On a Criterion for Symmetric Hadamard Matrices

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Abstract

In this paper, we improve the best known algorithm on symmetric equivalence of Hadamard matrices by considering the eigenvalues of symmetric Hadamard matrices. As a byproduct the eigenvalues of skew Hadamard matrices are also discussed.

1 Introduction

An Hadamard matrix of order n is an n by n matrix H with all of its entries either +1 or -1, satisfying $HH^T = nI$. It is well known that if there is an Hadamard matrix H of order n, then $n = 1$ or $n = 2$ or n is a multiple of 4, and the moduli of all eigenvalues of H are $\sqrt{n}$.

Two Hadamard matrices are called Hadamard equivalent (or simply H-equivalent) if one can be obtained from the other by a sequence of row and column permutations and negations.

If H is an Hadamard matrix of order n, then $HH^T = nI$ implies that H is nonsingular and has an inverse $n^{-1}H^T$, whence $H^TH = nI$, so H^T is also an Hadamard matrix. However, it is not necessary for H^T to be Hadamard equivalent to H.

An Hadamard matrix H is symmetric if $H^T = H$, and skew if $H = S - I$ where $S^T = -S$. We refer to this as a skew Hadamard matrix of type

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I. Some authors use $H = S + I$, which we shall call a skew Hadamard matrix of type II. (Clearly, a type II matrix can be obtained by negating a type I matrix and conversely.)

In [4] and [1], symmetric and skew equivalence of Hadamard matrices were considered because of the importance of symmetric and skew Hadamard matrices in the construction of Hadamard matrices of larger orders, of other designs, and of Hadamard tournaments. In addition, symmetric and skew matrices require about half the storage space of arbitrary Hadamard matrices.

In this paper, we improve the best known algorithm on symmetric equivalence of Hadamard matrices by considering the eigenvalues of symmetric Hadamard matrices. As a by product, the eigenvalues of skew Hadamard matrices are also discussed.

2 Eigenvalues of Symmetric and Skew Hadamard Matrices

The following lemmas are well-known.

Lemma 1: If H is an Hadamard matrix of order n, then the moduli of all its eigenvalues of H are $\sqrt{n}$.

Lemma 2: If $S^T = -S$, and S is a real matrix, then eigenvalues of S are pure imaginary numbers.

Theorem 1: If $H = S + I$ with $S^T = -S$ is a skew Hadamard matrix of type II of order n, then the eigenvalues of H are $\lambda = 1 + i\sqrt{n-1}$ and $\overline{\lambda} = 1 - i\sqrt{n-1}$ each with multiplicity $\frac{n}{2}$.

Proof; Let v be an eigenvector of S with eigenvalues λ_0. Since $S^T = -S$ and S is a real matrix, λ_0 is purely imaginary. Set $\lambda_0 = i\beta$. Then $Sv = (i\beta)v$ which implies that $Hv = (S + I)v = i\beta v + v = (1 + i\beta)v$. Hence v is an eigenvector of H with eigenvalue $\lambda = 1 + i\beta$.

By Lemma 1, $|\lambda| = \sqrt{\lambda\overline{\lambda}} = \sqrt{n}$ or $1 + \beta^2 = n$ or $\lambda = 1 + i\sqrt{n-1}$ or $\lambda = 1 - i\sqrt{n-1}$.

Since the sum of all the eigenvalues of H is equal to the sum of the diagonal elements of H which equals n, we have

$$t\left(1+i\sqrt{n-1}\right)+(n-t)\left(1-i\sqrt{n-1}\right)=n+(2t-n)\,i\sqrt{n-1}=n$$

or $2t-n=0$ or $t=\frac{n}{2}$, where t is the multiplicity of the eigenvalue $\lambda=1+i\sqrt{n-1}$. □

Note that if $H=S-I$ with $S^T=-S$ is a skew Hadamard matrix of type I of order n, then the eigenvalues of H are $\lambda=-1+i\sqrt{n-1}$ and $\bar{\lambda}=-1-i\sqrt{n-1}$ each with multiplicity $\frac{n}{2}$.

Corollary 1: The sum of diagonal elements of skew Hadamard matrix H of order n is $\pm n$.

Theorem 2: If H is a symmetric Hadamard matrix of order n $(n>1)$, and the eigenvalues of H are $\sqrt{n}$ and $-\sqrt{n}$ with multiplicities r and $n-r$ respectively, then $r=\frac{n}{2}$ when n is square free; $r=\frac{n+k}{2}$ when n is a square number where $k=0$, or ± 2, or ± 4, $\cdots$, or $\pm\sqrt{n}$.

Proof: Since H is symmetric, then all eigenvalues of H are real, and therefore by Lemma 1 are $\sqrt{n}$ and $-\sqrt{n}$.

