Structure and cardinality of the set of dependence systems on a given set

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ABSTRACT. A dependence system on a set S is defined by an operator f, a function on the power set of S which is extensive (A is included in $f(A)$) and monotone (if A is included in B, then $f(A)$ is included in $f(B)$). In this paper the structure of the set $F(S)$ of all dependence systems on a given set S is studied. The partially ordered set of operators ($f < g$ if for every set A, $f(A)$ is included in $g(A)$) is a bounded, complete, completely distributive, atomic and dual atomic lattice with an involution. It is shown that every operator is a join of transitive operators (usually called closure operators, operators which are idempotent $ff = f$). The study of the class of transitive operators that join-generate all operators makes it possible to express $F(n)$ (the cardinality of the set $F(S)$ of all operators on a set S with n elements) by the Dedekind number $D(n)$. The formula has interesting consequences for dependence system theory.

1. Introduction

A dependence system $\langle S, f\rangle$ is a structure defined on a set S by an operator f, an extensive and monotone function on the power set of $S(\ \forall A, B \subseteq S :$ $A \subseteq f(A)$ and $A \subseteq B \Rightarrow f(A) \subseteq f(B))$. The notion of a dependence system is a generalization of the notion of a closure space. Indeed, a dependence system is a closure space if f is idempotent, i.e. for every subset A of $S : ff(A) = f(A)$.

Motivations for this generalization and an extensive study of properties of dependence systems may be found in the author's dissertation [9] and a series of papers in which ideas of the dissertation have been developed [10–13]. In this paper we consider the structure of the set $\mathbf{F}(S)$ of all dependence systems on a given set S, in particular a partial order relation

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on $\mathbf{F}(S)$ defined by: for all f, g in $\mathbf{F}(S)$, $f \leq g$ if $\forall A \subseteq S : f(A) \subseteq g(A)$. This partially ordered set is a bounded, complete, completely distributive, atomic and dual atomic lattice with an involution (atomic fuzz according to the recently introduced terminology [1]). There is also a natural structure of a monoid on $\mathbf{F}(S)$ with respect to the composition of operators. The partial order is compatible with the monoid operation so that $\langle \mathbf{F}(S), \circ, \leq \rangle$ is a po-monoid, but not divisibility monoid. The connections between the ordering and monoid structure seem to be rather loose, therefore we will study mainly the former one.

The set of join-irreducible elements, essential in every complete, completely distributive lattice, consists of operators of a special type which we study in detail. Surprisingly, these operators are transitive, hence every operator is a join of transitive operators. Also, a more extensive class of operators of elementary type will be considered. We use the representation of operators by join-irreducible operators to find a formula for the cardinality $\mathbf{F}(n+1)$ of the set $\mathbf{F}(S)$ of all operators on an $(n+1)$ element set S in terms of the n-th Dedekind number $D(n)$ (the number of antichains in the power set of an n-element set ordered by inclusion): $\mathbf{F}(n+1) = [D(n)]^{n+1}$. The problem of finding an effective formula for $D(n)$ is still open (the formula given by Kisielewicz [7] requires more operations in calculations than the number itself, so can not be used even for $n = 8$), therefore the search for an alternative formula for $\mathbf{F}(n)$ indicates a new possible direction in attempts to solve the Dedekind problem. Recently $D(8)$ has been found [17], so $\mathbf{F}(n)$ is known now for $n \leq 9$. However, the formula for $\mathbf{F}(n)$ in terms of $D(n)$ can be used to solve the problem of determination of operators by families of subsets. The classical results of particular kinds of dependence systems (closure spaces, topological spaces, matroids etc.) show that those systems are uniquely determined by the choice of some family of subsets (closed subsets, open subsets, independent subsets, generating subsets, bases etc.). The problem was: what family of subsets (if any) determines a general operator. In our paper we show that for finite $n \geq 4$ operators on an n element set can not be determined uniquely by any family of subsets. Moreover, they can not be determined by any fixed number of families.

2. Preliminaries of dependence system theory

For brevity we will use the symbol $\mathrm{Fin}(A)$ for the family of all finite subsets of a set A. Also, superscript ^{c} will be used for the complementation in a universe set S when from the context it is clear which universe set is meant.

Definitions of lattice-theoretical concepts not given in this paper may be found in Birkhoff's monograph [2]. Here we will recall only that a strong orthocomplementation, called also an involution, on a poset P is defined as a mapping $* : P \longrightarrow P$ satisfying the following conditions:

$\forall x, y \in P : x \leq y \Rightarrow y^* \leq x^*$ and $x^{**} = x$. If P is a bounded lattice with the least and greatest elements 0 and 1, then a strong orthocomplementation * is an orthocomplementation if additionally $\forall x \in P : x \wedge x^* = 0$ and $x \vee x^* = 1$. We follow the literature of the subject in using this confusing terminology.

Definition 2.1: A pair $\langle S, f \rangle$ where S is a set and f is a function from the power set of S into the power set of S satisfying the following two conditions:

i) $\forall A \subseteq S : A \subseteq f(A)$,

ii) $\forall A, B \subseteq S : A \subseteq B \Rightarrow f(A) \subseteq f(B)$,

is called a dependence system on set S defined by operator f.

If f satisfies also: (I) $\forall A \subseteq S : f(f(A)) = f(A)$, then $\langle S, f \rangle$ is called a *transitive dependence system* and f is called a *transitive* or *idempotent operator*. Transitive dependence sytems are often called *closure spaces* and transitive operators, *closure operators*. The set of all operators on a set S will be denoted $\mathbf{F}(S)$, the set of all transitive operators $\mathbf{I}(S)$. We will consider also other classes of operators distinguished by some conditions. They will be denoted by capital letters possibly preceded by some small letters indicating variations of the property. If a given class is the intersection of some other classes, its name will be denoted by the juxtaposition of the names of the intersecting classes; for example the intersection of classes $\mathbf{abX}(S)$ and $\mathbf{cdY}(S)$ will be denoted by $\mathbf{abXcdY}(S)$. We will omit S and write $\mathbf{F}$, $\mathbf{I}$, $\mathbf{abXcdY}$ if no confusion is likely.

There is a partial order defined on $\mathbf{F}(S)$ by: $\forall f, g \in \mathbf{F}(S) : f \leq g$ *iff* $\forall A \subseteq S : f(A) \subseteq g(A)$. We can also define a semigroup structure on $\mathbf{F}(S)$ with respect to the composition of operators written as juxtaposition: $\forall f, g, h \in \mathbf{F}(S) : f = gh$ if $\forall A \subseteq S : f(A) = g(h(A))$. $\mathbf{F}(S)$ is a monoid with respect to composition as there exists an operator e defined by: $\forall A \subseteq S : e(A) = A$ which satisfies $\forall f \in \mathbf{F}(S) : ef = fe = f$. Moreover there exists a zero operator defined by: $\forall A \subseteq S : o(A) = S$ which satisfies $\forall f \in \mathbf{F}(S) : of = fo = o$.

