k-perfect $3k$-cycle systems

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ABSTRACT: The spectrum for k-perfect $3k$-cycle systems is considered here for arbitrary $k \not\equiv 0 \pmod 3$. Previously, the spectrum when $k = 2$ was dealt with by Lindner, Phelps and Rodger, and that for $k = 3$ by the current authors. Here, when $k \equiv 1$ or $5 \pmod 6$ and $6k + 1$ is prime, we show that the spectrum for k-perfect $3k$-cycle systems includes all positive integers congruent to $1 \pmod{6k}$ (except possibly the isolated case $12k + 1$). We also complete the spectrum for $k = 4$ and 5 (except possibly for one isolated case when $k = 5$), and deal with other specific small values of k.

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1 Introduction

In recent years a great deal of work has been done on decompositions of complete graphs into edge-disjoint cycles; see the splendid survey [15] for details. What we shall concentrate on here are certain so-called *perfect* cycle decompositions of the complete graph K_v; we start with some definitions.

An *m-cycle system* of K_v is an ordered pair (V, C) where V is the vertex set of K_v and C is a set of edge-disjoint cycles, each of length m, which partition the edge set of K_v. The *order* of the m-cycle system is $|V| = v$.

Suppose that we have an m-cycle system, (V, C), of order v. Take each cycle c in C, and replace it by the graph $c(i)$ formed by joining all the vertices of c at distance i in c; let $C(i) = \{c(i) \mid c \in C\}$. Then if $(V, C(i))$ is again a cycle system of K_v (but not necessarily an m-cycle system), we say that the original system (V, C) is an *i-perfect* m-cycle system of K_v. Moreover, if (V, C) is an i-perfect m-cycle system for all i, then we call the system a *Steiner* m-cycle system. Since our graphs are undirected, it is only necessary for an m-cycle system to be i-perfect for $1 \leq i \leq \lfloor m/2 \rfloor$ in order for it to be a Steiner m-cycle system.

It is immediate that, if the cycle length m is 3, all 3-cycle systems are Steiner; they are, of course, Steiner triple systems, since the cycle C_3 and the complete graph K_3 are the same!

If the cycle length m is 4, there are no 2-perfect 4-cycle systems. (The reader could draw a 4-cycle c and consider the graph $c(2)$ obtained from this!) So henceforth we shall assume that the cycle length is at least 5.

Work has chiefly concentrated on finding the *spectrum* for i-perfect m-cycle systems; this is the set of values v for which there exists an i-perfect m-cycle system of order v. It is straightforward to see that necessary conditions for existence of an i-perfect m-cycle system of order v are the same as the necessary conditions for existence of an m-cycle system of order v, namely, that:

$v \geq m$ (or $v = 1$);
v is odd (so each vertex of K_v has even degree); and
$2m$ divides $v(v-1)$ (so m divides the total number of edges).

Of the work done to date on i-perfect cycle systems, most has involved finding the spectrum for 2-perfect cycle systems. The case $m = 5$ was treated first ([16]), $m = 7$ in [17], and some general results for any odd m in [14]. For even m, see [13] for $m = 6$, [2] for $m = 8$, and for other small values of m, both even and odd, see [3]. Also, 2-perfect decompositions of λ copies of K_v have been considered; see [6] and [1].

Besides 2-perfect cycle systems, which have been of interest partly because of the associated quasigroup that arises (see [12]) and also because of statistical applications ([11]), if the cycle length m is a multiple of 3, say $m = 3k$, then k-perfect $3k$-cycle systems are of interest because the cycles $C(k) = \{c(k) \mid c \in C\}$, where (V, C) is a $3k$-cycle system, form a Steiner triple system.

Henceforth we shall concentrate on the problem of finding the spectrum for k-perfect $3k$-cycle systems. This has been done for $k = 2$ (see [13], and [6] for the two isolated cases missing from [13]) and for $k = 3$ (see [5]). We remark that there does in fact exist a 3-perfect 9-cycle system of order 9, so the spectrum in [5] should be extended to include the value 9. A 3-perfect 9-cycle system of order 9 is (V, C) where $V = Z_9$ and $C = \{(0,1,2,3,4,5,6,7,8), (0,2,4,7,1,8,5,3,6), (0,3,1,4,8,6,2,7,5), (0,4,6,1,5,2,8,3,7)\}$.

