the union of $A = \{a_1, a_2, \ldots, a_5\}$ and $B = \{b_1, b_2, b_3, b_4\}$ is a complete bipartite graph with bipartition $\{A, B\}$ in which a complete matching is deleted.

The adjacency of vertices in $W_1 \cup W_2 \cup W_3$ defined by (iv) is shown in Figure 11.

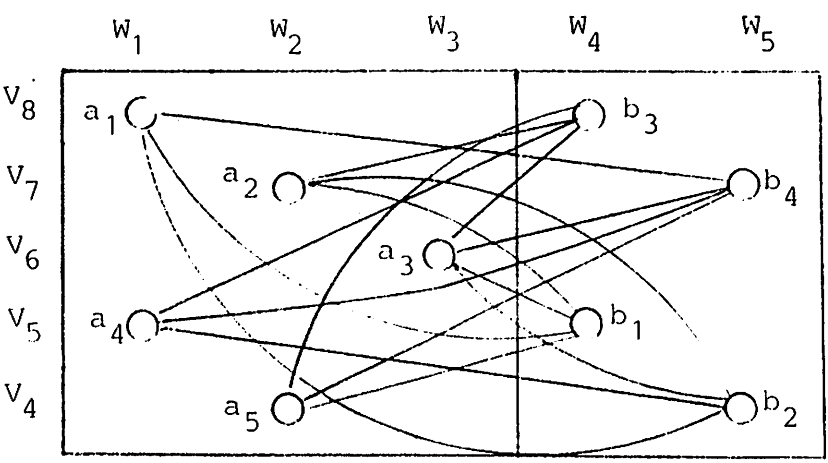


Figure 10

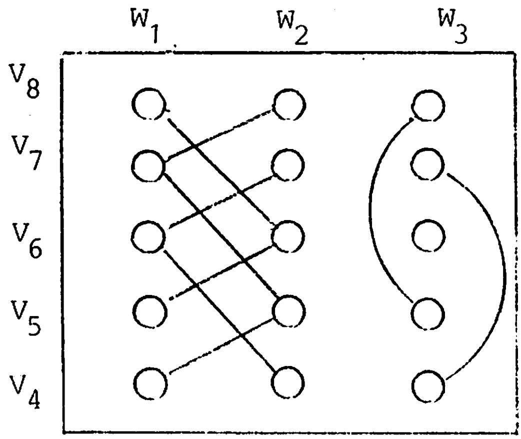


Figure 11

We shall now show that G is a desired graph. From (i), the vertices of W_0 must be coloured by different colours, say $v \in V_i \cap W_0$ is coloured by "i", $i = 1, 2, \ldots, k$. From (ii), for each $j, 1 \leq j \leq r$, the vertices of W_j have colours different from 1 to k. Likewise, from (ii) and (iii), colours of $v \in W_j (j > r)$ are different from 1 to k. By (iii) and (iv), every vertex in $G - W_0$ has exactly $q - k - 1$ neighbours in $G - W_0$ and hence $\chi^{(2)}(G - W_0) \geq q - k$, which implies that $\chi^{(2)}(G) \geq q$. We shall now show that each vertex $v \in V_i (i > k)$ is adjacent to exactly one vertex in V_j, where $j > k$ and $j \neq i$. Suppose $v \in W_s$. Observe that v is adjacent to a vertex $u \in V_j \cap W_{s^*}$ by (iii) only if $s + i \equiv s^* + j \pmod{r_2}$ and $s \leq r$ iff $s^* > r$. Thus, if $r_2 = r < s$ or $s \leq r < r_2$, then there is exactly one such

u for each $j \neq i$, while if $s \leq r = r_2$ or both s and $r_2 > r$, there is exactly one u if $s + i + j - 1 \equiv t \pmod{r_2}$, where $0 \leq t \leq r_1 - 1$ (note that $r_2 \leq q - k - 1$). Now, if $s+i+j-1$ has a remainder greater than $r_1 - 1$ on dividing by r_2, then and only then we have an adjacency using (iv). Thus, if we colour the vertices of V_i by colour i, we have a q-colouring of G, and so $\chi^{(2)}(G) \leq q$. Hence $\chi^{(2)}(G) = q$. Since each vertex in $G - W_0$ is of degree q in G, the graph G so constructed is q-critical.

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