Definition 2.2: For every dependence system $\langle S, f \rangle$ the *derived set operator* is the function from the power set of S into itself given by: $df : A \longrightarrow A^{df} = \{x \in S : x \in f(A \backslash \{x\})\}$. We will use either the traditional notation A^{df}, or the more convenient one: $df(A)$. The notion of a derived set operator provides an equivalent way dependence systems can be defined. For every function $d : 2^S \longrightarrow 2^S$ satisfying conditions:

i) $\forall A, B \subseteq S : A \subseteq B \Rightarrow d(A) \subseteq d(B)$,

ii) $\forall A \subseteq S \, \forall x \in S : x \in d(A)$ *iff* $x \in d(A \setminus \{x\})$, the function f defined by: $\forall A \subseteq S : f(A) = A \cup d(A)$ is an operator on S and $df = d$.

Definition 2.3: For every dependence system $\langle S, f \rangle$ the following function from the power set of S into itself constructed by the consecutive application of the operator df and the set-theoretical operations in the power set of S defined by: $\forall A \subseteq S : f^*(A) = A \cup A^{cdfc}$ is an operator called the *dual operator* of f.

Also, we say property X is dual to property $Y(X = Y^*)$ if for every operator f, f has property Y iff f^* has property X.

Definition 2.4: For every dependence system $\langle S, f \rangle$ the following families of subsets of S are distinguished: $\forall A \subseteq S$:

i) $A \in$ **f-Cl** $\subseteq 2^S$ *iff* $f(A) = A$ *iff* $A^{df} \subseteq A$, and we say that A is f*-closed*, or *closed* if no confusion is likely.

ii) $A \in$ **f-Ind** $\subseteq 2^S$ *iff* $A \cap A^{df} = \emptyset$ *iff* $\forall x \in A : x \notin f(A \setminus x)$ and we say that A is f*-independent* or simply *independent*.

iii) $A \in$ **f-Gen** $\subseteq 2^S$ *iff* $A \cup A^{df} = S$ *iff* $f(A) = S$ and we call A an f*-generating* set or simply a *generating* set.

iv) $A \in$ **f-Base** $\subseteq 2^S$ *iff* $A \in$ **f-Ind** $\cap$ **f-Gen** *iff* $A^{df} = A^c$ and we call A an f*-base* or simply a *base*.

The particular cases of dependence systems can be distinguished by some additional conditions imposed on the operator.

A topological space can be described by **INfA**-operator (i.e. operator which has the properties **I**, **N**, **fA**), where: **N** means the *normalization condition:* $f(\emptyset) = \emptyset$, **fA** means the *finite additivity:* $\forall A, B \subseteq S : f(A \cup B) = f(A) \cup f(B)$, and **I** is the *transitivity* (or *idempotence*) condition mentioned above which completes the set of assumptions for the standard closure operator. This condition can be formulated in a few eqivalent ways for instance by the formulas: $\forall A, B \subseteq S : A \subseteq f(B) \Rightarrow f(A) \subseteq f(B)$, or $\forall A, B \subseteq S : A \subseteq f(B) \Rightarrow f(A \cup B) \subseteq f(B)$.

An algebraic generating operator (subalgebra operator) is an **IfC**-operator, where **fC** means *finite character*: $\forall x \in S \; \forall A \subseteq S : x \in f(A) \Rightarrow \exists A_0 \in \text{Fin}(A) : x \in f(A_0)$.

A matroid can be defined as an **IwEfC**-operator where **wE** is *weak exchange:* $\forall x, y \; \forall A \subseteq S : x \notin f(A)$ and $x \in f(A \cup y) \Rightarrow y \in f(A \cup x)$.

The name *weak exchange* indicates the existence of more restrictive property called *exchange:* (E) : $\forall x \in S \quad \forall A, B \subseteq S : x \in f(A)$ and $x \notin f(A \setminus B) \Rightarrow \exists y \in B : y \in f(A \setminus y \cup x)$.

The notion of a geometry requires slightly more restrictive conditions. It is defined by **IwEfCNt$_1$S**-operator, where $\mathbf{t}_1 S$ denotes the well-known T_1 condition from topology, also known as the separation axiom of Frechet: $\forall x \in S : f(\{x\}) = \{x\}$.

The transitivity condition can be weakened in many ways. For instance *weak idempotence* **wI** is given by: $\forall A, B \subseteq S : A \in \text{Fin}(S) \Rightarrow [A \subseteq f(B) \Rightarrow f(A \cup B) \subseteq f(B)]$.

It appeared [6] that the *weak exchange* is the dual property to the *weak idempotence* i.e. $\mathbf{wE}(S) = \mathbf{wI}^*(S)$ (also $\mathbf{E}(S) = \mathbf{I}^*(S)$).

We can write the fundamental relations among the properties of operators using the symbol of inclusion of classes: $\mathbf{I}(S) \subseteq \mathbf{wI}(S)$, $\mathbf{wIfC}(S) \subseteq \mathbf{I}(S)$, $\mathbf{wEwIfC}(S) \subseteq \mathbf{IE}(S)$. Certainly, for a finite set S we have $\mathbf{wI}(S) = \mathbf{I}(S)$ and $\mathbf{wE}(S) = \mathbf{E}(S)$.

We will refer to the notion of a direct sum of dependence systems [13].

Definition 2.5: Let $\{\langle S_i, f_i\rangle : i \in I\}$ be a family of dependence systems defined on the disjoint family of sets $\{S_i : i \in I\}$. Let $S = \bigcup\{S_i : i \in I\}$. Then the dependence system $\langle S, f\rangle$, where f is defined by: $\forall A \subseteq S : f(A) = \bigcup\{f_i(A \cap S_i) : i \in I\}$ is called the direct sum of $\{\langle S_i, f_i\rangle : i \in I\}$. We will write $\langle S, f\rangle = \sum\{\langle S_i, f_i\rangle : i \in I\}$ or $f = \sum\{f_i : i \in I\}$.