Since the sum of all eigenvalues of H is equal to the sum of diagonal elements of H, we get

$$r\sqrt{n}-(n-r)\sqrt{n}=m$$

for some integer m, so

$$(2r-n)\sqrt{n}=m.$$

If n is square free, then

$$2r-n=0$$

because m is an integer and $\sqrt{n}$ is not. Therefore, $r=\frac{n}{2}$.

If n is a square number, then m is 0 or an even multiple of $\sqrt{n}$ since n is even. Let $m=k\sqrt{n}$ where k is even, we get

$$2r - n = k.$$

This implies that

$$r = \frac{n+k}{2},$$

where $k = 0$ or ± 2, or $\pm 4, \cdots$, or $\pm\sqrt{n}$, because $-\sqrt{n} \le k \le \sqrt{n}$. □

Corollary 2: The sum of diagonal elements of symmetric Hadamard matrix H of order n is zero when n is square free and $k\sqrt{n}$ when n is a square number where $k = 0$ or ± 2, or $\pm 4, \cdots$, or $\pm\sqrt{n}$.

Note that for the special case $k = \pm\sqrt{n}$ when n is a square number, a symmetric Hadamard matrix is called graphical, such matrices are of some interests (see, for example [6] and [7]).

If H and K are symmetric Hadamard matrices of orders n and m respectively, and $H \otimes K = [h_{ij}K]$ is the Kronecker product, then $H \otimes K$ is a symmetric Hadamard matrix of order nm and the sum of diagonal elements of $H \otimes K$ is hk, where h and k are the sums of diagonal elements of H and K respectively. Thus, Theorem 2 and Corollary 2 hold for $H \otimes K$, too.

3 Symmetric and Skew Equivalences

Usually, we consider the equivalence classes of Hadamard matrices under Hadamard equivalence. A class is called symmetric if its representatives are symmetric equivalent. This implies that the class contains a representative which is symmetric. Skew classes are defined analogously.

In [5], the author gave three criteria to determine the skew equivalence of a given Hadamard matrix of order 16 and 20. These results were further explored in [4], this paper also contained some new theoretical and computational results on symmetric and skew equivalence of Hadamard matrices of order 16, 20, 24. In addition, the authors in [1] gave computational results on symmetric and skew equivalence of Hadamard matrices of order 28.

In the algorithm in [4] on skew equivalence, to check if an Hadamard matrix H is skew, the sum of diagonal elements of H is to be checked which requires n operations. If the sum of diagonal elements of H is not $\pm n$, then by Corollary 1 H is not skew. If the sum is $\pm n$, then nondiagonal elements h_{ij} and h_{ji} are to be checked to see if $h_{ij} = -h_{ji}$ which requires $\frac{n^2-n}{2}$ operations. Thus using Corollary 1, the procedure of checking the sum of the diagonal elements makes the algorithm more efficient as it requires only n operation instead of $\frac{n^2-n}{2}$ for matrices that are not skew and we need to check as many as 8 $[(\frac{n}{4})!]^4$ representatives of H .

In the algorithm in [4] on symmetric equivalence, to check if an Hadamard matrix H is symmetric, nondiagonal elements h_{ij} and h_{ji} are to be checked to see if $h_{ij} = h_{ji}$. The sum of diagonal elements of H was not checked since the sum of diagonal elements of a symmetric Hadamard matrix was not known.We are interested in obtaining a similar result for the sum of the diagonal elements of a symmetric Hadamard matrix. If the sum of diagonal elements of a symmetric Hadamard matrix is known, then to check if an Hadamard matrix H is symmetric, the sum of diagonal elements of H is to be checked first which requires only n operations. If the sum does not satisfy the condition, then H is not symmetric. If the sum satisfies the condition, then nondiagonal elements h_{ij} and h_{ji} are to be checked to see if $h_{ij} = h_{ji}$ which requires $\frac{n^2-n}{2}$ operations. Thus, using Corollary 2 will improve the best known algorithm in [4] on symmetric equivalence and makes it more efficient since there are as many as 32 $[(\frac{n}{4})!]^4$ representatives of H.

According to the Corollary 2, the sum of diagonal elements of symmetric Hadamard matrix H of order n is $k\sqrt{n}$ when n is a square number where $k = 0$ or ± 2, or $\pm 4, \cdots$, or $\pm\sqrt{n}$.

Our examples show that for $n = 4$ and 16, and each possible value of k, there is a symmetric Hadamard matrix H of order n, such that the sum of diagonal elements H is $k\sqrt{n}$.

It is an open question whether for each square number $n(n > 1)$ and each $k, k = 0, \pm 2, \cdots, \pm n$ there exists a symmetric Hadamard matrix H of order n, such that the sum of diagonal elements of H is $k\sqrt{n}$.

Acknowledgement

This research was carried out while the second author was at the Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931. Authors thank referee for giving useful suggestions that led to an improvement and revision of this manuscript.

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