The Esther-Klein-Problem in the Projective Plane

Heiko Harborth and Meinhard Möller
Discrete Mathematik
Technische Universität Braunschweig
D-38106 Braunschweig, Germany

Abstract. Let $p(k)$ ($q(k)$) be the smallest number such that in the projective plane every simple arrangement of $n \geq p(k)$ ($\geq q(k)$) straight lines (pseudolines) contains k lines which determine a k-gonal region. The values $p(6) = q(6) = 9$ are determined and the existence of $q(k)$ ($\geq p(k)$) is proved.

In 1931 Esther Klein asked for the smallest number $f(k)$ such that in the Euclidean plane every set of $n \geq f(k)$ points, no three collinear, contains a convex k-gon. The existence of $f(k)$ can be deduced with Ramsey's theorem ([5]). The exact values $f(3) = 3$, $f(4) = 5$, and $f(5) = 9$ are known, but in general only

$$1 + 2^{k-2} \leq f(k) \leq 1 + \binom{2k-4}{k-2}$$

is proved (see[1], F8).

If straight lines are used instead of points one can ask for the smallest number $f_1(k)$ such that in the Euclidean plane every arrangement of $n \geq f_1(k)$ straight lines, no two parallel, and without multiple intersections, one can find k straight lines which contain the sides of a finite convex k-gon. From the unique arrangements of 3 and 4 straight lines it follows trivially $f_1(3) = 3$ and $f_1(4) = 4$. However, for $k \geq 5$, arrangements as in Figure 1 prove the nonexistence of $f_1(k)$ since any set of k lines determines only triangles and quadrilaterals.

Since duality of points and lines holds in the projective plane a subsequent question could be as follows. What is the smallest number $p(k)$ such that in the projective plane every set of $n \geq p(k)$ points, no three collinear, contains a convex k-gon where k points are considered as vertices of a convex k-gon if there exists a mapping of the projective plane onto itself such that the k points form a convex k-gon in the Euclidean sense, that is, the convex k-gon has no point in common with the line at infinity. Then by duality it is also guaranteed that $p(k)$ is the smallest number such that in the projective plane every simple arrangement of $n \geq p(k)$ straight lines (simple means, no multiple intersections) contains a simple arrangement of k straight lines such that a (convex) k-gon occurs among the regions the plane is partitioned into by this arrangement of k lines.

The existence of $f(k)$ implies trivially the existence of $p(k)$:

$$p(k) \leq f(k).$$

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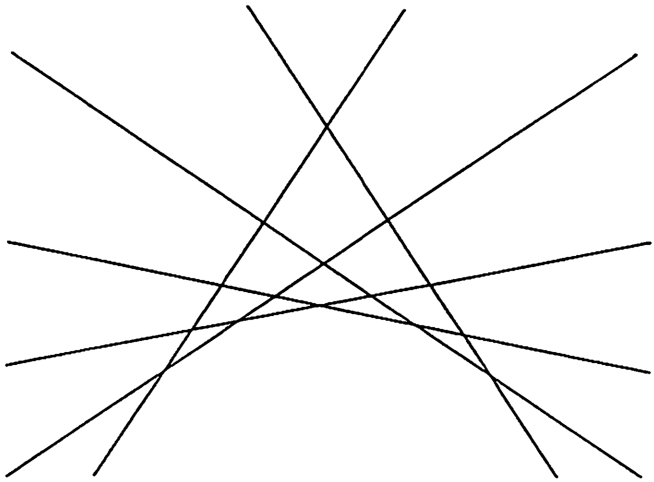

Figure 1

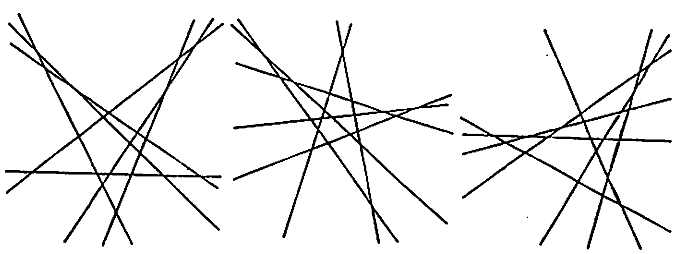

Figure 2

The numbers of nonisomorphic arrangements of 3, 4, and 5 straight lines in the projective plane are one in each case (see [4]). Since these unique arrangements determine a convex 3-gon, 4-gon, and 5-gon, respectively, it holds

$$p(k) = k \text{ for } k = 3,4,5.$$

Here we will determine the next value.