3. Basic properties of the structure of operators on a set S

In the preliminaries we defined a partial order on the set of operators on a given set $S : f \leq g$ if $\forall A \subseteq S : f(A) \subseteq g(A)$. With respect to this partial order $\langle \mathbf{F}(S), \leq\rangle$ is a complete, completely distributive lattice. This follows directly from the fact that meets and joins are given by intersections and unions of sets. Indeed, given a family of operators $\{f_i : i \in I\}$ on a set S, then $\forall A \subseteq S : \vee\{f_i : i \in I\}(A) = \cup\{f_i(A) : i \in I\}$ and $\wedge\{f_i : i \in I\}(A) = \cap\{f_i(A) : i \in I\}$. Therefore $\forall A \subseteq S : \wedge\{\vee\{f_{ij} : j \in J\} : i \in I\}(A) = \cap\{\cup\{f_{ij}(A): j \in J\} : i \in I\} = \cup\{\cap\{f_{ik(i)}(A) : i \in I\} : k \in J^I\} = \vee\{\wedge\{f_{ik(i)} : i \in I\} : k \in J^I\}(A)$ (J^I is the set of all functions from I to J.

$\langle \mathbf{F}(S), \leq\rangle$ is a lattice with least element the identity operator e ($\forall A \subseteq S : e(A) = A$) and greatest element o ($\forall A \subseteq S : o(A) = S$).

Recall that the operator dual to f is f^*, where $\forall A \subseteq S : A^{df^*} = A^{cdfc}$ Evidently $f^{**} = f$, and for all operators f and g on $S : f \leq g$ *iff* $\forall A \subseteq S : A^{df} \subseteq A^{dg}$ *iff* $\forall A \subseteq S : A^{cdf} \subseteq A^{cdg}$ *iff* $\forall A \subseteq S : A^{cdgc} \subseteq A^{cdfc}$ *iff* $g^* \leq f^*$. Therefore the duality on operators defines an involution (strong orthocomplementation) on the lattice $\langle \mathbf{F}(S), \leq\rangle$.

Now we can summarize:

Proposition 3.1. *The set of all operators on a given set S with respect to the ordering of operators is a bounded, complete, completely distributive lattice with an involution.*

We have also the structure of a noncommutative monoid with zero under composition of operators. The unity is the identity operator $\forall A \subseteq S: e(A) = A$, the zero operator is $\forall A \subseteq S : o(A) = S$. The monoid operation is compatible with the order: $\forall f, g, h \in \mathbf{F}(S) : g \leq h \Rightarrow [fg \leq fh$ and $gf \leq hf]$, so that $\langle \mathbf{F}(S), \circ, \leq \rangle$ is a po-monoid, but it is not a lattice ordered monoid as only one-sided distributivity condition is satisfied: $\forall A \subseteq S$: $\vee\{f_i : i \in I\}f_0(A) = \bigcup\{f_i f_0(A) : i \in I\} = \vee\{f_i f_0 : i \in I\}(A)$. When multiplying a join from the left we get only inequality: $\forall A \subseteq S : f_0 \vee \{f_i : i \in I\}(A) = f_0 \bigcup\{f_i(A) : i \in I\} \supseteq \bigcup\{f_0 f_i : i \in I\}(A) = \bigvee\{f_0 f_i : i \in I\}(A)$.

The following example shows that $\mathbf{F}(S)$ is not necessarily a divisibility monoid, i.e. there exist f, g on every sufficiently big set S, such that $f \leq g$, but there is no operator h on S, such that $g = hf$. Assume U, T, $W \subseteq S$, $T = U \cap W$. Write T as a disjoint union of two nonempty subsets T_U and T_W. Let $\emptyset \neq N_U \subseteq T_U$, $\emptyset \neq N_W \subseteq T_W$. Let $f(A) = T$ if $A \subseteq T$, and $= S$ otherwise.

$$
\begin{aligned}
\text{Let } g(A) &= U \text{ if } A \subseteq T_U, \\
&= W \text{ if } A \subseteq T_W, \\
&= U \cup W \text{ if } A \subseteq T \text{ and } A \cap T_U \neq \emptyset \text{ and } A \cap T_W \neq \emptyset \\
&= S \text{ otherwise}
\end{aligned}
$$

It is easy to show that $f \leq g$. Now suppose there exists an operator h, such that $g = hf$. Then $g(N_U) = U$, $g(N_W) = W$. But $T = f(N_U) = f(N_W)$, so $h(T) = U$ and $h(T) = W$, contradiction.

This shows that connections between the ordering and monoid structures seem to be rather loose, and we will focus our attention at the former one. An example of a self-dual matroid on a set with two elements given by Bondy and Welsh [3] shows that the involution defined on the lattice of operators by the duality need not be an orthocomplementation. We will show that actually for any set S with more than one element there exist self-dual operators. First, we have to recall a proposition:

Proposition 3.2. [13] *Let* $\langle S, f \rangle = \sum\{\langle S_i, f_i \rangle : i \in I\}$. *Then:* $\langle S, f^* \rangle = \sum\{\langle S_i, f_i^* \rangle : i \in I\}$.

Bondy and Welsh gave the following example of a self-dual matroid $\langle S_1, f_1 \rangle : S_1 = \{0, 1\}$, $f_1(\emptyset) = \emptyset$, $f_1(\{0\}) = f_1(\{1\}) = S_1$. It is easy to show that $f_1^* = f_1$. Now let us consider a dependence system $\langle S_2, f_2 \rangle : S_2 = \{a, b, c\}$, $f_2(\emptyset) = \emptyset$ and $f_2(\{a\}) = \{a, b\}$, $f_2(\{b\}) = \{b, c\}$, $f_2(\{c\}) = \{c, a\}$, $f_2(A) = S_2$ otherwise. Then $f_2^* = f_2$. Now, every set of cardinality greater than one can be considered as a disjoint union of sets of cardinality two and three. Therefore using the direct sum we can construct from operators f_1 and f_2 a self-dual operator on an arbitrary set of cardinality greater than one. So we get:

Corollary 3.3. *If the cardinality of S is greater than 1, then there exists a self-dual operator on a set S.*

4. Join-irreducible elements

Before we proceed to the more detailed study of the structure of the set of all dependence systems we will distinguish a class of operators of a very elementary form.

Definition 4.1: Let $A \subseteq S$, $x \in S \setminus A$. Then $f_{A,x}$ is an operator on S defined by: $\forall B \subseteq S : f_{A,x}(B) = B \cup \{x\}$ if $A \subseteq B$, $f_{A,x}(B) = B$ otherwise. It will be shown that $f_{A,x} = f_{B,y}$ iff $A = B$ and $x = y$. We will write $\mathbf{F}_x(S)$ for the set $\{f_{A,x} : A \subseteq S \setminus \{x\}\}$. The class of all operators of this form will be written $\mathbf{F}_{irr}(S)$, i.e. $\mathbf{F}_{irr} = \bigcup\{\mathbf{F}_x : x \in S\}$.