In Section 2 we give some well-known but necessary lemmas and describe the now standard construction that we use. Section 3 then deals in detail with the case $k = 4$, Section 4 with $k = 5$, while Section 5 gives some general results for arbitrary k *not* divisible by 3. Section 6 then applies these results further, and summarises the situation for small k.

2 Some lemmas and the Construction

LEMMA 2.1 *(i) When* $2n \equiv 0$ *or* 2 *(mod* 6*),* $2n \geq 6$*, then there exists a group divisible design with* n *groups of size* 2 *and blocks of size* 3.
(ii) When $2n \equiv 4$ *(mod* 6*),* $2n \geq 10$*, then there exists a group divisible design with* one *group of size* 4 *and the rest of size* 2*, and with blocks of size* 3.

Proof: *(i)* This first appeared in Hanani [10], Lemma 6.3; such group divisible designs also arise by taking any Steiner triple system of order $2n + 1$ (which is 1 or 3 (mod 6)) and deleting one point. The groups of size 2 arise from the blocks that contained the one deleted point, and the blocks of size 3 are those blocks of the Steiner triple system not meeting the deleted point.
(ii) This is essentially done in [18], page 276. Wilson gives a pairwise balanced design with number of elements congruent to 5 (mod 6), with one block of size 5 and the rest of size 3. Deletion of one point from the block of size 5 yields a suitable group divisible design with one group of size 4, the rest of size 2, and blocks of size 3. □

Henceforth a group divisible design (GDD) on s elements with block size 3 and group sizes in $\{m_i\}$ (and index $\lambda = 1$) will be denoted GD(3, $\{m_i\}$; s). We frequently use the main theorem in [9] to ensure that a suitable GDD with block size 3 exists.

We shall also use the following well-known results (see [14], Lemmas 2.1 and 2.2).

LEMMA 2.2 *When m is prime, there exists a Steiner m-cycle system of K_m.*

LEMMA 2.3 *When m is odd and $v \equiv 1 \pmod{2m}$ is a prime power, then there exists a Steiner m-cycle system of K_v.*

THE CONSTRUCTION Let $v = \alpha s + \epsilon$ for α, $\epsilon \geq 0$; generally in our constructions of k-perfect $3k$-cycle systems, α will be either k or $3k$, and is sometimes referred to as the number of "layers". Vertices of K_v will be

$$\{(i,j) \mid 1 \leq i \leq s,\ \ 1 \leq j \leq \alpha\} \cup \{\infty_i\}_{i=1}^{\epsilon}.$$

On the set $\{(i,j) \mid 1 \leq i \leq s\}$ we take a GDD with blocks of size 3, and various group sizes. For each block $\{(x,j),(y,j),(z,j)\}$ of the GDD we take a k-perfect $3k$-cycle decomposition of $K_{\alpha,\alpha,\alpha}$ on the vertices

$$\{(x,j) \mid 1 \leq j \leq \alpha\} \cup \{(y,j) \mid 1 \leq j \leq \alpha\} \cup \{(z,j) \mid 1 \leq j \leq \alpha\}.$$

Now suppose groups of the GDD are $\{(i_1,j),(i_2,j),\ldots,(i_g,j)\}$. If $\epsilon = 0$, for each group of the GDD, on the vertex set $\{(i_1,j),\ldots,(i_g,j) \mid 1 \leq j \leq \alpha\}$, place a k-perfect $3k$-cycle system of order $g\alpha$.

For $\epsilon = 1$, for each group of the GDD, place on $\{\infty_1\} \cup \{(i_1,j),\ldots,(i_g,j) \mid 1 \leq j \leq \alpha\}$ a system of order $g\alpha + 1$.

