Orthogonal Diagonal Latin Squares with Orthogonal Diagonal Subsquares

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Abstract. It is proved in this paper that for any integer $n \geq 100$, a (v,n)-IODLS (incomplete orthogonal diagonal Latin squares) exists if and only if $v \geq 3n+2$. Results for $n = 2$ are also mentioned.

1. Introduction

A *Latin square* of order n is an $n \times n$ array such that every row and every column is a permutation of a n-set. A *transversal* in a Latin square is a set of positions, one per row and one per column among which the symbols occur precisely once each. A *transversal Latin square* is a Latin square whose main diagonal is a transversal. It is easy to see that the existence of a transversal Latin square is equivalent to the existence of an idempotent square. A *diagonal Latin square* is a transversal Latin square whose back diagonal also forms a transversal.

Two Latin squares of order n are *orthogonal* if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. *t pairwise orthogonal (transversal, diagonal) Latin squares* of order n, denoted briefly by t POLS(n) (POILS(n), PODLS(n)) are t pairwise orthogonal Latin squares each of which is a (transversal, diagonal) Latin square of order n. We let OLS(n) (OILS(n), ODLS(n)) denote 2 POLS(n) (POILS(n), PODLS(n)).

The spectrum of orthogonal diagonal Latin squares was finally determined by Brown, Cherry, Most, Most, Parker and Wallis [3].

Theorem 1.1. *An orthogonal diagonal Latin squares of order n exists if and only if $n \neq 2, 3,$ or 6.*

The problem we study in this paper is the orthogonal diagonal Latin squares analogue of the Doyen-Wilson theorem [5]. We begin with some definitions. If two orthogonal dialgonal Latin squares have subsquares occupying the central positions in each, the subsquares themselves must be orthogonal dialgonal Latin squares. We refer to them as *orthogonal dialgonal Latin subsquares.* We denote by ODLS(v, n) a pair of orthogonal dialgonal Latin squares of order v with orthogonal dialgonal subsquares of order n. It is easy to see the existence of an ODLS(v, n) required that $v - n$ is even. In particular, any ODLS(v) is an ODLS(v, 1) when v is odd. In view of Theorem 1.1, no orthogonal dialgonal Latin squares can contain orthogonal dialgonal subsquares of order 2, 3, or 6. However, we can construct orthogonal dialgonal Latin squares missing subsquares of these orders. We have the following more general definition.

A (v, n)-IODLS (*incomplete orthogonal dialgonal Latin squares*) is a pair of $v \times v$ arrays which satisfy the following:

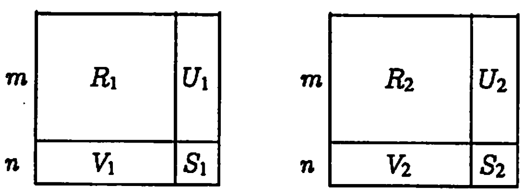
(1) they are OLS(v) with sub-OLS(n) missing
(2) the first $v - n$ elements in the main diagonal of each square are distinct and different from the missing elements
(3) the elements in the cells $(1, v-n), (2, v-n-1), \ldots, (v-n, 1)$ of each square are also distinct and different from the missing elements.

If the condition (3) is missing, the resulting pair of latin squares is called (v, n)-IOILS (*incomplete orthogonal transversal Latin squares*).

We refer to the subsquares as the *hole.* Observe that the hole can be filled in with any orthogonal diagonal Latin squares of order n (provided $n \neq 2, 3$, or 6), thereby constructing orthogonal diagonal Latin squares of order v containing orthogonal diagonal subsquares of order n when $v - n$ is even.

Theorem 1.2. *If there exists a (v, n)-IODLS, then $v \geq 3n + 2$.*

Proof. Write $v = n + m$ and put the (v, n)-IODLS L_1 and L_2 as follows:



Suppose S_1, S_2 are based on the elements $m, m+1, \ldots, v-1$ and L_1, L_2 are based on the elements $0, 1, \ldots, m-1, m, m+1, \ldots, v-1$. Notice U_1, U_1, V_1, V_2 only contain the elemts $0, 1, \ldots, m-1$, the ordered pairs (i, j), (j, i), $i = 0, 1, \ldots, m-1, j = m, m+1, \ldots, v-1$ must be included in the superposition (R_1, R_2) and cannot appear in the cells $(0,0), (1,1), \ldots, (m-1, m-1)$ and $(0, m-1), (1, m-2), \ldots, (m-1, 0)$. We than have

$$2nm + 2m \leq m^2 \quad \text{or} \quad 2nm + 2m - 1 \leq m^2$$

so

$$2n + 2 \leq m \quad \text{or} \quad 2n + 2 \leq m + 1/m$$

then

$$v \geq 3n + 2$$

IODLS have been studied by several researchers. Some applications to the construction of other types of designs are as follows: orthogonal diagonal Latin squares, incomplete self-orthogonal Latin squares and magic squares with magic subsquares.

In this paper, we prove the necessary condition is also sufficient for $n \geq 100$.