Proposition Proposition 4.2. *For all subsets A, B of S and all elements x, y such that $x \in S \setminus A$, $y \in S \setminus B : f_{A,x} \leq f_{B,y}$ iff $B \subseteq A$ and $x = y$.*

Proof: ($\Leftarrow$) Suppose $B \subseteq A \subseteq S$ and $x \in S \setminus A$. Then $\forall C \subseteq S : A \subseteq C \Rightarrow B \subseteq C$. Therefore for every subset C of S, if $A \subseteq C$, then $f_{A,x}(C) = C \cup \{x\} = f_{B,x}(C)$; if A is not included in C, then $f_{A,x}(C) = C \subseteq f_{B,x}(C)$.

($\Rightarrow$) Suppose $x = y \in S \setminus A$ and B is not included in A. Then $f_{A,x}(A) = A \cup \{x\}$ is not included in $A = f_{B,x}(A)$. Now $x \in S \setminus A$ and $y \in S \setminus B$, hence $f_{A,x}(A) = A \cup \{x\} \neq A$ and $f_{B,y}(A) \subseteq A \cup \{y\}$. If $x \neq y$, then $x \notin A \cup \{y\}$, and therefore $f_{A,x}(A)$ is not included in $f_{B,y}(A)$.

Proposition 4.3. *For every operator g on S different from the least operator e on S there exists a subset A of S and element x of $S \setminus A$ such that $f_{A,x} \leq g$.*

Proof: $g \neq e$, so $\exists A \subseteq S : A \neq g(A)$, hence $\exists A \subseteq S \exists x \in S \setminus A : x \in g(A)$. If A is not included in B, then $f_{A,x}(B) = B \subseteq g(B)$. So assume $A \subseteq B$. Then $f_{A,x}(B) = B \cup \{x\}$, but $x \in g(A) \subseteq g(B)$, so $B \cup \{x\} \subseteq g(B)$.

Proposition 4.4. *The atom space $At(\mathbf{F}(S))$ of $\langle \mathbf{F}(S), \leq \rangle$ consists of operators from the set $\{f_{A,x} : x \in S, A = S \setminus \{x\}\}$. Moreover the lattice $\langle \mathbf{F}(S), \leq \rangle$ is atomic, i.e. every nonzero element is greater than some atom.*

Proof: For every $x \in S$, $f_{A,x} \neq e$ as $f_{A,x}(S \setminus \{x\}) = S \neq S \setminus \{x\}$. If $g \leq f_{S\setminus\{x\},x}$ and $g \neq f_{S\setminus\{x\},x'}$, then $\forall B \subseteq S : g(B) \subseteq f_{S\setminus\{x\},x}(B)$ and for every $B \neq S \setminus \{x\}$, $g(B) = B$. If $g(S \setminus \{x\}) = S$, then $g = f_{S\setminus\{x\},x'}$, hence $\forall B \subseteq S : g(B) = B$. Therefore we have $\{f_{A,x} : x \in S, A = S \setminus \{x\}\} \subseteq At(\mathbf{F}(S))$. Now suppose $g \neq e$. We will show that there exists x in S such that $f_{S\setminus\{x\},x} \leq g$. From Prop. 4.3 we know about the existence of $f_{A,x}$ such that $f_{A,x} \leq g$, but by Prop. 4.2 we have $f_{S\setminus\{x\},x} \leq f_{A,x} \leq g$. Therefore we get $\{f_{A,x} : x \in S, A = S \setminus \{x\}\} = At(\mathbf{F}(S))$ and the second statement in the proposition.

Corollary 4.5. *$\langle F(S), \leq\rangle$ is atomic and dual atomic.*

Proof: The dual atoms (in the lattice theoretical sense) are operators dual to atoms (in the sense of dependence system duality).

Proposition 4.6. *Let g be an operator on S, $A \subseteq S$, and $x \in S\setminus A$. Then: $x \in g(A)$ iff $f_{A,x} \leq g$.*

Proof: $(\Rightarrow)$ Suppose $f_{A,x}$ is not less than g, i.e. $\exists B \subseteq S : A \subseteq B$ and $x \notin g(B)$. But $x \in g(A) \subseteq g(B)$, contradiction.
$(\Leftarrow)$ $f_{A,x} \leq g$ *iff* $\forall B \subseteq S : f_{A,x}(B) \subseteq g(B)$, so $A \cup \{x\} \subseteq g(A)$.

Proposition 4.7. *For every operator g on $S : g = \vee\{f_{A,x} : f_{A,x} \leq g\}$.*

Proof: Certainly $f = \vee\{f_{A,x} : f_{A,x} \leq g\} \leq g$. Suppose $f \neq g$, i.e. $\exists B \subseteq S \exists x \in g(B) : x \notin f(B)$ and $x \in g(B)$. But then by Prop. 4.6 we have $f_{B,x}$ is less than g, but not less than f, contradiction.

The class of operators $\mathbf{F}_{irr}(S)$ join-generates $\mathbf{F}(S)$. We will show that all operators of the form $f_{A,x}$ are completely join irreducible, so this representation is unique.

Proposition 4.8. *For every subset A of S and every element of $S \setminus A$, $f_{A,x}$ is completely join-irreducible, i.e. for every family of operators $\{g_i \in F(S) : i \in I\}$, if $f_{A,x} = \vee\{g_i \in F(S) : i \in I\}$, then there exists j in I such that $f_{A,x} = g_j$.*

Proof: Certainly $\forall i \in I : g_i \leq f_{A,x}$. Now $\forall B \subseteq S : f_{A,x}(B) = \bigcup\{g_i(B) : i \in I\}$. Therefore, if A is not included in B, then $\forall i \in I : g_i(B) = B = f_{A,x}(B)$. If $A \subseteq B$, then $\exists j \in I : x \in g_j(B)$, so for this j we have $f_{A,x} = g_j$.

Corollary 4.9. *$F_{irr}(S)$ is the set of all completely join-irreduciirrble elements of the complete lattice $\langle \mathbf{F}(S), \leq\rangle$.*

Proof: We have to justify only why every completely join-irreducible element belongs to $\mathbf{F}_{irr}(S)$. But by Prop. 4.7 every operator is a join of elements from $\mathbf{F}_{irr}(S)$, so by irreducibility it belongs to $\mathbf{F}_{irr}(S)$.

Recall that in a poset $\langle P, \leq\rangle$ a poset ideal Q is a subset of P such that if $x \in Q$ and $y \leq x$, then $y \in Q$.