If $\epsilon > 1$, choose one group $\{(i_1^*,j),\ldots,(i_{g^*}^*,j)\}$ of the GDD and on

$$\{\infty_i\}_{i=1}^{\epsilon} \cup \{(i_1^*,j),\ldots,(i_{g^*}^*,j) \mid 1 \leq j \leq \alpha\}$$

place a system of order $g^*\alpha + \epsilon$. Then for all remaining groups, on

$$\{\infty_i\}_{i=1}^{\epsilon} \cup \{(i_1,j),\ldots,(i_g,j) \mid 1 \leq j \leq \alpha\}$$

place a k-perfect $3k$-cycle decomposition of $K_{g\alpha+\epsilon} \setminus K_\epsilon$. (Here $K_a \setminus K_b$ refers to the graph on a vertices with b vertices singled out, and all the $\binom{b}{2}$ edges between these b vertices removed; this is sometimes referred to as a "hole" of size b in K_a.)

3 The case $k = 4$

The necessary conditions for existence of a 4-perfect 12-cycle system of order v are that $v \equiv 1$ or 9 (mod 24), and of course $v \geq 12$, so $v = 25$ is the smallest possible order. We start with some relatively small examples.

EXAMPLE 3.1 (Z_{25}, C) is a 4-perfect 12-cycle system of order 25, where
$C = \{(0+i, 1+i, 4+i, 12+i, 14+i, 5+i, 16+i, 11+i, 17+i, 24+i, 9+i, 21+i) \mid 0 \leq i \leq 24\}$, with addition in Z_{25}.

EXAMPLE 3.2 (Z_{49}, C) is a 4-perfect 12-cycle system of order 49, where $C = \{(0+i, 1+i, 3+i, 6+i, 2+i, 7+i, 13+i, 5+i, 14+i, 24+i, 10+i, 21+i), (0+i, 7+i, 19+i, 35+i, 13+i, 47+i, 23+i, 5+i, 41+i, 18+i, 1+i, 30+i) \mid 0 \leq i \leq 48\}$, with addition in Z_{49}.

EXAMPLE 3.3 $(Z_{11} \times Z_3, C)$ is a 4-perfect 12-cycle system of order 33, where C is obtained from the following four "starter" cycles, by cycling mod $(11, -)$; this yields 44 cycles altogether.
$((0,1),(1,1),(0,3),(0,2),(1,3),(3,3),(1,2),(2,1),(3,2),(2,2),(4,1),(4,2))$,
$((0,1),(2,1),(4,2),(1,1),(3,3),(6,1),(1,3),(5,2),(10,3),(7,1),(1,2),(7,2))$,
$((0,1),(5,1),(10,3),(6,3),(7,2),(10,2),(4,1),(1,1),(8,1),(5,2),(8,3),(9,3))$,
$((0,2),(2,2),(10,3),(9,1),(2,3),(5,3),(0,3),(0,1),(7,3),(1,2),(5,2),(9,3))$.

EXAMPLE 3.4 $(Z_{19} \times Z_3, C)$ is a 4-perfect 12-cycle system of order 57, where C is obtained from the following seven starter cycles, mod $(19, -)$:
$((0,1),(2,2),(9,2),(10,1),(11,1),(17,2),(12,2),(5,1),(1,1),(3,1),(0,2),(7,1))$,
$((0,1),(3,2),(11,2),(5,2),(8,2),(10,1),(6,2),(4,2),(0,3),(1,3),(0,2),(8,1))$,
$((0,1),(4,2),(9,1),(0,3),(0,2),(3,3),(3,1),(6,1),(15,1),(2,1),(10,2),(9,2))$,
$((0,1),(5,2),(11,1),(0,3),(2,3),(7,1),(8,3),(11,3),(12,1),(18,3),(4,3),(16,3))$,
$((0,1),(10,2),(0,2),(15,2),(4,3),(13,3),(9,1),(4,1),(17,3),(5,1),(14,3),(3,3))$,
$((0,1),(0,2),(6,3),(8,1),(15,3),(13,1),(18,3),(2,2),(14,3),(1,3),(6,2),(11,3))$,
$((0,1),(1,2),(18,3),(7,2),(6,3),(18,2),(9,3),(13,3),(9,2),(3,3),(13,2),(15,3))$.

EXAMPLE 3.5 There exists a decomposition of $K_{33} \setminus K_9$ into 4-perfect 12-cycles.

