Balanced Ouchterlony Neighbour Designs and Quasi Rees Neighbour Designs

D. A. Preece

Institute of Mathematics and Statistics, Cornwallis Building
The University, Canterbury, Kent CT2 7NF, U.K.

Abstract. This paper concerns neighbour designs in which the elements of each block are arranged on the circumference of a circle. Most of the designs considered comprise a general class of balanced Ouchterlony neighbour designs which include the balanced circuit designs of Rosa and Huang [30], the neighbour designs of Rees [29], and the more general neighbour designs of Hwang and Lin [13]. The class of Rees neighbour designs includes schemes given in 1892 by Lucas [22] for round dances. Isomorphism, species and adjugate set are defined for balanced Ouchterlony neighbour designs, and some seemingly new methods of constructing such designs are presented. A new class of quasi Rees neighbour designs is defined to cover a situation where Rees neighbour designs cannot exist but where a next best thing may be needed by experimental scientists. Even-handed quasi Rees neighbour designs and even-handed balanced Ouchterlony neighbour designs are defined too, the latter being closely related to serially balanced sequences. This paper does not provide a complete survey of known results, but aims to give the flavour of the subject and to indicate many openings for further research.

1. Definition of a balanced Ouchterlony Neighbour Design

We define a *balanced Ouchterlony neighbour design* (BOND) to be an arrangement where the members of a set S of v distinct elements are disposed in b blocks so that

(i) each block contains k elements ($k > 2$) that are drawn from S but are not necessarily all distinct;
(ii) the elements in each block are arranged on the circumference of a circle so that each of these elements has 2 neighbours;
(iii) each member of S appears exactly r times throughout the arrangement;
(iv) no element from S ever has itself as a neighbour;
(v) every element from S has each other element of S as a neighbour exactly λ' times throughout the arrangement.

The parameters v, r, b, k, λ' of a BOND clearly satisfy the equations

$$vr = bk \tag{1.1}$$

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and

$$\lambda' = 2r/(v-1). \tag{1.2}$$

In general, the parameter k may be less than, equal to, or greater than v; the upper limit for k is $vr = bk$, which is attained for a BOND with just one block. If $k = v$, each block may contain each member of S exactly once, or may have some element(s) of S repeated (i.e. present twice or more than twice) and therefore some other element(s) of S absent. Even if $k < v$, a block of a BOND may have some element(s) of S repeated.

A BOND with $(v, b, \lambda') = (5, 2, 1)$, and so with $k = v$, is as follows:

$$\begin{array}{ccccc} & & 1 & & \\ 5 & & & & 2 \\ & 4 & & 3 & \end{array} \qquad\qquad \begin{array}{ccccc} & & 1 & & \\ 4 & & & & 3 \\ & 2 & & 5 & \end{array} \tag{1.3}$$

whereas a BOND with $(v, b, \lambda') = (5, 1, 1)$, and so with $k = vr$, is the following:

$$\begin{array}{ccccc} & & 1 & & \\ & 4 & & 2 & \\ 2 & & & & 3 \\ 5 & & & & 4 \\ & 3 & & 5 & \\ & & 1 & & \end{array} \tag{1.4}$$

In an obvious linear notation, these two designs can be written respectively as

$$(1,2,3,4,5) \quad (1,3,5,2,4) \tag{1.5}$$

and

$$(1,2,3,4,5,1,3,5,2,4), \tag{1.6}$$

each block being read clockwise. A BOND with $(v, b, \lambda') = (7, 7, 1)$, and so with $k < v$, is the following:

$$(2,3,5)(3,4,6)(4,5,7)(5,6,1)(6,7,2)(7,1,3)(1,2,4). \tag{1.7}$$

The examples (1.5), (1.6) and (1.7) were given by Hwang and Lin [13, p. 304].

Equation (1.2) for a BOND can be rearranged as

$$v = 1 + 2r/\lambda'. \tag{1.8}$$

Thus, if λ' is odd, so is v. In particular, if $\lambda' = 1$, then v is odd and

$$r = (v-1)/2, \qquad b = vr/k \quad = v(v-1)/2k. \tag{1.9}$$

If $\lambda' = 1$ and $k = 3$, then $b = v(v-1)/6$, which restricts v to the values $7, 9, 13, 15, 19, 21, \ldots$; indeed a BOND with $k = 3$ and $\lambda' = 1$ is merely a Steiner triple system with the elements of each block arranged on the circumference of a circle.

If a Steiner triple system contains the block (a, b, c), this block can be used to produce either the block (a, b, c) or the block (a, c, b) in a corresponding BOND. Therefore, unlike some previous authors, we do not regard a Steiner triple system as being the same as a BOND with $k = 3$ and $\lambda' = 1$. Indeed, we regard the *circumference of a circle*, which features in condition (ii) of our definition of a BOND, as having a *direction* that can be reversed by reading anticlockwise (*backward*) instead of clockwise (*forward*). [We nevertheless regard the orientation of a block as immaterial; i.e. a block is unchanged by rotation; e.g. the block (a, b, c) is the same as the block (b, c, a). For convenience of exposition, we regard the elements of each block as being positioned at equidistant points on the circumference of a circle.] A practical motivation for taking account of the direction of each block is discussed in Section 2 below.

Modern study of BOND's stems from a 1967 paper by Rees [29], who gave examples of and constructions for BOND's with $\lambda' = 1$; for these designs he suggested the name *neighbour designs* on his p. 787, but the present paper refers to BOND's with $\lambda' = 1$ as *Rees neighbour designs* (RND's). Subsequent to Rees's paper, Lawless [19], Hwang [10], Dey and Chakravarty [8], Hwang and Lin [12, 13, 14], Nair [25], and Chandak [5] used the name *neighbour designs* more generally to cover designs with $\lambda' > 1$ too. Nowadays, though, the mathematical and statistical literature contains so many types of design having neighbour properties of one sort or another, perhaps in a linear sense or in two dimensions (see Street and Street [33, Chapter 14]), that a distinctive name is needed for designs where neighbours, defined as above, are on the circumferences of circles; we therefore now adopt the terminology *balanced Ouchterlony neighbour designs* as above, to reflect the practical background that gave rise to the designs (see Section 2 below).