The set $\mathbf{F}_{irr}$ inherits a partial order from $\mathbf{F}(S)$. By Prop.4.7 there is a bijective correspondence between poset ideals of $\langle \mathbf{F}, \leq\rangle$ and operators of $\mathbf{F}(S)$. Certainly, every subset of $\mathbf{F}(S)$ generates some operator, but the correspondence between subsets of $\mathbf{F}_{irr}$ and operators in $\mathbf{F}(S)$ is of many-to-one type. However, we have:

Proposition 4.10. *Let $F_1(S)$, $F_2(S)$ be two antichains in $\langle F_{irr}, \leq\rangle$ (no two elements are comparable), $f = \vee F_1(S)$, $g = \vee F_2(S)$. Then $f = g \Rightarrow F_1(S) = F_2(S)$. The correspondence is one-to-one, but not necessarily onto if S is infinite.*

Proof: Suppose $f = g$ but $\mathbf{F}_1(S) \neq \mathbf{F}_2(S)$ (we will write $\mathbf{F}_1, \mathbf{F}_2$). We can assume that there exists $f_{A,x} \in \mathbf{F}_1 \setminus \mathbf{F}_2$. Then $x \in f(A)$, i.e. $f_{A,x} \leq f = g = \vee\mathbf{F}_2$, hence $x \in \vee\mathbf{F}_2(A)$, so $\exists f_{B,y} \in \mathbf{F}_2 : f_{A,x} \leq f_{B,y'}$, and finally $x = y$ and $B \subseteq A$. But $f_{A,x} \neq f_{B,y'}$ so $B \subseteq A$ and $B \neq A$. Now $f_{A,x}(B) = B$ and $f_{B,x}(B) = B \cup \{x\}$. Hence $f(B) = g(B) = B \cup \{x\}$, so $\exists f_{C,x} \in \mathbf{F}_1 : C \subseteq B$, but then $f_{A,x} \leq f_{C,x'}$, a contradiction. This proves that the correspondence between the antichains of $\langle \mathbf{F}_{irr}, \leq \rangle$ and operators in $\mathbf{F}(S)$ is one-to-one. Now if $f = \vee\mathbf{F}_1$ where $\mathbf{F}_1$ is an antichain in $\mathbf{F}_{irr}$, and $f_{A,x} \leq f$, then $f_{A,x} = f_{A,x} \wedge f = f_{A,x} \wedge (\vee\mathbf{F}_1) = \vee\{f_{A,x} \wedge f_{B,x} : f_{B,x} \in \mathbf{F}_1\}$. But $f_{A,x} \in \mathbf{F}_{irr}$, so there exists $f_{B,x} in \mathbf{F}_1$ such that $f_{A,x} = f_{A,x} \wedge f_{B,x'}$, i.e. $f_{A,x} \leq f_{B,x}$. Therefore, $\mathbf{F}_1$ is a set of all maximal elements of the poset ideal in $\mathbf{F}_{irr}$ that corresponds to f. To show that the correspondence is not onto for infinite S suffices to find an operator f whose ideal of irreducible elements does not have minimal elements. Consider an operator f defined by: $f(A) = A$ if $A \in \mathrm{Fin}(S)$, $f(A) = S$ otherwise. The ideal of all irreducible operators less than f consists of all the operators $f_{B,x}$ such that $x \in S$ and B is an infinite subset of $S \setminus \{x\}$. Certainly this ideal does not have maximal elements as we do not have minimal infinite sets. This concludes the proof.

On the other hand if S is finite, so is $\mathbf{F}_{irr}$, and every ideal $\mathbf{F}_{irr}$ is generated by its maximal elements. Therefore we have:

Corollary 4.11. *If S is finite, then there is a bijective correspondence between antichains in $\langle \mathbf{F}_{irr}, \leq \rangle$ and operators in $\mathbf{F}(S)$.*

Proposition 4.12. *Every operator of the form $f_{A,x}$ is transitive, i.e. $\mathbf{F}_{irr}(S) \subseteq \mathbf{I}(S)$.*

Proof: $C \subseteq f_{A,x}(B)$ iff ($A \subseteq B$ and $C \subseteq B \cup \{x\}$) or (A is not included in B and C is included in B). Therefore if $A \subseteq B$, then $C \cup \{x\} \subseteq B \cup \{x\}$, so $f_{A,x}(C) \subseteq f_{A,x}(B)$. Now, if A is not included in B, then also it is not included in C, and $f_{A,x}(C) = C \subseteq B = f_{A,x}(B)$.

Corollary 4.13. *Every operator on a set S is a join of transitive operators on S. Also, every operator on a set S is a meet of operators having exchange property.*

Proof: The second statement follows from the fact that $\mathbf{E} = \mathbf{I}^*$.

We already considered the set of dual atoms which consists of operators dual to atoms and the set of completely meet irreducible elements of $\mathbf{F}(S)$ which consists of operators dual to operators in $\mathbf{F}_{irr}$. Now we will find the explicit form of those operators.

Proposition 4.14. *The set of all completely meet irreducible elements of $\mathbf{F}(S)$ consists of the operators defined by: For every subset A of S and every $x \notin A$, $\forall B \subseteq S : f^*_{A,x}(B) = S \setminus \{x\}$ if $A \cup \{x\} \subseteq S \setminus B$, and $f^*_{A,x}(B) = S$ otherwise. Therefore the set of all dual atoms consists of the operators* *defined by: For every element x of S, $\forall B \subseteq S : f^*_{S\setminus\{x\},x}(B) = S \setminus \{x\}$ if $B = \emptyset$, $f^*_{S\setminus\{x\},x}(B) = S$ otherwise.*

Proof: $df_{A,x}(B) = \{x\}$ if $A \subseteq B$, and $df_{A,x}(B) = \emptyset$ otherwise. So $df^*_{A,x}(B) = df_{A,x}(B^c) = S \setminus \{x\}$ if $A \cap B = \emptyset$, and equals S otherwise. Finally we get $f^*_{A,x}(B) = B \cup df^*_{A,x}(B)$.

All operators in $\mathbf{F}_{irr}$ are transitive. Some joins of these operators preserve transitivity.

Proposition 4.15. *Let $\{A_j : j \in J\}$ be a family of subsets of $S \setminus \{x\}$. Then $\vee\{f_{A_j,x} : j \in J\}$ is a transitive operator.*

Proof: Let $f = \vee\{f_{A_j,x} : j \in J\}$ and $D \subseteq f(C)$. Then $D \subseteq f(C) = C \cup \{x\}$ if $\exists j \in J : A_j \subseteq C$, $f(C) = C$ otherwise. If $f(C) = C$, then $D \subseteq C$ and $f(D) = D \subseteq f(C)$. If $A_j \subseteq C$ for some j, then $f(D) \subseteq D \cup \{x\} \subseteq C \cup \{x\} = f(C)$. Therefore if $D \subseteq f(C)$, then $f(D) \subseteq f(C)$, i.e. $f \in \mathbf{I}(S)$.