Theorem 1. $p(6) = 9$.

Proof. Two arrangements are called isomorphic if there exists a one-to-one incidence-preserving correspondence between their intersection points, line segments, and regions. There exist 11 nonisomorphic simple arrangements of 7 lines (see [4,

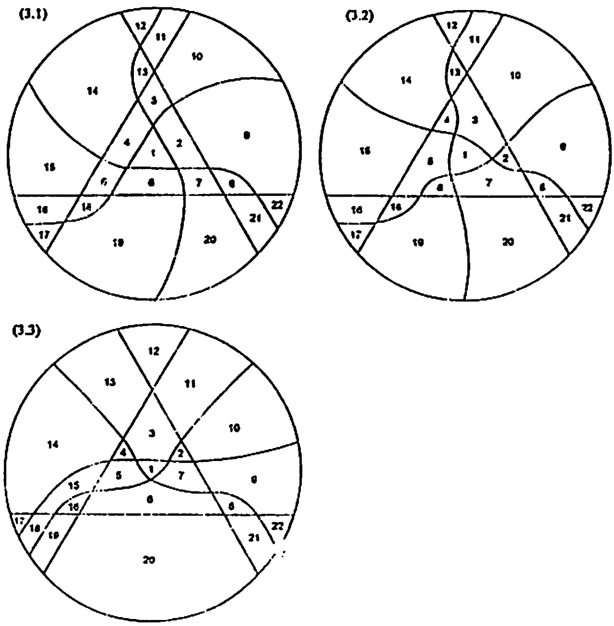

Figure 3

p.395]). Only 3 of them do not contain a simple arrangement of 6 straight lines such that one of its regions is a hexagon. These 3 arrangements are shown in Figure 2. In Figure 3 the same arrangements are represented by the corresponding pseudolines where one line is mapped to the line at infinity, represented by the circle. It may be noted that for n pseudolines with $n \leq 8$ every arrangement is stretchable (see [2]), that means, there exists an isomorphic arrangement of straight lines.

Now an eighth pseudoline will be added to the arrangement $(3i)$, $i = 1,2,3$ in all possible ways avoiding that this additional line together with five lines of $(3i)$ determines a hexagonal region. If the regions are labelled as in Figure 3 and if symmetry is considered then all possibilities are covered by the rows in Tables 1 to 3 where the eighth line always starts with the intersection of the line at infinity. Only those rows with (4.1) or (4.2) at the end guarantee an eighth line which does not determine a hexagonal region together with five of the other pseudolines. All possible arrangements are isomorphic to the two pseudoline arrangements (4.1) and (4.2) in Figure 4. The corresponding straight line arrangements are represented in Figure 5. These arrangements prove $p(6) \geq 9$.

The same procedure as above for the arrangements (4.1) and (4.2) leads to Tables 4 and 5. Since it comes out that no row can be completed cyclically it is proved that every arrangement of nine pseudolines contains six which determine a hexagonal region, and $p(6) \leq 9$ is proved. ∎