Let the vertex set of $K_{33} \setminus K_9$ be $\{(i,j) \mid 0 \leq i \leq 2, 0 \leq j \leq 7\} \cup \{A, B, C, D, E, F, G, H, I\}$. Here the vertices $\{A, \ldots, I\}$ correspond to the hole, and remain fixed. The 41 cycles are given in two parts:

First the following 13 starters are cycled mod $(3, -)$ (with the elements in the hole remaining fixed):
$(A,(0,0),(1,0),E,(0,5),(0,4),B,(1,2),(2,0),F,(2,4),(0,7))$,
$(C,(0,0),(0,1),G,(1,0),(2,1),D,(2,0),(1,1),H,(2,2),(1,2))$,

$(A,(0,3),(1,3),B,(1,1),(2,4),C,(0,1),(2,2),D,(1,4),(0,2))$,
$(A,(0,1),(1,3),C,(1,7),(2,3),F,(2,2),(0,3),I,(1,2),(1,5))$,
$(A,(0,6),(1,5),G,(2,6),(2,7),F,(2,5),(0,4),H,(1,7),(1,4))$,
$(B,(0,6),(2,6),H,(0,3),(1,2),G,(1,3),(1,6),I,(1,1),(1,5))$,
$(D,(0,7),(2,1),F,(1,6),(2,4),E,(1,1),(0,5),I,(1,0),(0,6))$,
$(B,(0,7),(0,3),D,(0,5),(1,3),E,(2,7),(1,7),G,(1,4),(2,0))$,
$(C,(0,5),(2,2),E,(1,6),(1,0),H,(2,5),(2,7),I,(2,4),(2,6))$,
$((0,0),(0,4),(0,2),(0,6),(1,2),(2,7),(0,5),(2,0),(1,7),(2,1),(1,4),(1,3))$,
$((0,1),(1,5),(1,3),(2,7),(0,6),(2,1),(1,1),(1,2),(0,4),(1,4),(2,5),(2,6))$,
$((0,2),(2,7),(2,0),(2,5),(1,3),(1,1),(1,7),(1,2),(0,0),(2,3),(1,4),(0,3))$,
$((0,3),(1,6),(1,1),(1,4),(0,0),(2,5),(1,5),(2,2),(0,6),(2,0),(0,7),(2,6))$.

Then the following two cycles are taken (not cycled).

$((0,0),(0,3),(1,1),(2,2),(2,0),(2,3),(0,1),(1,2),(1,0),(1,3),(2,1),(0,2))$,
$((0,4),(1,6),(0,5),(1,7),(2,4),(0,6),(2,5),(0,7),(1,4),(2,6),(1,5),(2,7))$.

We also need the following crucial "building block" for the constructions in this case, as there is no 4-perfect 12-cycle decomposition of $K_{4,4,4}$.

LEMMA 3.6 *The tripartite graph $K_{12,12,12}$ has an edge-disjoint decomposition into 4-perfect 12-cycles.*

Proof: Consider the following idempotent quasigroup of order 12, obtained from the direct product of

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

and

1	3	2
3	2	1
2	1	3

.

Let the quasigroup operation be denoted $\circ$.

∘	1	2	3	4	5	6	7	8	9	10	11	12
1	1	4	2	3	9	12	10	11	5	8	6	7
2	7	6	8	5	3	2	4	1	11	10	12	9
3	12	9	11	10	8	5	7	6	4	1	3	2
4	2	3	1	4	10	11	9	12	6	7	5	8
5	9	12	10	11	5	8	6	7	1	4	2	3
6	3	2	4	1	11	10	12	9	7	6	8	5
7	8	5	7	6	4	1	3	2	12	9	11	10
8	10	11	9	12	6	7	5	8	2	3	1	4
9	5	8	6	7	1	4	2	3	9	12	10	11
10	11	10	12	9	7	6	8	5	3	2	4	1
11	4	1	3	2	12	9	11	10	8	5	7	6
12	6	7	5	8	2	3	1	4	10	11	9	12