Street and Street [33, Section 14.2] used the name *circular block design* for a BOND, but we recommend allowing this name its natural and much more general meaning, to cover (a) designs such as the quasi Rees neighbour designs of Section 8 below, which do not satisfy condition (v) of the definition of a BOND; (b) designs that do not satisfy the equal-replication condition (iii) of the BOND definition; and (c) designs where the number of elements on the circumference can differ from one block to another. Furthermore, but much less importantly, the name *circular block design* invites confusion with the *circular* repeated measurements designs of Magda [23] and Kunert [18], described as *designs balanced for circular residual effects* despite their circularity being artificial.

Hwang and Lin [13] and Street and Street [33, Section 14.2] permitted $k = 2$ in BOND's, but inclusion of this degenerate possibility leads to unnecessary complications that we prefer to avoid. Hwang and Lin [13, 14] and others used the notation λ , not λ' , for the number of times that any two members of S are neighbours in a BOND, but we prefer to follow Lawless [19], Street [32] and Street and Street [33, Section 14.2] in using λ' , thereby avoiding confusion with the parameter λ of a balanced incomplete block design (BIBD).

Rosa and Huang [30], Hwang and Lin [13], and Bermond, Huang and Sotteau [4] used the term *balanced circuit design* for a BOND in which no block has a repeated element. Thus $k \leq v$ for a balanced circuit design. We therefore use the terminology *balanced circuit RND* for an RND that is also a balanced circuit design. In the terminology of Lindner, Phelps and Rodger [20], Adams, Billington and Lindner [1] and others, a balanced circuit RND with parameters v and k is a *k-cycle system of order* v.

Keedwell [16] considered *2-fold perfect circuit designs*, these being balanced circuit designs whose neighbour properties apply not only to immediate neighbours but also to neighbours that are 2 places apart. Likewise, Lindner, Phelps and Rodger [20] considered *2-perfect k-cycle systems of order* v, i.e. balanced circuit RND's whose neighbour properties hold both for immediate neighbours and for 2-places-apart neighbours. However, 2-perfect designs, and indeed the more general i-perfect designs with $i > 2$ (see, for example, Adams, Billington and Lindner [1]), do not concern us in this paper, except in passing. Nor do we concern ourselves with the neighbour designs of Colbourn, Lindner and Rodger [7], where each block has $k + 1$ treatments, of which k are arranged round the circumference whilst the remaining one is at the centre of the circle.

2. The Ouchterlony Double Diffusion Test

In laboratory research on viruses, *the Ouchterlony method, Ouchterlony gel diffusion test*, or *Ouchterlony double diffusion test* uses plates or Petri dishes containing a suitable medium (e.g. an agar gel) through which antigens (e.g. viruses) and antiserum can diffuse. On each plate, antigens are arranged at equidistant points on the circumference of a circle centered on the point where an antiserum has been placed; the experimenter subsequently observes what occurs at the meeting points of the diffusion areas of the antiserum and two adjacent (neighbouring) antigens. For a particular set of antigens and a single antiserum, a complete story will emerge only if each antigen has each other antigen as a neighbour. This provides the practical motivation for finding BOND's, particularly those with $k = 5$ or 6, each of these numbers being appropriate for the number of antigens per plate.

As the Ouchterlony test is not run as an *experiment* in the usual statistical sense of that term, there may be no strong need to *replicate* each pairing of two antigens as neighbours, so a design with $\lambda' = 1$ may well be judged to be adequate. Some of the standard statistical arguments for some *randomisation* of the design may nevertheless still be relevant. The randomisation procedures to be considered for BOND's will not be discussed in detail here but are as follows:

(i) For the required v, k and λ', select a BOND at random from a set Z of

BOND's having these parameter-values.

(ii) Take the blocks of the selected BOND in random order.

(iii) For each block of the selected BOND, select an orientation at random.

(iv) Allocate the antigens at random to the elements of S.

Procedure (i) provides a motivation for enumeration of BOND's for any given set of values (v, k, λ'). Some or all of the members of the set Z might differ from one another merely in the directions of some of the blocks, so we see that procedure (i) can be equivalent to taking at random the direction of each block of a BOND. Such randomisation might be thought desirable if the experimental technique were to require every block to be set up in a systematic clockwise (or anticlockwise) way that might introduce a temporal/directional effect on the experimental material. Depending on Z, the above procedures (i) to (iv) are equivalent to or wider than the procedures (i) to (iv) given by Azaïs [3] for a radically different practical background from that considered in the present paper.

The Ouchterlony method, as outlined above, is named after Professor Örjan Ouchterlony, of the Department of Bacteriology, University of Gothenburg [Göteborg], Sweden, but is only one of several methods that he described and discussed. Full background and details were given by Ouchterlony [26, 27, 28].

In practice, the "antigens" in a particular Ouchterlony gel diffusion test might include successively weaker dilutions of a bacteriological preparation; the BOND's of the present paper are not suited for use in such circumstances.

3. A "Round-dance" problem, including the "Ramsgate Sands" Problem

A general request for a balanced circuit RND with $v = k$ was made in 1892 by Lucas [22, pp. 162-166], as part of his *Sixième récréation* (Tôme II) entitled *Les Jeux de Demoiselles* [*Games for Young Ladies*]. As the writings of Édouard Lucas have been much overlooked in English-speaking lands, his own statement of the problem is worth quoting here:

> LES RONDES ENFANTINES [CHILDREN'S ROUND-DANCES]
>
> Des enfants dansent en rond en se tenant par la main; on demande comment il faut disposer les enfants, de telle sorte que chacun d'eux se trouve successivement voisin de tous les autres, soit à droite, soit à gauche, mais ne puisse l'être qu'une seule fois.

Lucas, followed by Kraitchik [17, pp. 227–229] and by Sainte-Laguë [31, pp. 163–168), gave a solution that is equivalent to Method B of Section 7 below; the solution is for any odd number of children. Lucas [22, p. 162] indicated that his exposition included *les solutions simples et ingénieuses de M[onsieur] Walecki*, but whether the Lucas solution to *les Rondes Enfantines* is due to Walecki is not clear from Lucas's text.