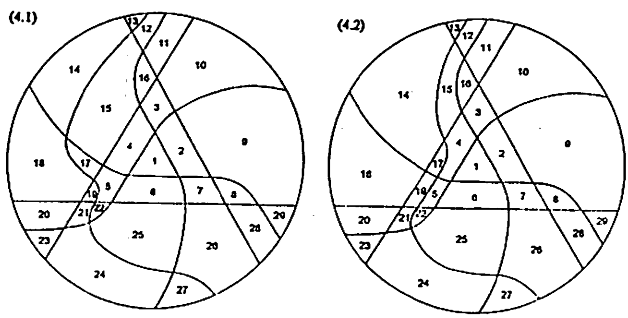

Figure 4

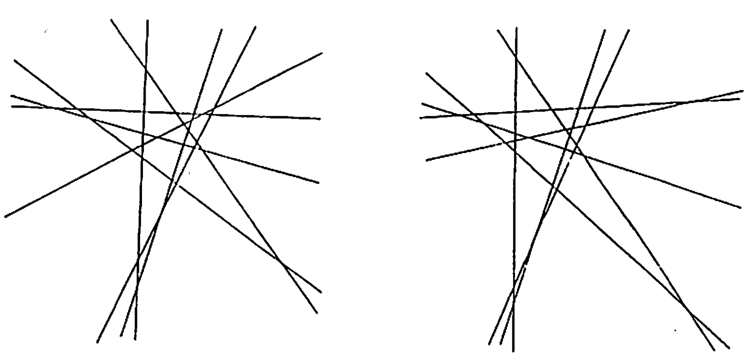

Figure 5

(3.1): 19 11 13 14 15 16 17
3 2 7 6
20
10 9 8 7 6
20
21 20
22 21 20

20 12 14 15 16 17 19
18
5 18 19 (4.1)
6 19
7 (4.2)
4 5 18 19 (4.2)
6 19
7 (4.1)
1 6 19
2
11 10 9 8 7
21
22 21

Table 1

(3.2): 19 11 13 14 15 16 17
3 1 5 18 (4.1)
7 20
2 7 6 (4.1)
20
10 9 8 7 6
21
22 21 20

20 12 14 15 16 17 19
5 6 19
7
4 5 18 19
6 19
7
11 10 9 8 7
21
22 21

Table 2

(3.3): 22 14 15 16 19 20
6 8 21
5 6 8 21
1 7 9 (4.2)
4 5 6 20 21 (4.2)
8 9
21
3 2 7
10 9 (4.2)
11

21 13 3 1 5 6 20
7 9 22
2 7 6 20
10 9 22 (4.1)

20 12 13 14 17 18 19
15 18 19
16 19
11 10 9 8 6
21
22 21

19 11 3 4 14 17 18 (4.2)
5 6 20
1 5 15 18
6 20
7 6

18 10 2 3 13 14 17 (4.1)
4 14 17 (4.2)
5
7 6 16 15
19
20 19 (4.1)
9 8 6 5
16 15
19
21 20

17 9 7 1 3 13 14
5 15 18 (4.2)
6 16 15 14
8 6 5 4 14
15 18 (4.1)
16 15 14
19 18
21 20

Table 3

```
(4.1) 29 18 19 21 24                28 14 15 16 11                26 13 14 18 19  5  6  7
               5  4  3 10  9                 3  2  7 26                     21 24 27
                        2  9              4  1  2  9 29                  20 23 24 25
                  6  7 26                       6 25                              27
                        8  9                                        12 11 10  9 29 28
                           28                                                   8  7
            17 15 16 11                                                           28
                     3  2
               5  6  7  8  9
                           28
                        26
                     25

      27 12 15  4  1  2                 24 11 16 15 14 18 20 23       23 10  3  4  5 19 18
                     6 25 24                         17  5 22 25                     21 24
                  5 22 21                                     21                  22 21 20
                        25 26                            18 20 23                     25
               17  5  6  7 26                      3  4  5 22 25
                        25                                    21
                     22 21 24                         2  7 26 27
                        25 26                   10  9 29 28 26 25
                                                              27
                                                      8  7  6
                                                         28

      20  9  2  1  4 15 14
                  6 25 24 23
               7 26 27 24 21
                           23
            8  7  6  5 17 18
                        19 18
                           21
               28
```

Table 4

For $k = 7$ the arrangement of 13 lines in Figure 6 does not contain 7 lines which determine a heptagon. This proves $p(7) \geq 14$.