A decomposition of $K_{12,12,12}$ on the vertex set $\{i_1\} \cup \{i_2\} \cup \{i_3\}$, $1 \leq i \leq 12$, into triangles, is given by

$$\{(x_1, y_2, (x \circ y)_3) \mid 1 \leq x \leq 12, 1 \leq y \leq 12\}.$$

We form 36 12-cycles from these 144 triangles. First we group the triples into certain sets of four, and for each set

$$(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3),$$

we take the 12-cycle

$$(a_1, b_2, c_3, d_1, a_2, b_3, c_1, d_2, a_3, b_1, c_2, d_3).$$

The 36 sets of four triples are as follows; here $x = 1, 5, 9$ and $y = 1, 2, \ldots, 12$:

$$\{((x+i)_1, (y+i)_2, ((x+i) \circ (y+i))_3) \mid i = 0, 1, 2, 3\};$$

if y lies in $\{1,2,3,4\}$, or in $\{5,6,7,8\}$, or in $\{9,10,11,12\}$ then addition of i is to be such that $y+i$ remains in that set; so, for example, if $y=6$, then $y+i$ equals 6, 7, 8, 5 for $i=0,1,2,3$. A straightforward check now shows that this is indeed a 4-perfect 12-cycle decomposition of $K_{12,12,12}$. □

The Construction described in Section 2 above is now applicable here, with $\alpha=12$, $\epsilon=1$ or 9; so $v=24n+\epsilon$ with $s=2n$.

If $2n\equiv 0$ or 2 (mod 6) we take a GDD on $\{(i,j)\mid 1\le i\le 2n\}$ with groups all of size 2, while if $2n\equiv 4$ (mod 6) there is one group of size 4. The required 4-perfect 12-cycle systems are all given above: $K_{12,12,12}$, K_{25}, K_{49} (since the GDD does not exist when $n=2$), K_{57}, $K_{33}\setminus K_9$ and K_{33}.

In summary, we have:

THEOREM 3.7 *The spectrum for 4-perfect 12-cycle systems is the set of all $v\equiv 1$ or 9 (mod 24), $v\ge 25$.*

4 The case $k=5$

The necessary conditions for existence of a 5-perfect 15-cycle system of K_v are that $v\equiv 1$, 15, 21 or 25 (mod 30), and $v\ge 15$.

We start with some examples.

EXAMPLE 4.1 Since 15 is odd, and 31 and 61 are prime, there exist Steiner 15-cycle systems of orders 31 and 61. (Lemma 1.3 above.)

There is also a 5-perfect 15-cycle decomposition of $K_{5,5,5}$ (see Theorem 5.1 below). So, taking a GDD with n groups of size 6 (for $n\ge 3$), and blocks of size 3, using the Construction in Section 2 above we obtain a 5-perfect 15-cycle system of K_v when $v=30n+1$. (Use $\alpha=5$, $s=6n$, $\epsilon=1$.)

When $v=30n+15=5(6n+3)$, we use the Construction with $\epsilon=0$ and $\alpha=5$. Take a Kirkman triple system of order $6n+3$ as our GDD; so only one case is needed:

EXAMPLE 4.2 (V,C) is a 5-perfect 15-cycle system of K_{15} where $V=(Z_7\times Z_2)\cup\{\infty\}$ and the seven cycles in C come from the one starter cycle:

$$(\infty,(5,1),(5,0),(3,0),(4,1),(0,0),(6,1),(4,0),(1,0),(2,0),(0,1),$$
$$(3,1),(1,1),(2,1),(6,0))\quad \text{cycled modulo } (7,-) \text{ (with } \infty \text{ fixed).}$$

When $v=30n+21=5(6n+4)+1$, we use a GDD$(3,\{6,4^*\};6n+4)$, which exists for $n\ge 3$. (The asterisk means *one* group of size 4.) Use the Construction with $\epsilon=1$, $\alpha=5$ and $s=6n+4$. Then 5-perfect 15-cycle systems of orders 21, 31, 51 (and the isolated case of order 81) are needed. We give ones of orders 21 and 51 below; for order 31 see Example 4.1.