Dudeney [9, pp. 147 and 242, Miscellaneous Puzzle 100], seemingly inspired by Lucas, posed a special case of Lucas's problem, with the title "On Ramsgate Sands":

> Thirteen youngsters were seen dancing in a ring on the Ramsgate sands. ... How many rings may they form without any child taking twice the hand of any other child — right hand or left? That is, no child may ever have a second time the same neighbour.

This is a request for a balanced circuit RND with $v = k = 13$. Dudeney gave the solution obtained by the Method A described in Section 7 below.

4. Isomorphism of balanced Ouchterlony Neighbour Designs

The literature to date seems to contain no definition of isomorphism of BOND's. We therefore repair the omission, to provide a basis for enumeration of non-isomorphic BOND's for any particular set of values (v, b, λ'). Other new definitions for BOND's are given in Sections 5 and 6 below.

In this paper, two BOND's with the same set of values (v, b, λ') are said to be *isomorphic* to one another if either can be obtained from the other by a succession of the following operations:

(i) Permutation of the v distinct elements;
(ii) Permutation of the b blocks;
(iii) Rotation of a block through an angle that is a multiple of $2\pi/k$ radians, different angles being used, if necessary, for different blocks.

If such a succession of operations maps a BOND onto itself, the succession is an *automorphism* of the BOND. The automorphisms of a BOND clearly constitute a group in the mathematical sense.

5. Adjugate sets of non-isomorphic balanced Ouchterlony Neighbour Designs

The *reflection* of a block $(a, b, c, \ldots, y, z)$ of a BOND is the block $(a, z, y, \ldots, c, b)$. Here, *reflection* is used in the sense of a *mirror-image*, otainable by reading the original block anti-clockwise (backward) instead of clockwise (forward).

A BOND D1 that has b blocks is a member of a set C of 2^b BOND's, each obtained by copying the blocks of D1 one by one except that each copied block may be replaced by its reflection. For example, the BOND (1.5) is the first member of the following such set of 4 BOND's:

$$\text{D1}: \quad (1,2,3,4,5) \quad (1,3,5,2,4) \tag{5.1}$$

$$\text{D2}: \quad (1,2,3,4,5) \quad (1,4,2,5,3) \tag{5.2}$$

$$\text{D3}: \quad (1,5,4,3,2) \quad (1,3,5,2,4) \tag{5.3}$$

$$\text{D4}: \quad (1,5,4,3,2) \quad (1,4,2,5,3) \tag{5.4}$$

In general, the members of C might all be isomorphic to one another, or none of them might be isomorphic to any other, or some (but not all) might be isomorphic to others. (In the above example, D1, D2, D3 and D4 are all isomorphic to one another.) Thus the set C contains a subset C^* whose cardinality c^* satisfies $1 \leq c^* \leq 2^b$ and whose members are such that none is isomorphic to any other, whilst any member of $C - C^*$ is isomorphic to some member of C^*. In vocabulary taken from the theory of Latin squares, two distinct members of C^* are *adjugates* of one another, and the members of C constitute a *species* of BOND's. Any particular member of C, and all other members of C that are isomorphic to it, constitute an *adjugate set* of BOND's. Thus a species comprises c^* adjugate sets. When the randomisation procedure (i) of Section 2 above is used, it might well be appropriate for the set Z to comprise one representative of each adjugate set for each species having the required set of values (v, k, λ') .

6. Forming BOND's by Concatenation, Intramutation and Intermutation

We now describe a method that permits us, in certain circumstances, to use a BOND with block size k to construct a BOND with block size mk , where m is an integer greater than 1.

Suppose a BOND with block size k contains the blocks

$$\left.\begin{array}{c}(a, b, c, \ldots, f, g, h, \ldots, y, z) \\ (A, B, C, \ldots, F, g, H, \ldots, Y, Z)\end{array}\right\} \quad (6.1)$$

that have the common element g. Then the following block, of size $2k$, has the same pairs of neighbouring elements as do, jointly, the two blocks (6.1):

$$(a, b, c, \ldots, f, g, H, \ldots, Y, Z, A, B, C, \ldots, F, g, h, \ldots, y, z) \quad (6.2)$$

Adopting vocabulary of Street and Street [33, Chapter 14], we say that the block (6.2) has been formed by *concatenation* from the blocks (6.1), the second of the blocks (6.1) having been *concatenated* within the first. Similarly, any further block of size k that contains an element that is present in block (6.2) could be concatenated within the block (6.2) without changing the overall pairings of neighbours.

Suppose now that we have a BOND D0 with parameters (v, b, λ') such that $b = mb^*$, where m is an integer greater than 1 and b^* is an integer greater than or equal to 1. Then a BOND with parameters (v, b^*, λ') can be constructed by concatenation so long as the blocks of D0 can be grouped into b^* non-overlapping sets each containing m blocks that can be concatenated into one another. Trivially, the BOND (1.4) is obtainable in this way when the BOND (1.3) is used as D0. More generally, this method of construction is available if D0 has the *handcuffing* property (more concisely, property H), namely that the blocks can be arranged in a sequence such that blocks i and $i+1$ (where $i = 1, 2, \ldots, b-1$) have at least one element in common (see Street and Street [33, p. 321]). Street and Street [33, pp.

322–324] gave existence theorems for two classes of BOND's that have property H.

Whenever a block is concatenated as just described, its reflection could have been concatenated instead. Likewise, the concatenating block could have been replaced by its own reflection.

These comments lead naturally to a more general result, namely this: If an element x is repeated within a block of a BOND, as in

$$(\ldots, d, e, x, g, h, \ldots, p, q, x, s, t, \ldots),$$

the design remains a BOND if the ordering of the elements between the two occurrences of x is reversed, as in

$$(\ldots, d, e, x, q, p, \ldots, h, g, x, s, t, \ldots)$$

or

$$(\ldots, t, s, x, g, h, \ldots, p, q, x, e, d, \ldots).$$

We call such a reversal an *intramutation.* As a BOND obtained by means of an intramutation may well not be isomorphic to the starter design, intramutation provides a method of generating designs where at least one block has at least one repeated element.