In general

$$p(k) \geq 1 + 2^{(k+1)/2}$$

can be deduced from $f(r) > 2^{r-2}$ since any Euclidean set of points without a convex r-gon does not contain a convex $(2r-1)$-gon in the corresponding projective plane.

If in the definition of $p(k)$ pseudolines are used instead of straight lines then for the corresponding function $q(k)$ holds

$$q(k) \geq p(k).$$

```
(4.2)  29 18 19  5  6 25
                     7 26
                        8  9
                          28
                   4  3 10
             14 15 16 11
                     3  2
```

```
28 14 15 16 11
            3  2  7 26
      4  1  2  9 29
               6 25
17  4  3 10  9 29
            2  9 29
                  7 26
         1  2  9 29
               6 25
      19 21 24 27 26
```

```
26 13 12 15  4  1  2
                  6 25
      11 10  9  8  7
                    28
                 29 28
   14 17 19 21 24 27
            4  1  2
                  6 25
      18 19  5  6 25
                     7
            21 24 27
         20 23 24 25
                    27
```

```
27 12 15  4  5  6  7 26
                  25 24
               22 25 26
                  21
            1  2
               6 25 24
```

```
24 11 16 15 14 18 20 23
         3  2  7 26 27
   10  9  8  7  6 25
               26 27
         28
      29 28 26 25
                  27
```

```
23 10  3  4 17 14
            19 21 24
         5 19 18
               21 24
            22 25
               21 20
                  24
```

```
20  9  2  1  4 15 14
               17 14
                  19 18
                     21
            6 25 24 23
         7 26 27 24 23
   8  7  6  5 19 18
                     21
              25 24 23
           26 27 24 23
      28
```

Table 5

From the preceding arguments $q(k) = p(k)$ for $k \leq 6$ follows immediately. However, in general there are more arrangements of pseudolines than of straight lines and thus the existence of $q(k)$ remains questionable.

Theorem 2. *There exists a smaller number $q(k)$ such that in the projective plane every simple arrangement of $n \geq q(k)$ pseudolines contains k pseudolines which determine a k-gonal region.*

Proof. At first we prove the following Lemma.