Proposition 4.16. *Let A be a subset of S, $X \subseteq S \setminus A$. Then an operator $f_{A,x} = \vee\{f_{A,x} : x \in X\}$ is transitive.*

Proof: Let $D \subseteq f_{A,x}(C) = \vee\{f_{A,x} : x \in X\}(C) = \bigcup\{f_{A,x}(C) : x \in X\} = C \cup X$ if $A \subseteq C$ and is equal C otherwise. Then $f_{A,x}(D) \subseteq D \cup X \subseteq C \cup X = f_{A,x}(C)$ if $A \subseteq C$. If A is not included in C, then $D \subseteq f_{A,x}(C) = C$ and therefore $f_{A,x}(D) \subseteq f_{A,x}(C)$.

Observe that an operator $f_{A,x}$ can be defined as follows: Assume that $X \subseteq S \setminus A$. Then $\forall B \subseteq S : f_{A,x}(B) = B \cup X$ if $A \subseteq B$ and $f_{A,x}(B) = B$ otherwise. The operators of the form $f_{A,x}$ have several properties in $\mathbf{F}_{irr}$. We will provide some examples without proof.

Proposition 4.17. *Let A, B, $C \subseteq S$, $X \subseteq S \setminus A$, and $Y \subseteq S \setminus B$. Then:*

i) $f_{A,X} = f_{B,Y}$ *iff* $A = B$ *and* $X = Y$,

ii) $f_{A,X} \leq f_{B,Y}$ *iff* $B \subseteq A$ *and* $X \subseteq Y$,

iii) If f is an operator on S, then $f_{A,X} \leq f$ iff $X \subseteq f(A)$,

iv) If f is an operator on S, then $f = \vee\{f_{A,X} : f_{A,X} \leq f\}$,

v) If f is an operator on S, then:
$f(B) = C$ *iff* $f_{B,C\setminus B} \leq f$ *and* $(f_{B,C\setminus B} \leq f_{B,Y} \leq f \Rightarrow Y = C \setminus B)$.

Although we decided not to consider the monoid structure of operators in this paper, let us write explicitly the formula for the composition of operators from $\mathbf{F}_{irr}$. Recall that if f and g are transitive operators, then fg is transitive if $fg = gf$. Also if fg is transitive, then $f \vee g = fg$ [16].

Proposition 4.18. *Let $x \in S \setminus A$ and $y \in S \setminus B$. Then:*

$$\begin{aligned} f_{A,X} f_{B,Y}(C) &= C \cup \{x, y\} \text{ if } B \subseteq C \text{ and } A \subseteq C \cup \{y\}, \\ &= C \cup \{y\} \text{ if } B \subseteq C \text{ and } A \text{ is not included in } C \cup \{y\}, \\ &= C \cup \{x\} \text{ if } B \text{ is not included in } C \text{ and } A \subseteq C, \\ &= C \text{ if neither } B, \text{ nor } A \text{ is in } C. \end{aligned}$$

If $x, y \notin A$ and $x, y \notin B$, then: $f_{A,x} f_{B,y} = f_{B,y} f_{A,x} = f_{A,x} \vee f_{B,y}$.

Proposition 4.19. *Let $X \subseteq S \setminus (A \cup C)$. Then: $f_{A,X} f_{C,X} = f_{C,X} f_{A,X}$. Therefore $f_{A,X} f_{C,X}$ is transitive and $f_{A,X} \vee f_{C,X} = f_{A,X} f_{C,X}$.*

Proposition 4.20. *Let $X \cup Y \subseteq S \setminus (A \cup C)$. Then $f_{A,X} f_{C,Y} = f_{C,Y} f_{A,X} = f_{A,X} \vee f_{C,Y}$ and $f_{A,X} f_{C,Y}$ is transitive.*

Proof: In general when $X \subseteq S \setminus A$ and $Y \subseteq S \setminus C$:

$$\begin{aligned} f_{A,X} f_{C,Y}(B) &= B \cup X \cup Y \text{ if } A \subseteq B \text{ and } C \subseteq B \cup X, \\ &= B \cup X \text{ if } A \subseteq B \text{ and } C \text{is not included in } B \cup X, \\ &= B \cup Y \text{ if } A \text{ is not included in } B \text{ and } C \subseteq B, \\ &= B \text{ otherwise.} \end{aligned}$$

Hence when $X \cup Y \subseteq S \setminus (A \cup C)$:

$$\begin{aligned} f_{A,X} f_{C,Y}(B) &= B \cup X \cup Y \text{ if } A \cup C \subseteq B, \\ &= B \cup X \text{ if } A \subseteq B \text{ and } C \text{ is not included in } B, \\ &= B \cup Y \text{ if } A \text{ is not included in } B \text{ and } C \subseteq B, \\ &= B \text{ otherwise.} \end{aligned}$$

5. Cardinalities of classes of operators for infinite S

It happens very often that trivial problems for finite structures are very difficult for an infinite case, and simple proofs for infinite structures do not suggest how to prove or disprove extremely difficult finite versions of problems. The problem of finding the cardinality of classes of dependence systems of specified properties is a good example of the latter situation. First we will study the cardinalities of the basic classes of dependence systems on an infinite set S.

Let $|S| = m$, an infinite cardinal. Then certainly $|\mathbf{F}(S)| \leq 2^{2^m}$, as every operator is a function on the power set of S which has the cardinality 2^m. Sierpinski proves the following [14]: For every infinite set S of power m there exists a family $\mathcal{A}$ composed of 2^m subsets of the set S none of which is a subset of any other. The proof is based on the formula $2m = m$ for infinite cardinal numbers. From this theorem we can deduce [15]: The total number of different hereditary families of subsets of a given infinite set S of power m is 2^{2^m}.

Every hereditary family of subsets of a set S is closed with respect to arbitrary intersections, therefore the family C of families closed with respect to arbitrary intersections has its cardinality greater than or equal to 2^{2^m}, but the cardinality is bounded above by the cardinality of the set of all operators, so the cardinality of C is exactly 2^{2^m}.

Larson and Andima [8] quote the result of Frohlich [5] which states that if S is an infinite set of cardinality m, then $|\mathbf{INfA}(S)| = 2^{2^m}$.