A further method of generating additional BOND's is available when we have a BOND where a block contains the sequence

$$(\ldots, x, a_1, a_2, \ldots, a_n, y, \ldots)$$

and another block, or a separate (i.e. non-overlapping) section of the same block, contains the sequence

$$(\ldots, x, b_1, b_2, \ldots, b_n, y, \ldots).$$

Then, if the entry a_i is swopped with the entry b_i for all $i = 1, 2, \ldots, n$, the resultant design will still be a BOND. Likewise, swapping is possible, with reversal of the order of the swapped elements, if the 2 initial sequences are

$$(\ldots, x, a_1, a_2, \ldots, a_n, y, \ldots)$$

and

$$(\ldots, y, b_n, b_{n-1}, \ldots, b_1, x, \ldots).$$

We use the term *intermutation* for either of these processes of swapping.

Sometimes, when sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_1, \ldots, b_n$ are found to lie between the elements x_1 and y_1 in blocks B1 and B2 respectively, they are also found to lie between the elements x_2 and y_2 in blocks B2 and B1 respectively.

Then, after the intermutations for both (x_1, y_1) and (x_2, y_2) have been made, each of blocks $B1$ and $B2$ will contain the same elements as at the start. An example with $n = 1$ is provided by the balanced circuit RND (7.5) given in Section 7 below, where the elements 4 and 8 lie between 3 and 5 in blocks 1 and 3 respectively, and between 7 and 9 in blocks 3 and 1 respectively; after both intermutations have been made, the design remains a balanced circuit RND. Making both such intermutations is akin to *intercalate reversal* in Latin squares, i.e. to interchanging the two symbols of a 2×2 Latin square embedded in a larger Latin square.

7. Rees Neighbour Designs

Table 1 lists parameter sets for RND's with $v \leq 15$, $k \leq 15$, and indicates how at least one RND can be obtained for each parameter set. This Section discusses some of these RND's. We do not discuss the balanced circuit RND's in full detail, but refer readers instead to the review papers of Alspach, Bermond and Sotteau [2] and Lindner and Rodger [21].

We first present general methods A and B that produce balanced circuit RND's with $v = k$.

Method A for $v = k =$ an odd prime (see Sainte-Laguë [31, p. 176], Rees [29, p.782] and Street [32, p.122]):

The blocks are as follows, except that every entry greater than v must be reduced modulo v to give an entry from the set $S = \{1, 2, \ldots, v\}$:

$$\begin{array}{lcccr}
(1, & 2, & 3, & \ldots, & v\ ) \\
(1, & 3, & 5, & \ldots, & 2v-1\ ) \\
(1, & 4, & 7, & \ldots, & 3v-2\ ) \\
\vdots & & & & \\
(1, & (v+1)/2, & v, & \ldots, & 1+(v-1)^2/2\ )
\end{array}$$

Method B for $v = k =$ any odd integer (see Rees [29, p.783] and Street [32, p.122]):

The blocks are as follows:

$$\begin{array}{lcccccccr}
(v, & 1, & 2, & v-1, & 3, & v-2, & & \ldots, & \frac{(v+1)}{2}\ ) \\
(v, & 2, & 3, & 1, & 4, & v-1, & & \ldots, & \frac{(v+3)}{2}\ ) \\
(v, & 3, & 4, & 2, & 5, & 1, & & \ldots, & \frac{(v+5)}{2}\ ) \\
\vdots & & & & & & & & \\
(v, & \frac{(v-1)}{2}, & \frac{(v+1)}{2}, & \frac{(v-3)}{2}, & \frac{(v+3)}{2}, & \frac{(v-5)}{2}, & & \ldots, & v-1\ )
\end{array}$$

Sainte-Laguë [31, p. 176] presented Method A geometrically as the *regular polygons solution*. The *Compass-needle Method* of Lucas [22, pp. 162–164] and of Sainte-Laguë [31, pp. 163–168 and 174–177] is merely a geometrical way of presenting Method B. Also equivalent to Method B is the construction given by Kraitchik [17, pp. 227–229], with

$$a_0 = v, \quad a_1 = 1, \quad a_{2i} = i + 1 \ (i = 1, 2, \ldots, (v-1)/2),$$

$$a_{2i+1} = v - i \ (i = 1, 2, \ldots, (v-3)/2).$$

Method B is also the basis of a method for constructing *serially balanced sequences* (see Street and Street [33, p. 331]). With its first element omitted, the first block from Method B is the basis of a construction given by Williams [34, p. 152] for a row-complete Latin square of order $(v-1)$, with v odd.

Any RND with $v = k = 5$ must be a balanced circuit design. It follows almost immediately that, for $v = k = 5$, all RND's are isomorphic to (5.1). For $v = k = 5$, Method A gives (5.1) immediately, whereas Method B can quickly be shown to give an RND isomorphic to (5.1). The isomorphism here between the outcomes of methods A and B arises merely because v is so small. For $v = k = 7$, the RND's obtained by Methods A and B are, respectively,

$$(1,2,3,4,5,6,7) \quad (1,3,5,7,2,4,6) \quad (1,4,7,3,6,2,5) \qquad (7.1)$$

and

$$(7,1,2,6,3,5,4) \quad (7,2,3,1,4,6,5) \quad (7,3,4,2,5,1,6) \qquad (7.2)$$

If, starting in any position, we take the successive elements of any block of (7.1), these are seen to be the pth, $(p+i)$th, $(p+2i)$th, ... elements of any other block, for some p and i, with reduction modulo 7; this is not true of (7.2), so the RND's (7.1) and (7.2) are from different species. Indeed, for $v = k =$ any odd prime greater than 5, the RND's obtained by Methods A and B come from different species; this can readily be shown by examining 2-places-apart neighbours, as the RND's obtained by Method A are 2-perfect whereas those from Method B are not.

For $v = k = 7$, there are, apart from the balanced circuit RND's (7.1) and (7.2), the following other RND's where, to illustrate such structure as the designs possess, symbols other than $1, 2, \ldots, 7$ have been used for the set S:

$$(I, x, y, c, b, y, a) \quad (I, y, z, a, c, z, b) \quad (I, z, x, b, a, x, c) \qquad (7.3)$$

and

$$(3,4,5,x,y,1,2) \quad (5,1,x,4,2,x,3) \quad (1,4,y,2,5,y,3) \qquad (7.4)$$

In (7.3), each block has a duplicated element (respectively y, z and x), and each block can be obtained from either of the others by use of the cyclic permutation

$(abc)(xyz)$ once or twice. In (7.4), the second and third blocks each have a duplicated element (respectively x and y), but the first block does not.