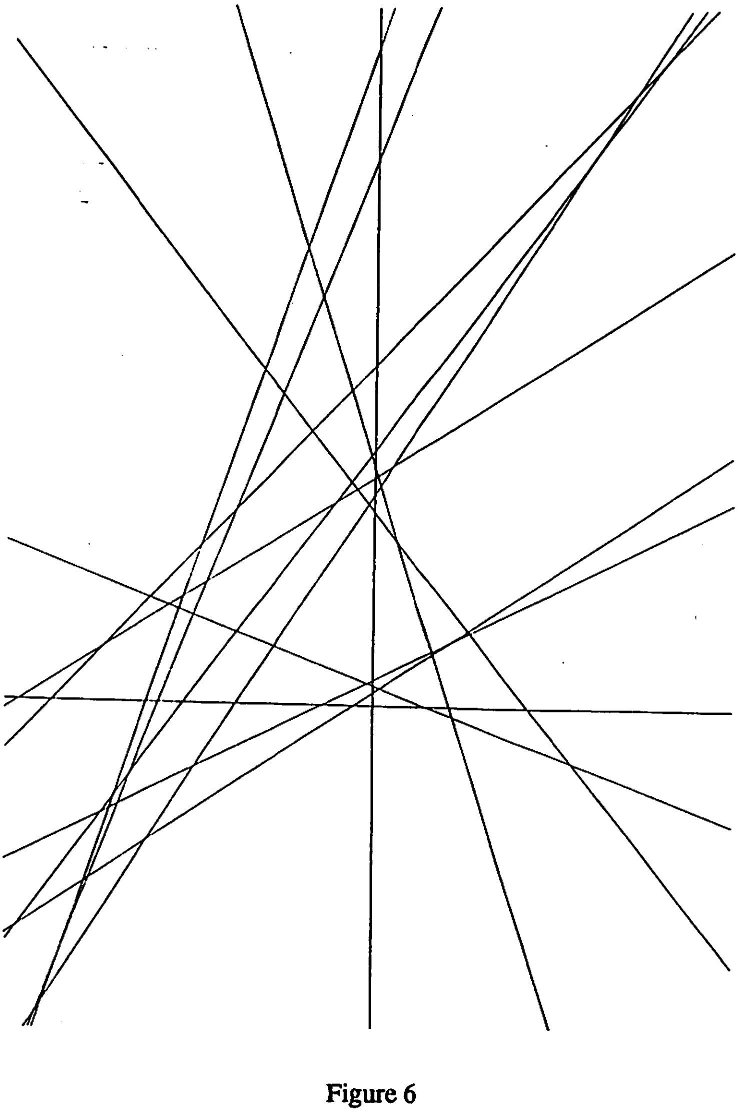

Figure 6

Lemma. *In the projective plane an arrangement of k pseudolines determines a k-gonal region if and only if every arrangement of six of the k pseudolines determines a hexagonal region.*

Proof of the Lemma. One part of the proof follows immediately.

If every set of six lines determines a hexagon then we assume that $s < k$ is the largest cardinality of a subset of the k pseudolines L_i, $1 \leq i \leq 5$, which determines an s-gon R_s. Consider one of the $k - s \geq 1$ additional lines, say L. If L intersects R_s then L does not intersect consecutive sides of R_s since s is a maximum. Thus three of the s lines can be chosen which together with the two lines of the intersected sides of R_s determine a pentagon such that L does not intersect consecutive sides of the pentagon. Since there exists only one pentagon in the unique arrangement of five lines (see[4]) a hexagon can be originated only if the sixth line intersects consecutive sides of the pentagon. Thus the five lines of the pentagon together with L determine six lines without a hexagon which contradicts the assumption.

It remains that L does not intersect R_s. Consider those neighbored triangles T_i of R_s which are determined by the lines of three consecutive sides of R_s. If L intersects one T_i then L instead of L_i together with the remaining $s-1$ sides of R_s determine an s-gon R'_s which is intersected by L_i as above. If L does not intersect any of the T_i then deletion of one line L_s leaves $s - 1$ lines which determine an R_{s-1} by the union of R_s and T_s. Since L does not intersect R_{s-1} the arguments can be repeated as long as a pentagon R_5 remains which is not intersected by L. Then, however, six lines exist which do not determine a hexagon in contradiction to the assumption. This proves the Lemma.

To continue the proof of Theorem 2 the existence of the Ramsey numbers $R_6(9, k)$ (see [3]) can be used. All six-tuples of n pseudolines are partitioned into two classes those which determine a hexagon and those which do not. Then for $n \geq R_6(9, k)$ either 9 pseudolines can be found such that no six-tuple determines a hexagon, or k pseudolines exist such that every six-tuple determines a hexagon. Since the first case contradicts Theorem 1 the second case remains, and together with the Lemma k pseudolines determine a k-gonal region. ∎

Although the existence of $q(k)$ is guaranteed by Theorem 2 no example is known so far where $q(k)$ exceeds $p(k)$.

References

[1] H.T. Croft, K.J. Falconer and R.K. Guy, "Unsolved Problems in Geometry", Springer-Verlag, 1991.

[2] J.E. Goodman and R. Pollack, *Proof of Grünbaum's Conjecture on the stretchability of certain arrangements of pseudolines*, J. Combinatorial Theory (A) 29 (1980), 385–390.

[3] R.L. Graham, B.L. Rothschild, and J.H. Spencer, "Ramsey Theory", J.Wiley, 1980.

[4] B. Grünbaum, "Convex Polytopes", J. Wiley, 1967.

[5] M. Hall, "Combinatorial Theory", Blaisdell, 1967.