EXAMPLE 4.3 (V, C) is a 5-perfect 15-cycle system of order 21 where $V = Z_7 \times Z_3$ and C consists of 14 cycles, obtained cyclically mod $(7, -)$ from the following two starter cycles:

$$((0,0),(1,0),(3,0),(6,0),(0,1),(5,0),(1,1),(6,1),(5,1),(0,2),$$
$$(2,1),(3,2),(4,2),(2,2),(6,2)),$$
$$((0,1),(4,1),(5,0),(2,1),(2,0),(3,2),(6,0),(2,2),(6,1),(1,0),$$
$$(1,2),(3,0),(5,2),(1,1),(0,2)).$$

EXAMPLE 4.4 (V, C) is a 5-perfect 15-cycle system of order 51, where $V = Z_{17} \times Z_3$ and C consists of 85 cycles obtained cyclically mod $(17, -)$ from the following five starter cycles:

$$((0,1),(1,1),(0,2),(1,2),(0,3),(2,1),(2,2),(4,1),(1,3),(2,3),(4,3),$$
$$(3,2),(7,1),(3,1),(5,1)),$$
$$((0,1),(3,1),(0,2),(2,2),(1,1),(7,1),(9,2),(5,1),(8,2),(0,3),(3,3),$$
$$(8,3),(6,1),(15,2),(9,1)),$$
$$((0,1),(7,1),(0,2),(9,1),(16,2),(4,1),(15,2),(10,1),(15,3),(11,1),(11,3),$$
$$(1,1),(10,3),(14,1),(3,3)),$$
$$((0,2),(5,2),(9,2),(6,2),(15,2),(8,2),(2,2),(10,3),(2,1),(9,3),(8,1),$$
$$(7,3),(1,3),(1,2),(13,3)),$$
$$((0,3),(10,3),(15,1),(9,3),(16,2),(6,3),(2,2),(8,3),(6,2),(11,3),(7,3),$$
$$(15,3),(1,2),(12,3),(14,2)).$$

When $v = 30n + 25 = 5(6n + 5)$, we use a GDD$(3, \{3, 5^*\}; 6n + 5)$, which exists for $n \geq 2$. We then need decompositions of $K_{5,5,5}$ (Theorem 5.1 below), K_{15} (Example 4.2 above) and K_{25} (Example 4.5 below), and use the Construction with $\epsilon = 0$, $\alpha = 5$ and $s = 6n + 5$.

(The isolated case of order 55 is also needed to complete this case.)

EXAMPLE 4.5 (V, C) is a 5-perfect 15-cycle system of order 25 where $V = Z_5 \times Z_5$ and C consists of 20 cycles obtained cyclically mod $(5, -)$ from the following four starter cycles:

$$((0,1),(1,1),(3,1),(0,2),(2,1),(1,2),(3,2),(2,2),(0,3),(4,1),(2,3),$$
$$(1,3),(4,3),(4,4),(3,5)),$$
$$((0,1),(0,2),(4,1),(1,3),(1,1),(0,4),(2,1),(2,4),(2,2),(1,5),(3,3),$$
$$(1,4),(4,2),(2,5),(4,5)),$$
$$((0,1),(4,3),(0,5),(3,2),(3,5),(1,1),(1,5),(2,3),(2,2),(3,4),(2,4),$$
$$(0,4),(3,1),(4,5),(1,4)),$$
$$((0,2),(1,3),(2,4),(2,5),(0,3),(3,2),(2,3),(1,4),(3,5),(3,3),(0,4),$$
$$(1,5),(0,5),(4,2),(3,4)).$$

Summarising, we have:

THEOREM 4.6 *The spectrum for 5-perfect 15-cycle systems is the set of all $v \equiv 1$, 15, 21 or 25 (mod 30), $v \geq 15$, except possibly the isolated case $v = 55$.*

5 A general result

We have a decomposition of $K_{k,k,k}$ into k-perfect $3k$-cycles for any odd k not divisible by 3.