For each of the parameter sets $v = k = 9$ and $v = k = 11$, Sainte-Laguë [31, p. 167] gave a second balanced circuit RND, namely

$$(1,2,3,4,5,6,7,8,9) \quad (1,7,3,5,9,2,4,6,8)$$
$$(1,3,8,5,2,7,4,9,6) \quad (1,4,8,2,6,3,9,7,5) \tag{7.5}$$

and

$$(1,2,3,4,5,6,7,8,9,10,11) \quad (1,6,4,7,10,3,5,11,9,2,8)$$
$$(1,3,7,11,8,4,9,5,2,6,10) \quad (1,4,10,5,7,2,11,6,8,3,9)$$
$$(1,5,8,10,2,4,11,3,6,9,7) \tag{7.6}$$

Sainte-Laguë claimed that each of (7.5) and (7.6) comes from a different *system* from that of the corresponding balanced circuit RND obtainable by Method B, but his exposition falls short of a formal proof that he was using *system* in the sense of *species*. In fact, examination of 2-places-apart neighbours readily shows that, for $v = k = 9$, design (7.5) and a Method B design come from different species, and for $v = k = 11$, the design (7.6), a Method A design and a Method B design all come from different species. For the parameter set $v = k = 13$, Sainte-Laguë [31, p. 177) similarly gave an example of a third *system* (i.e. other than the systems obtainable by Methods A and B):

$$(1,2,3,4,5,6,7,8,9,10,11,12,13)(1,4,9,12,2,8,13,10,5,7,3,6,11)$$
$$(1,8,3,5,9,7,4,11,13,2,6,10,12)(1,6,4,13,5,12,3,9,2,11,8,10,7)$$
$$(1,5,11,7,12,6,8,4,2,10,3,13,9)(1,3,11,9,6,13,7,2,5,8,12,4,10) \tag{7.7}$$

Again, examination of 2-places-apart neighbours establishes that the 3 systems are indeed 3 species.

For $v = k = 7$ and 9, Colbourn [6] showed that there are, respectively, 2 and 122 species of balanced circuit RND's. For $v = k = 11$, he showed that there are 3140 such species that have non-trivial automorphism groups.

For $v = 5$, $k = 10$, we have the RND (1.6), obtained by concatenation from the RND (1.5):

$$(1,2,3,4,5,1,3,5,2,4). \tag{7.8}$$

Applying the permutation $(25)(34)$ to its elements gives

$$(1,5,4,3,2,1,4,2,5,3); \tag{7.9}$$

rotating this through π radians gives

$$(1,4,2,5,3,1,5,4,3,2), \tag{7.10}$$

i.e. the reflection of (7.8), which can thus be described as *self-adjugate*. Another self-adjugate RND with $v = 5$, $k = 10$ and obtained by intramutation from (7.8) is

$$(1,2,3,4,5,3,1,5,2,4); \qquad (7.11)$$

this RND, however, has no element that is in positions p and $p + 5$, for some p, and so it cannot be obtained by concatenation from an RND having two blocks of equal size.

For an RND with $k = 3$, the parameter v must, as shown in Section 1 above, take one of the values 7, 9, 13, 15 within the range of Table 1. To write down an RND with $k = 3$, we merely need a Steiner triple system (BIBD with $k = 3$, $\lambda = 1$) with the same values of v, r and b; the blocks of the BIBD can be used for the blocks of the RND, each block (x, y, z) of the BIBD giving rise to either (x, y, z) or (x, z, y) in the RND. An RND with $v = 7$, $k = 3$ is the following, already given as (1.7):

$$(2,3,5)\ (3,4,6)\ (4,5,7)\ (5,6,1)\ (6,7,2)\ (7,1,3)\ (1,2,4). \qquad (7.12)$$

Applying the permutation $(235)(764)$ to the elements of (7.12), rotating each block through $2\pi/3$ radians, and appropriately reordering the blocks gives us (7.12) back again; so too, of course, does the permutation (1234567) and a re-ordering of the blocks. Thus the RND, like the corresponding BIBD, is rich in automorphisms. So too is the adjugate RND obtained from (7.12) by taking the reflection of every block. However, other adjugates of (7.12), obtained by reflecting some blocks but not others, lack these automorphisms.

The RND (7.12) is one of many RND's that can be developed cyclically from one or more initial blocks. For such RND's we henceforth use an obvious succinct notation that is used also for cyclically developed BIBD's; in this notation, (7.12) is written

$$(2,3,5) \text{ modulo } 7. \qquad (7.13)$$

The blocks of cyclically developed RND's may or may not come from the blocks of a BIBD. Thus, for example, there is no BIBD with $v = b = 9$, $r = k = 4$, yet we have Rees's own RND for these parameters, as follows:

$$(1,4,9,8) \text{ modulo } 9. \qquad (7.14)$$

Likewise there is no BIBD for $v = 15$, $r = 7$, $k = 5$, $b = 21$, yet Rees [29] and Hwang and Lin [13, p. 307] were able to give an RND for these parameters as follows:

$$\begin{aligned} &(1,2,15,4,9) \text{ modulo } 15 \\ &(1,4,7,10,13) \text{ modulo } 15 \text{ (partial cycle, of length 3)} \\ &(1,7,13,4,10) \text{ modulo } 15 \text{ (partial cycle, of length 3)} \end{aligned} \qquad (7.15)$$

For $v = b = 11, r = k = 5$, Rees [29] and Street [32, p. 127] gave the following 2-perfect RND, whose blocks come from those of a BIBD:

$$(1,4,5,9,3) \text{ modulo } 11. \tag{7.16}$$

Another RND with the same parameters is

$$(1,4,8,9,3) \text{ modulo } 11, \tag{7.17}$$

which is not 2-perfect nor do its blocks come from those of a BIBD. Thus (7.16) and (7.17) come from different species.