This can be summarized as follows:

Proposition 5.1. *Let S be an infinite set of cardinality m. Then:*

i) $|fC(S)| = |IfC(S)| = 2^m$.

ii) $|F(S)| = |I(S)| = |E(S)| = |fA(S)| = |wI(S)| = |wE(S)| = 2^{2^m}$

Proof: i) We have $m = |S| = |\mathrm{Fin}(S)|$. Every operator f in $\mathbf{fC}(S)$ is determined by the action on the finite subsets of S. Therefore $|\mathbf{fC}|$ is less than the number of all relations between S and $\mathrm{Fin}(S)$ (f is determined by a relation $R \subseteq S \times \mathrm{Fin}(S)$ defined by: xRA *iff* $x \in f(A)$), i.e. $|\mathbf{fC}| \leq 2^{mm} = 2^m$. Certainly, $|\mathbf{IfC}(S)| \leq |\mathbf{fC}(S)|$ and we have the following example of a family of $\mathbf{IfC}(S)$ operators of cardinality greater than 2^m. Consider the family of the operators described in Prop. 4.17 defined by: $\{f_{A,X} : A \in \mathrm{Fin}(S)$ and $X \subseteq S \backslash A\}$. These operators are transitive and also they belong to $\mathbf{fC}(S)$ as if $y \in f_{A,X}(B)$, than for y in B we have $y \in f_{A,X}(\{y\})$, for $y \notin B$ we have $A \subseteq B$ and $y \in X$, so $y \in f_{A,X}(A)$. Certainly the cardinality of this family is greater or equal 2^m. So $2^m \leq |\mathbf{IfC}(S)| \leq |\mathbf{fC}(S)| \leq 2^m$. ii) Recall that $\mathbf{I}(S) \subseteq \mathbf{wI}(S)$, so $|\mathbf{I}(S)| \leq |\mathbf{wI}(S)|$, and $\mathbf{wE}(S) = \mathbf{wI}^*(S)$, so $|\mathbf{wE}(S)| = |\mathbf{wI}(S)|$. Also, by the duality $|\mathbf{I}(S)| = |\mathbf{E}(S)|$. Every operator in $\mathbf{I}(S)$ is determined by its set of closed subsets (family of subsets closed with respect to arbitrary intersections and containing S) so we have $2^{2^m} \leq |\mathbf{I}(S)| \leq |\mathbf{wI}(S)| \leq |\mathbf{F}(S)| \leq 2^{2^m}$. Also, $2^{2^m} = |\mathbf{INfA}(S)| \leq |\mathbf{fA}(S)| \leq |\mathbf{F}(S)|$. This concludes the proof.

6. Number of dependence systems on a set with n elements

The problem of enumeration of all dependence systems on a set with n elements is equivalent to the classical, almost one hundred year old problem of Dedekind: What is the number $D(n)$ of all antichains in the power set of an n-element set.

We will write $\mathbf{X}(n)$ for the cardinality of all operators on a set with n elements having the property $\mathbf{X}$. For instance $\mathbf{F}(n)$ is the number of all dependence systems on a set with n elements.

Proposition 6.1. *i)* $F(n) = [D(n-1)]^n$, *ii)* $N(n) = [D(n-1)-1]^n$.

$D(n)$ is the n-th Dedekind number.

Proof: First observe that for every x in S, $\mathbf{F}_x$ with the ordering inherited from $\mathbf{F}(S)$ is isomorphic to the Boolean algebra of all subsets of $S \setminus \{x\}$ ordered by inclusion (notice: $f_{A,x} \vee f_{B,x} \notin \mathbf{F}_x$ when join is in the lattice $\langle \mathbf{F}(S), \leq \rangle$; in $\langle \mathbf{F}_x, \leq \rangle$, $f_{A,x} \vee f_{B,x} = f_{C,x'}$, where $C = A \cap B$). Also $f_{A,x}$ and $f_{B,y}$ are incomparable when $x \neq y$. Therefore every antichain in $\mathbf{F}_{irr}$ consists of (possibly empty) antichains in particular $\mathbf{F}_x$'s. So we have in each $\mathbf{F}_x$ (Boolean algebra of subsets of $n-1$ elements) $D(n-1)$ antichains. Therefore in $\mathbf{F}_{irr}$ we have $[D(n-1)]^n$ antichains which together with Cor. 4.11 proves i). For ii) observe that f is normalized ($f(\emptyset) = \emptyset$} iff $\forall x \in S\ \forall A \subseteq S \setminus \{x\} : A = \emptyset \Rightarrow f_{A,x}$ is not less than f. Therefore counting the number of the normalized operators we have to exclude antichains containing operators $f_{A,x}$ for $A = \emptyset$. In each $\mathbf{F}_x$ there is only one such an antichain $\{f_{A,x} : A = \emptyset\}$. Therefore in each $\mathbf{F}_x$ we have $[D(n-1)-1]$ antichains. This concludes the proof.

There is only one class of operators for which we can give the explicit numerical value of its cardinality:

Proposition 6.2. *i)* $fA(n) = [2^{n-1}+1]^n$, *ii)* $NfA(n) = 2^{(n-1)n}$.

Proof: Every fA-operator is determined by the choice of its action on one-element sets. For NfA-operators we have for each of n one-element sets 2^{n-1} choices as the given element has to belong to the image of the operator on its one-element set). This gives us $(2^{n-1})^n$ choices of operators.

If we admit non-normalized operators, then we have to consider separately every choice of $f(\emptyset)$. Suppose $|f(\emptyset)| = k$. We have $\binom{n}{k}$ different choices of $f(\emptyset)$. Then for $n-k$ one-element subsets of $S \setminus f(\emptyset)$ we have 2 choices of images of the operator f (subsets of an $(n-k-1)$-element set because k elements of $f(\emptyset)$ already belong to the image and also the given one-element set is included in its image.) For k one-element subsets of $f(\emptyset)$ we have 2^{n-k} choices Therefore for given k we have $\binom{n}{k}2^{(n-k-1)(n-k)}2^{(n-k)k}$ different operators. So $fA(n) = \sum_{k=0}^{n}\binom{n}{k}2^{(n-k-1)(n-k)}2^{(n-k)k} = \sum_{k=0}^{n}\binom{n}{k}2^{(n-1)(n-k)} = \sum_{k=0}^{n}\binom{n}{k}[2^{(n-1)}]^{(n-k)} = [2^{n-1}+1]^n$.