THEOREM 5.1 *Let $k \equiv 1$ or $5 \pmod 6$. There is a k-perfect $3k$-cycle decomposition of $K_{k,k,k}$.*

Proof: Let the vertices of $K_{k,k,k}$ be

$$\bigcup_{j=1}^{3}\{(i,j) \mid 1 \le i \le k\}.$$

First, let $k = 6s+1$. We take k cycles of length $3k$ as follows. For the j$^{\text{th}}$ cycle (for $1 \le j \le k$) we take:

$((1,1),(j,2),(k-j+2,3),(2,1),(j+1,2),(k-j+3,3),(3,1),\ldots,(2s,1),$
$(j+2s-1,2),(k-j+2s+1,3),(2s+1,1),(j+2s,2),(k-j+2s+2,3),\ldots$
$\ldots,(4s+1,1),(j+4s,2),(k-j+4s+2,3),(4s+2,1),\ldots,$
$\ldots,(k,1),(j-1,2),(k-j+1,3))$.

Consider the edges at distance 1:
$(i,1)$ is adjacent to $(j+i-1,2)$, for $1 \le j \le k$;
$(i,1)$ is adjacent to $(k-j+i,3)$, for $1 \le j \le k$.
Also $(j+i-1,2)$ is adjacent to $(k-j+i+1,3)$ for $1 \le j \le k$ and $1 \le i \le k$; that is, (letting $x = j+i-1$), we have $(x,2)$ is adjacent to $(k-x+2i,3)$, for $1 \le x \le k$ and $1 \le i \le k$. Now as i varies between 1 and k, so $k-x+2i$ takes all k values between 1 and k. (This is where we need k odd!)

Next, we check the k-perfect requirement. To do this, we list the k triangles that arise from the j$^{\text{th}}$ k-cycle given above. Recall that $k = 6s+1$.

Position in cycle	Vertices of triangle
$1, k+1, 2k+1$	$(1,1),(j+2s,2),(k-j+4s+2;3)$
$2, k+2, 2k+2$	$(j,2),(k-j+2s+2,3),(4s+2,1)$
$3, k+3, 2k+3$	$(k-j+2,3),(2s+2,1),(j+4s+1,2)$
$\vdots$	$\vdots$
$k, 2k, 3k$	$(2s+1,1),(j+4s,2),(k-j+1,3)$.

Summarising this, we have the triangles $((i,1),(j+2s+i-1,2),(k-j+4s+i+1,3))$, for $1 \le i \le 6s+1 = k$ (where addition is modulo k). As j varies between 1 and k, we see that $(i,1)$ occurs with $(x,2)$ for $1 \le x \le k$, and also with $(x,3)$ for $1 \le x \le k$.

In the case $k = 6s + 5$, we have a similar result. The k cycles of length $3k$ are as follows, for $1 \leq j \leq k$:

$$((1,1),(j,2),(k-j+2,3),(2,1),(j+1,2),(k-j+3,3),(3,1),\dots$$
$$\dots,(2s,1),(j+2s-1,2),(2s-j+1,3),(2s+1,1),(j+2s,2),\dots$$
$$\dots,(4s+1,1),(j+4s,2),(4s-j+2,3),\dots,(6s+5,1),(j-1,2),(k-j+1,3)).$$

Again, consider edges at distance 1. The vertex $(i,1)$ is adjacent to $(j+i-1,2)$ for $1 \leq j \leq k$, and also to $(i-j,3)$ for $1 \leq j \leq k$.

Also the vertex $(j+i,2)$ is adjacent to $(k-j+i+2,3)$; so (letting $x = i+j$) we have $(x,2)$ adjacent to $(k-2j+x+2,3)$ for $1 \leq j \leq k$ (and k is odd, so that as j varies, $k-2j+x+2$ takes all values (mod k) between 1 and k).

The edges at distance k in the above (j^{th}) k-cycle give rise to the following triangles. (Recall that here $k = 6s+5$.)

Position in cycle	Vertices of triangle
$1, k+1, 2k+1$	$(1,1),(2s-j+3,3),(j+4s+3,2)$
$2, k+2, 2k+2$	$(j,2),(2s+3,1),(4s-j+5,3)$
$\vdots$	$\vdots$
$k, 2k, 3k$	$(j+2s+1,2),(4s+4,1),(k-j+1,3).$

That is, we have the triangles $((i,1),(j+4s+2+i,2),(2s-j+2+i,3))$ for $1 \leq i \leq 6s+5 = k$, addition modulo k. For $1 \leq j \leq k$, clearly $(i,1)$ occurs with all $(x,2)$ and $(x,3)$, $1 \leq x \leq k$ (since k is odd). □

6 Further values of k

6.1 The case $k = 7$

Since $k \equiv 1 \pmod 6$, we have (Theorem 5.1) a 7-perfect 21-cycle decomposition of $K_{7,7,7}$. The necessary conditions for existence of a 7-perfect 21-cycle system of order v are $v \equiv 1$, 7, 15 or 21 (mod 42).