Comparison between (7.16) and (7.17) shows how one cyclically developed RND can be constructed from another. For this, we copy Rees [29], Street [32] and Street and Street [33, p. 322] by defining the *forward differences* of an initial block to be the quantities

$$e_{i+1} - e_i \text{ modulo } n, \qquad i = 1,2,\ldots,k$$

where e_i is the i'th element of the block ($e_0 = e_k$) and where the cyclic development is done modulo n (the integer n not necessarily being equal to v); we similarly define the *backward differences* to be

$$e_i - e_{i+1} \text{ modulo } n, \qquad i = 1,2,\ldots,k.$$

For an RND having just a single initial block that is developed with $n = v$, the following must be satisfied:

(a) $k = r - (v-1)/2$;
(b) the $(v-1)$ forward and backward differences must together constitute the set $\{1,2,\ldots,v-1\}$;
(c) the forward differences (and therefore the backward ones too) must sum to v modulo v.

For (7.16) the differences are as follows:

forward:	3	1	4	5	9	
backward:	8	10	7	6	2	(7.18)

For (7.17), the forward differences (and therefore the backward ones too) are the same except in their ordering:

forward:	3	4	1	5	9	
backward:	8	7	10	6	2	(7.19)

Other orderings are possible, for example the ordering

forward: 1 9 4 5 3

obtained when the general *half cycle* solution given by Azaïs [3, p. 338] for $v = 2k + 1 = 3$ (modulo 8) is used for $v = 11$; however, checking all the corresponding RND's for isomorphism or non-isomorphism would need care. A change of ordering of the forward differences in the first initial block of (7.15) gives the alternative initial block $(1,2,6,11,3)$, also produced by a construction of Hwang and Lin [14, p. 114]. A change of ordering of the forward differences in the reflection of the initial block of (7.14) gives the alternative initial block $(1,2,8,3)$, obtained when the general Azaïs solution for $v = 2k + 1 = 1$ (modulo 8) is used for $v = 9$.

For RND's with $v = b, r = k = (v-1)/2$, the scope for obtaining designs increases as v increases. We make no attempt here to give an exhaustive account of possibilities; we merely give examples within the range of Table 1. For $v = b = 13, r = k = 6$, Rees [29] and Lindner, Phelps and Rodger [20] respectively gave the non-isomorphic RND's

$$(1,11,2,7,13,12) \text{ modulo } 13 \tag{7.20}$$

and

$$(5,2,8,7,9,13) \text{ modulo } 13 \tag{7.21}$$

- the second of these being 2-perfect, but not the first - whereas the RND

$$(1,2,5,1,3,8) \text{ modulo } 13 \tag{7.22}$$

can readily be obtained by concatenation from an RND with $v = 13$, $k = 3$. A change of the ordering of the forward differences in the reflection of the initial block of (7.21) gives the alternative initial block $(1,2,13,3,11,5)$ obtained when the general solution given by Azaïs [3, p. 338] for $v = 2k + 1 = 5$ (modulo 8) is used for $v = 13$. For $v = b = 15, r = k = 7$, Rees [29] gave the RND

$$(1,3,6,11,5,9,2) \text{ modulo } 15, \tag{7.23}$$

whose blocks come from those of a BIBD but which is not 2-perfect, whereas the 2-perfect balanced circuit RND

$$(1,4,10,8,13,6,5) \text{ modulo } 15 \tag{7.24}$$

given by Manduchi [24, p. 106] has blocks that do not come from those of a BIBD, and the seemingly unrelated balanced circuit RND

$$(1,2,15,4,14,5,13) \text{ modulo } 15 \tag{7.25}$$

obtained from the general Azaïs solution for $v = 2k + 1 = 7$ (modulo 8) is not 2-perfect nor do its blocks come from a BIBD. Two further RND's for this set of parameters are

$$(1,2,4,1,5,15,9) \text{ modulo } 15, \tag{7.26}$$

where each block has a single duplicated element (see Hwang [10, p. 788], Street [32, pp. 123–125] and Street and Street [33, p. 320]), and

$$(1,2,4,1,12,2,8) \text{ modulo } 15, \tag{7.27}$$

where each block has two duplicated elements.

For $v = 9$, $k = b = 6$, we offer 3 RND's. For each of these we take $S = \{1,2,3,A,B,C,a,b,c\}$ and give only 2 initial blocks, the remaining blocks being obtained by applying the permutation $(123)(ABC)(abc)$ once and twice to the initial blocks:

$$(1,2,A,B,a,b) \quad (1,A,b,2,C,c); \tag{7.28}$$

$$(1,2,A,B,a,b) \quad (1,A,a,2,C,a); \tag{7.29}$$

$$(B,2,A,B,a,b) \quad (1,3,a,2,C,a). \tag{7.30}$$

In the RND obtained from (7.28) no block has a duplicated element, in the RND obtained from (7.29) exactly 3 blocks have a single duplicated element, and in the RND obtained from (7.30) every block has a single duplicated element; the RND obtained from (7.30) can be obtained by concatenation from a design with $k = 3$. An RND with $v = 9$, $k = 12$ can clearly be obtained by concatenation in many ways from RND's above.

8. Quasi Rees Neighbour Designs

For practical applications, e.g. in the microbiological laboratory, the non-existence of RND's for even values of v might be thought vexatious, especially when no need is perceived for replicating each pairing of two antigens as neighbours. To suggest blocks of different sizes may provide no solution of the difficulty, as practical considerations may well require each block to have the same geometry. As the *next best thing* to an RND, we therefore define a *quasi Rees neighbour design* (QRND) exactly as an OND save that condition (v) (see Section 1 above) is replaced by the following:

(v*) every element from S has each other element as a neighbour exactly once, except that it has just one of the other elements as a neighbour exactly twice.

This definition implies a partitioning of the set $S = \{1, 2, \ldots, v-1, v\}$ into pairs such that the two elements within any pair are neighbours twice, whereas any two elements from different pairs are neighbours just once. The parameters v, r, b, k of a QRND satisfy the equations

$$vr = bk \tag{8.1}$$

and

$$r = v/2, \tag{8.2}$$

with v required to be even.

A QRND does not correspond to any of the *Jeux de Demoiselles* of Lucas [22, pp. 161–197], nor are we aware of any corresponding problem in the literature of recreational mathematics. The closest problem, given by Lucas [22], seems to be where condition (v) for a BOND is replaced by

(v**) every element from S has each other element as a neighbour exactly once, except that just one of the other elements is never its neighbour.

