A Note on a Generating Set for N_5^∞

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Abstract. In this note, we construct a $(39, \{5, 6, 7\}, 1)$-PBD. Thus we have a finite generating set for the PBD-closed set N_5^∞ with at most three inessential elements, where $N_5^\infty = \{1\} \cup \{v : v \geq 5\}$.

The theory of PBD-closure was developed by R. M. Wilson in a series of papers (see [5], [6], [7] and [8]). Given a set K of positive integers, we denote by $B(K)$, the *closure* of K, the set $\{v : a\ (v, K, 1)\text{-PBD exists}\}$. A set is a *PBD-closed set* if $B(K) = K$.

Wilson showed

Theorem 1. [8] *If K is a PBD-closed set, then there exists a finite set $J \subseteq K$ such that $K = B(J)$.*

Such a set J is called a *finite generating set* for the PBD-closed set K. Wilson also observed that each PBD-closed set K has a unique minimal generating set which is contained in every finite generating set of K. An element $x \in K$ is called *essential* in K if and only if $x \notin B(K - \{x\})$, or equivalently, $x \notin B(\{y \in K : y < x\})$.

It can be quite difficult to determine whether or not a particular element is not inessential. The following lemma can be of help.

Lemma 2. [4] *Let P be a PBD whose smallest block size is at least s and largest block size is m. Then P contains at least $m(s-1)+1$ elements.*

Let $N_k^\infty = \{1\} \cup \{v : v \geq k\}$. Hanani proved

Theorem 3. [3] *Let* $K_3 = \{3, 4, 5, 6, 8\}$, $K_4 = \{4, 5, \ldots, 12, 14, 15, 18, 19, 23, 27\}$, *and* $K_5 = K_5^\circ \cup \{32, 33, 34, 39\}$ *where* $K_5^\circ = \{5, 6, \ldots, 20, 22, 23, 24, 27, 28, 29\}$. *Then K_k is a finite generating set for N_k^∞ for each $k = 3, 4$ and 5.*

The fact that K_3 is the minimal finite generating set for N_3^∞ is an immediate consequence of Lemma 2. It is also well known (see, for example, [1] and [2]) that $K_4 - \{27\}$ is the minimal generating set for N_4^∞. But for N_5^∞, we do not know whether the elements of $\{32, 33, 34, 39\}$ are inessential.

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It is the purpose here to construct a $(39, \{5, 6, 7\}, 1)$-PBD, which implies that $K_5^\circ \cup \{32, 33, 34\}$ is a finite generating set for N_5^∞ which contains at most 3 inessential elements.

We assume familarity with standard terminlogy and results in design theory ([1]).

Theorem 4. *A* $(39, \{5, 6, 7\}, 1)$ *-PBD exists.*

Proof. Remove 6 points from two groups respectively of a TD$(7, 8)$ to get a $\{6, 7, 8\}$-GDD of type $1^2 7^6$ which contains a unique block of size 8. Remove 5 points from a block of size 6 which intersects the unique block of size 8 at one of these 5 removed points. Then we obtain a $\{5, 6, 7\}$-GDD of type $1^2 6^5 7^1$, which implies the existence of a $(39, \{5, 6, 7\}, 1)$-PBD.

Thus we have

Theorem 5. *Let* $K_5^! = K_5^\circ \cup \{32, 33, 34\}$, *where* $K_5^\circ = \{5, 6, \ldots, 20, 22, 23, 24, 27, 28, 29\}$. *Then* $K_5^!$ *is a finite generating set for* N_5^∞ *which contains at most three inessential elements of* 32, 33 *and* 34.

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References

[1] T. Beth, D. Jungnickel and H. Lenz, *Design Theory.* Bibliographiisches Institut, Zurick, 1985.

[2] S. Furino, *α-Resolvable structure.* Ph.D. Thesis, Univ. of Waterloo, 1990.

[3] H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math. **11** (1975), 255–369.

[4] R. C. Mullin, *On a generating set problem of Rosa*, Ars Combinatoria **27** (1989), 75–84.

[5] R. M. Wilson, *An existence theory for balanced designs I*, J. Combin. Th. (A) **13** (1972), 220–245.

[6] R. M. Wilson, *An existence theory for balanced designs II*, J. Combin. Th. (A) **13** (1972), 246–273.

[7] R. M. Wilson, *An existence theory for balanced designs III*, J. Combin. Th. (A) **18** (1975), 71–79.

[8] R. M. Wilson, *Constructions and uses of pairwise balanced designs*, Math. Centre Tracts **55** (1974), 18–41.