For $v \equiv 1 \pmod{42}$, let $v = 42n+1$ and use the Construction with $\alpha = 7$, $s = 6n$ and $\epsilon = 1$. There is a GD$(3,6;6n)$ for $n \geq 3$; since 43 is prime and 21 is odd we have a 7-perfect 21-cycle system of order 43 (Lemma 2.3). Also there is a 7-perfect 21-cycle system of order 85, to deal with the case $n = 2$.

EXAMPLE 6.1.1 Let (V,C) be given by $V = Z_{85}$ and C as follows:

$C = \{(0+i,1+i,3+i,6+i,2+i,7+i,13+i,4+i,11+i,19+i,5+i,15+i,26+i,8+i,21+i,9+i,25+i,40+i,57+i,14+i,33+i),$

$(0+i,20+i,41+i,1+i,23+i,46+i,2+i,37+i,74+i,44+i,72+i,12+i,70+i,17+i,76+i,38+i,67+i,28+i,52+i,18+i,49+i) \mid 0 \leq i \leq 84\}.$

Summarising, we have:

THEOREM 6.1.2 *The spectrum for* 7*-perfect* 21*-cycle systems includes the set of all* $v \equiv 1 \pmod{42}$*.*

6.2 Results for some general k

As a consequence of Theorem 5.1 and Lemma 2.3, we have:

THEOREM 6.2.1 *When* $k \equiv 1$ *or* 5 $\pmod 6$ *and* $6k+1$ *is prime, then there exists a* k*-perfect* $3k$*-cycle system of order* $1 \pmod{6k}$*, except possibly one of order* $12k+1$*.*

Proof: Lemma 1.3 ensures that a k-perfect $3k$-cycle system of order $6k+1$ exists. Let $v \equiv 1 \pmod{6k}$, so say $v = 6kn+1$. Then we use the Construction in Section 2 above with $\epsilon = 1$, $\alpha = k$, $s = 6n$, and GDD$(3,6;6n)$ for $n \geq 3$. This leaves possibly the isolated case (when $n = 2$) of order $12k+1$. □

COROLLARY 6.2.2

(i) When $k = 11$*, since* 67 *is prime, the spectrum for* 11*-perfect* 33*-cycle systems contains all* $v \equiv 1 \pmod{66}$ *except possibly* 133.

(ii) When $k = 13$*, since both* 79 *and* 157 *are prime, the spectrum for* 13*-perfect* 39*-cycle systems contains all* $v \equiv 1 \pmod{78}$*.*

(iii) When $k = 17$*, since* 103 *is prime, the spectrum for* 17*-perfect* 51*-cycle systems contains all* $v \equiv 1 \pmod{102}$ *except possibly* 205.

We summarise our results in an easy-to-read table.

The spectrum for k-perfect $3k$-cycle systems for small k

k	$3k$	Spectrum includes		Undecided
2	6	1 or 9 (mod 12), NOT 9	[13]	
3	9	1 or 9 (mod 18)	[5]	
4	12	1 or 9 (mod 24)		
5	15	1, 15, 21 or 25 (mod 30)		55
7	21	1 (mod 42)		7,15,21 (mod 42)
11	33	1 (mod 66)		33,45,55 (mod 66), 133
13	39	1 (mod 78)		13,27,39 (mod 78)
17	51	1 (mod 102)		51,69,85 (mod 102), 205

Clearly, much work remains to be done, in particular for cases with $k \equiv 0 \pmod 3$. At present, no suitable 6-perfect 18-cycle decomposition is known of $K_{6,6,6}$, or of $K_{18,18,18}$. However, the authors have further partial results, in particular, 6-perfect 18-cycle systems of orders 37 and 73, and 8-perfect 24-cycle systems of orders 33, 49, 81 and 97.

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