A QRND with $k = 3$ must have

$$b = vr/3 = v^2/6, \tag{8.3}$$

which restricts v to the values $v = 6, 12, \ldots$. A QRND with $v = 6$ and $k = 3$ is the following:

$$(1,2,3)\ (1,2,4)\ (3,4,5)\ (3,4,6)\ (5,6,1)\ (5,6,2). \tag{8.4}$$

A QRND with $k = 4$ must have

$$b = vr/4 = v^2/8, \tag{8.5}$$

which restricts v to the values $v = 4, 8, 12, \ldots$. A QRND with $v = 4$ and $k = 4$ is

$$(1,2,3,4) \quad (1,2,4,3). \tag{8.6}$$

Method B for constructing balanced circuit RND's can readily be modified to produce QRND's in which the pairs duplicated as neighbours are the pairs in the set $\{(1,v),(2,v-1),\ldots,(v/2,(v+2)/2)\}$. The blocks of the QRND produced by this modified method are as follows:

$$\begin{array}{lccccccc}
(v, & 1, & 2, & v-1, & 3, & v-2, & \ldots, & \frac{(v+2)}{2}\ ) \\
(v, & 2, & 3, & 1, & 4, & v-1, & \ldots, & \frac{(v+4)}{2}\ ) \\
(v, & 3, & 4, & 2, & 5, & 1, & \ldots, & \frac{(v+6)}{2}\ ) \\
\vdots & & & & & & & \\
(v, & \frac{v}{2}, & \frac{(v+2)}{2}, & \frac{(v-2)}{2}, & \frac{(v+4)}{2}, & \frac{(v-4)}{2}, & \ldots, & 1\ )
\end{array}$$

If the last block of this modified construction is omitted, condition (v^{**}) of Lucas [22] is met.

No systematic attempt has been made to construct, study or enumerate QRND's. However, QRND's can be given definitions of isomorphism, sepecies, adjugate set, etc., that are analogous to the corresponding definitions for BOND's.

Preliminary investigations suggest that QRND's have more mathematical appeal and interest than is initially apparent. Consider, for example, the following three QRND's for $v = k = 6$:

$$(1,2,4,5,6,3) \quad (1,2,6,5,3,4) \quad (1,5,2,3,4,6) \tag{8.7}$$

$$(1,2,4,5,6,3) \quad (1,2,6,4,3,4) \quad (1,5,2,3,5,6) \tag{8.8}$$

and

$$(1,2,3,1,2,4) \quad (3,4,5,3,4,6) \quad (5,6,1,5,6,2) \tag{8.9}$$

In each of these three QRND's, the pairs that are duplicated as neighbours are the pairs in the set $P = \{(1,2),(3,4),(5,6)\}$. Design (8.7) has each element exactly once in each block, but the neighbouring pairs in blocks 1, 2 and 3 include, respectively, 2, 3 and 1 of the pairs in P; design (8.7) can thus be said to be irregular in structure. This can be said even more strongly of design (8.8), which has element 4 twice in block 2 (the occurrences being 2 places apart) and element 5 twice in block 3 (the occurrences being 3 places apart). Contrariwise, design (8.9), which has two repeated elements in each block, is very regular in structure, each block being obtained from either of the others by use of the permutation (135)(246) once or twice. In an obvious extension of earlier terminology, design (8.9) can also be said to be *consistently 2-perfect*, the consistency arising because the pairs that are duplicated as immediate neighbours are the same as those duplicated as 2-places-apart neighbours.

A QRND with $v = k = 8$ is as follows, where P, defined as above, is now $P = \{(1,2),(3,4),(5,6),(7,8)\}$:

$$\begin{array}{ll}(1,2,3,6,8,7,5,4) & (1,3,4,8,2,5,6,7) \\ (1,2,4,6,5,3,7,8) & (1,5,8,3,4,7,2,6)\end{array} \tag{8.10}$$

Once again, despite the appearance of each element of S in each block, examination of the positionings of the pairs from P as neighbours shows the structure to be irregular.

Further mathematical interest can be introduced into the study of QRND's by requiring that, if p and q are any 2 elements that are neighbours twice, then p should occur once as a forward neighbour of q (i.e. as a *left neighbour*, if we think of children dancing in a ring whilst facing the centre of the ring) and once as a backward neighbour (*right neighbour*). We use the terminology *even-handed QRND* for a QRND with this property. An example obtained from (8.8) by first taking the reflection of block 1, then the other two blocks unchanged, is the following:

$$(1,3,6,5,4,2) \quad (1,2,6,4,3,4) \quad (1,5,2,3,5,6) \qquad (8.11)$$

9. Balanced Ouchterlony neighbour designs having $\lambda' > 1$

Hwang and Lin [11, 12, 13, 14] established that a BOND exists for any set of parameters v, r, b, k, λ' satisfying equations (1.1) and (1.2).

Literature sources already referred to and listed at the end of this article give a few constructions for BOND's with $\lambda' > 1$, but much remains to be discovered about such designs. The foregoing sections of the present paper suggest clearly that, for any particular parameter set, enumeration of designs will generally pose greater problems than mere construction of a single example.

Further mathematical interest can be introduced into the study of BOND's having an even value of λ' by requiring that, if p and q are any 2 elements from S, then p should occur $\lambda'/2$ times as a forward neighbour of q and $\lambda'/2$ times as a backward neighbour. We describe a BOND with this further property as an *even-handed BOND*. Consider, for example, the following two BOND's obtained by, respectively, Dey and Chakravarti [8, p. 103] and Chandak [5, p.3] for $v = 13$, $k = 4, \lambda' = 2$:

$$(1,2,4,8) \quad (3,6,12,11) \quad (4,8,3,6) \qquad \text{all modulo } 13 \qquad (9.1)$$

and

$$(1,8,12,5) \quad (2,3,11,10) \quad (4,6,9,7) \qquad \text{all modulo } 13. \qquad (9.2)$$

Here, design (9.2) is an even-handed BOND, but the BOND (9.1) is not even-handed and cannot be made even-handed by reflection of one or more of the initial blocks. The blocks of each of (9.1) and (9.2) come from those of a BIBD.