Effective formulas have not been found for $D(n)$, nor for $\mathrm{I}(n)$, $\mathrm{INfA}(n)$ (number of topological spaces), $\mathrm{INfAt_0S}(N)$ (number of partially ordered sets), $\mathrm{IwEfC}(n)$, $\mathrm{IwEfCNt_1S}(n)$ (number of geometries). However, there is quite a rich literature on these issues [cf. 4, 7, 17 for references] and some revitalization of interests in enumeration and estimation of these numbers can be recently observed. The Dedekind numbers are known for $n \leq 8$, which gives us values of $\mathbf{F}(n)$ for $n \leq 9$. However, the formula expressing $\mathbf{F}(n)$ by $D(n)$ can be useful in proving properties of dependence systems on sets of arbitrary finite cardinality.

One of the earliest results about closure systems (transitive operators) states that there is a bijective correspondence between closure operators on a set S and families of subsets of S containing S and closed with respect to arbitrary intersections (Moore families). Certainly, we have also similar connection between closures and families of subsets containing the empty set and closed with respect to arbitrary unions. For matroids we have also a connection between families of independent subsets and closures (actually matroids are usually defined by a family of independent subsets, and topological spaces by a family of open sets). We can uniquely characterize a matroid by a family of bases, minimal non independent sets (circuits), etc. The old folklore problem was: Is it possible to characterize a dependence system by a family (or families) of subsets. The answer is 'no' for a finite set S with more than 3 elements. The proof follows from the following proposition.

Proposition 6.3. *For $n \geq 4$, 2^{2^n} is strictly less than $\mathbf{F}(n)$.*

Proof: Using values of $D(3) = 20$, $D(4) = 168$, $D(5) = 7581$, we get $\mathbf{F}(4) = 160000$, $\mathbf{F}(5) = 2^{15}3^{5}7^{5}$, $\mathbf{F}(6) = 3^{6}7^{6}19^{12}$. Certainly these numbers are greater than 65536, 2^{32}, 2^{64} respectively. Now we will show that for $n \geq 7$, 2^n is strictly less than $n\binom{n-1}{k}$ where $k = [(n-1)/2]$ (the greatest integer less than $(n-1)/2$). We have $2^7 = 128$ and $7\binom{6}{3} = 140$. To complete proof by induction we have to show that from the inequality above for n (≥ 7) follows: $2^{n+1} < (n+1)\binom{n}{[n/2]}$. We have $\binom{n+1}{k+1} = \binom{n}{k}(n+1)/(k+1)$ and $\binom{n+1}{k} = \binom{n}{k}(n+1)/(n+1-k)$. Now let n be even. Then $[n/2] = [(n-1)/2]+1$, and therefore $\binom{n}{[n/2]} = \binom{n-1}{[(n-1)/2]}n/[n/2] = 2\binom{n-1}{[(n-1)/2]}$, so we have $2^{n+1} < 2n\binom{n-1}{[(n-1)/2]} = n\binom{n}{[n/2]} < (n+1)\binom{n}{[n/2]}$. Now let n be odd. Then $[n/2] = [(n-1)/2]$, so $\binom{n}{[n/2]} = \binom{n-1}{[(n-1)/2]}n/[(n-1)/2] = (2-2/(n+1))\binom{n-1}{[(n-1)/2]}$, so that $2^{n+1} < 2n\binom{n-1}{[(n-1)/2]} = 2n/(2-2/(n+1))\binom{n}{[n/2]} = (n+1)\binom{n}{[n/2]}$. This concludes the induction part of the proof. By Sperner's Lemma we have $\log_2(\binom{n-1}{[(n-1)/2]}) < \log_2(D(n-1))$, and as we have shown $n < \log_2(n\binom{n-1}{[(n-1)/2]})$ for $n \geq 7$, so $2^{2^n} < \mathbf{F}(n) = (D(n-1))^n$.

As a corollary we get:

Theorem 6.4. *Let S be a finite set of cardinality $n \geq 4$. Then the number of all operators on S is greater than the number of all families of subsets of S, i.e. an operator can not be uniquely determined by a family of subsets of S. Moreover, if k is any fixed positive integer, then for S with big enough cardinality an operator can not be uniquely determined by k families of subsets of S.*

Proof: Only the second statement requires an explanation. For even n we have $2^n < n\binom{n-1}{[(n-1)/2]} - \binom{n-1}{[(n-1)/2]}$, so $2^{2^n} < (D(n-1))^n/m$ where

$m = \binom{n-1}{[(n-1)/2]}$. Therefore $2^{2^n}2^m < \mathrm{F}(n)$. Therefore the ratio of the number of operators to the number of all families of subsets is an increasing function of n. $\mathrm{F}(n)$ is a monotone function of n, hence the conclusion is valid also for big enough odd n.

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References

[1] G. Artico, R. Moresco, *On the structure of fuzzes*, Math. Proc. Camb. Phil. Soc. **104** (1988), 235–251.

[2] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. 25, third ed., American Mathematical Society, Providence, (1967).

[3] J. A. Bondy, D. J. A. Welsh, *Some results on transversal matroids and constructions for identically self-dual matroids*, Quart. J. Math. Oxford (2) **22** (1971) 435–451.

[4] M. Erné, K. Stege, *Counting finite posets and topologies*, Order **8** (1991) 247–265.

[5] O. Fröhlich, *Das Halbordnungssystem der topologischen Raume auf einer Menge*, Math. Ann. **156** (1964) 79–95.

[6] D. A. Higgs Matroids and duality, Colloq. Math. **20** (1969) 215–220.

[7] A. Kisielewicz, *A solution of Dedekind's problem on the number of isotone Boolean functions*, J. reine angew. Math. **386** (1988) 139–144.

[8] R. E. Larson, S. J. Andima, *The lattice of topologies: A survey*, Rocky Mountains J. Math. **5** (1975) 177–198.

[9] M. J. Schroeder, *Dependence systems*, Doctoral dissertation, Southern Illinois University at Carbondale, (1991).

[10] M. J. Schroeder, *Dependence systems with the operator-image exchange property*, to appear in Discrete Mathematics.

[11] M. J. Schroeder, *Cryptomorphic characterizations of dependence systems*, submitted.

[12] M. J. Schroeder, *Representations of poset dependence systems*, submitted.

[13] M. J. Schroeder, *Dependence systems: mappings, congruences, constructions*, submitted.

[14] W. Sierpinski, *Cardinal and ordinal numbers*, Monografie Matematyczne PWN, Warszawa (1958).

[15] A. Tarski, *Sur les classes d'ensembles closes par rapport à certaines opérations élémentaires*, Fund. Math. **16** (1930) 181–304.

[16] M. Ward, *The closure operators of a lattice*, Ann. of Math. **43** (1942) 191–196.

[17] D. Wiedemann, *A computation of the eighth Dedekind number*, Order **8** (1991) 5–6.