Even-handed BOND's are closely related to *serially balanced sequences* (see Street and Street [33, Section 14.5]). Indeed, the following even-handed BOND with $v = k = 5, \lambda' = 2$ is obtainable directly from the first serially balanced sequence in Table 14.10 of Street and Street [33, p. 331]:

$$(1,2,4,3,5)\ (2,3,1,4,5)\ (3,4,2,1,5)\ (4,1,3,2,5) \qquad (9.3)$$

With the element 5 printed first instead of last in each block, the construction is seen to be that of Method B of Section 7 above, extended to $v-1$ blocks. More generally, extending either Method A or Method B to $v-1$ blocks (v odd) gives an even-handed BOND with $\lambda' = 2$.

The even-handedness of (9.3) holds not only for immediate neighbours but also for 2-places-apart neighbours. So, in an obvious extension of earlier terminology, design (9.3) is a *2-perfect even-handed BOND*.

A general construction similar to Method B, except that an extra block is added at the end, can be used to produce BOND's with v even, $k = v - 1$ and $\lambda' = 2$ (see Street [32, p. 125]). The BOND's produced by this construction are not even-handed, as is illustrated by the following example for $v = 6$:

$$(6,1,5,2,4)\ (6,2,1,3,5)\ (6,3,2,4,1)\ (6,4,3,5,2)\ (6,5,4,1,3)$$
plus the block $(1,2,3,4,5)$. (9.4)

The blocks of (9.4) come from those of a BIBD.

The *total cycles* of Azaïs [3, pp. 337–338] provide even-handed BOND's with v odd, $k = v - 1$ and $\lambda' = 2$, whose blocks come from those of a BIBD. Examples for $v = 5, 7, 9$ and 11 are, respectively, the following:

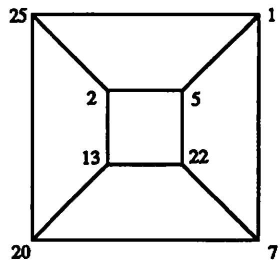
$$(1,2,5,4) \text{ modulo } 5 \qquad (9.5)$$
$$(1,2,7,4,6,5) \text{ modulo } 7 \qquad (9.6)$$
$$(1,2,9,3,8,5,7,6) \text{ modulo } 9 \qquad (9.7)$$
$$(1,2,11,3,10,5,9,6,8,7) \text{ modulo } 11 \qquad (9.8)$$

These examples all arise as special cases from the general construction (2) given by Azaïs for any odd value of $v > 3$.

10. Openings for further research

Much remains to be done in the study of BOND's and of QRND's. Despite some recent relevant work by Iqbal [15], there is a dearth of general methods of construction. Studies of non-isomorphism for particular parameter-sets are conspicuous by their almost total absence from the literature, as are enumerations of non-isomorphic designs for particular parameter-sets. Tabulations of known designs with particular properties are lacking. Theorems on possible patterns for repeated elements in the blocks of BOND's with $\lambda' > 1$ are not available, nor are many results on even-handed BOND's. If we compare present knowledge about BIBD's with that about BOND's and QRND's, we see that study of BOND's and QRND's is still in its early stages.

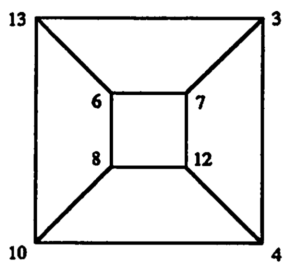
Also, as Rees himself has pointed out in private correpondence with the present author, many other openings for further research are revealed when we consider 3-dimensional analogues of RND's and BOND's. Suppose, for example, that a design has elements located on spheres. If there are 8 locations on each sphere and these are at the corners of a cube, we might define two locations to be neighbouring if they are joined by an edge of a cube. Then the following design [Rees, private communication] is analogous to an RND:



modulo 25

(10.1)

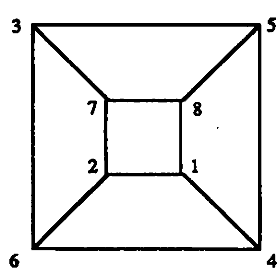
(We use the Schafli representation of a cube.) Likewise, the following is analogous to a BOND with $\lambda' = 2$:



modulo 13

(10.2)

whereas the following is analogous to a BOND with $\lambda' = 3$:



modulo 9

(10.3)

In an obvious sense, analogous to that for 2-dimensional designs, none of (10.1), (10.2) or (10.3) is 2-perfect, although similar 2-perfect designs might well be expected to exist. The 3-dimensional designs seem to offer a field of reseach as fertile as that for 2-dimensional designs.

Table 1

Parameter sets for RND's with $v \leq 15, k \leq 15$

v	$r = (v-1)/2$	k	$b = vr/k$	Possible method of construction
5	2	5	2	Method A or B
		10	1	Concatenation (from design with $k = 5$), or (7.11) in text
7	3	3	7	BIBD (Steiner triple system)
		7	3	Method A or B, or (7.3) or (7.4) in text
9	4	3	12	BIBD (Steiner triple system)
		4	9	Rees's differences [29] or see Azaïs [3, p. 338]
		6	6	Concatenation (from design with $k = 3$), or (7.28) or (7.29) in text
		9	4	Method B, or see Sainte-Laguë [31, p. 167] or concatenation (from design with $k = 3$)
		12	3	Concatenation (from design with $k = 3$ or 6)
11	5	5	11	Suitably ordered BIBD difference set (Rees [29]), or (7.17) in text, or see Azaïs [3, p. 338]
		11	5	Method A or B, or see Sainte-Laguë [31, p. 167]
13	6	3	26	BIBD (Steiner triple system)
		6	13	Rees's differences [29], or (7.21) in text, or see Azaïs [3, p. 338], or concatenation (from design with $k = 3$)
		13	6	Method A or B
15	7	3	35	BIBD (Steiner triple system)
		5	21	Rees's differences [29] (despite non-existence of BIBD)
		7	15	Suitably ordered BIBD difference set (Rees [29]), or (7.24), (7.25), (7.26) or (7.27) in text
		15	7	Method B, or concatenation (from design with $k = 3$ or 5)

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