Theorem 1.3. *For any positive integer $n \geq 100$, then there exists a (v,n)-IODLS if and only if $v \geq 3n+2$.*

2. Direct construction

First we state a starter-adder type construction for (v,n)-IOILS. The main idea is to generate each square under a cyclic group of order $v-n$, from its first row and from the last n elemnts of the first column. Let $X = \{0,1,\ldots,v-n-1\} \cup Y$, where $Y = \{x_1, x_2, \ldots, x_n\}$. Suppose L is a square based on X with a hole indexed by Y. We shall denote by $e_L(i,j)$ the entry in the cell (i,j) of the array L. The first row is given by the vectors $e = (e_L(0,0),\ldots,e_L(0,v-n-1))$ and $f = (e_L(0,v-n),\ldots,e_L(0,v-1))$, and the last n elements of the first column are given by the vector $g = (e_L(v-n,0),\ldots,e_L(v-1,0)$. The L is constructed modulo $v-n$ in the range $\{0,1,\ldots,v-n-1\}$, where the x_i's act as "infinity" elements as follows:

(1) $e_L(s+1,t+1) = e_L(s,t)$ if $e_L(s,t) = x_i$, and $e_L(s+1,t+1) \equiv e_L(s,t)+1 \pmod{v-n}$ otherwise, where $0 \leq s,t < v-n-1$

(2) $e_L(s+1,v-n-1+t) \equiv e_L(s,v-n-1+t)+1 \pmod{v-n}$, where $1 \leq t \leq n, 0 \leq s < v-n-1$

(3) $e_L(v-n-1+t,s+1) \equiv e_L(v-n-1+t,s)+1 \pmod{v-n}$, where $1 \leq t \leq n, 0 \leq s \leq v-n-1$.

We remark that there are obviously conditions which the vectors e, f, g must satisfy in order to produce the (v,n)-IOILS, but we shall not concern ourselves with that, the reader may see [11].

Lemma 2.1. *Suppose there exists a (v,n)-IOILS constructed by the starter-adder method, $v-n$ is even and the $(1+(v-n)/2)$-st element in the starter set e is not infinity element. Then there exists a (v,n)-IODLS.*

Proof. We begin with the (v,n)-IOILS and permute rows and columns with permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & (v\text{-}n)/2 & (v\text{-}n)/2+1 & (v\text{-}n)/2+2 & \cdots & v\text{-}n \\ 1 & 2 & & (v\text{-}n)/2 & v\text{-}n & v\text{-}n\text{-}1 & & (v\text{-}n)/2+1 \end{pmatrix}$$

Then we obtain the required design.

Lemma 2.2. *For $(v,n) \in F$, there exists a (v,n)-IOILS constructed by the starter-adder method such that the $(1+(v-n)/2)$-st element in the starter set is not an infinity element, where*

$$F = \{(t,2),(3t+6,t): \quad t \in E_1 = \{8,10,12,14,16,18\}\}$$

Proof. For $(v,n) = (8,2)$, see [11], for the other cases $(v,2)$, see [13]; and for the cases $(v,n) = (3t+6,t)$, see [20].

Combining Lemmas 2.1 and 2.2 we have

Lemma 2.3. *There exists a* (v, n)*-IODLS for* $(v, n) \in F$.

In the remaining of this paper we shall assume that the reader is familiar with the various methods of constructing (v, n)-IOLS starting with an OLS(n) (see, for example, [2,4]), and starting with an (n, k)-IOLS (see, for example, [8,22]). We shall also assume that the reader is familiar with the various techniques of constructing ODLS(v) from OLS(v) by permuting rows and columns (see, for example, [7,14,17]).

3. The case n even

A Latin square is *self-orthogonal* if it is orthogonal to its transpose. A Self-orthogonal Latin square (SOLS) of order v will be denoted by SOLS(v). We also denote by (v, n)-ISOLS an *incomplete self-orthogonal Latin square.*

A Latin square is symmetric if it is equal to its transpose. We denote by USOLSSOM(v) a self-orthogonal Latin square of order v with a constant main diagonal symmetric orthogonal mate. It is easy to see that the existence of an USOLSSOM(v) required that v is even.

Lemma 3.1. *([6,18]) If even* $n \notin E_2$, *then there exists an USOLSSOM(n), where*

$$E_2 = \{2, 6, 10, 14, 46, 54, 58, 62, 66, 70\}$$

Lemma 3.2. *If there exists a USOLSSOM(n), then there exists an ODLS(n) which posesses n disjoint common transversals including the main diagonal and the back diagonal.*

Proof. Suppose C is a constant main diagonal symmetric orthogonal mate. By applying a permutation simultaneously to the rows and columns, as Wallis did in [14], we can produce a Latin square with constant main diagonal and constant back diagonal. We do the same permutation to self-orthogonal Latin squares A, A', and obtain required ODLS(n).

Lemma 3.3. *If there exists an ODLS(n) with k disjoint common transversals including the main diagonal and the back diagonal, then there exists a* $(3n+k, n)$*-IODLS,* $2 \leq k \leq n$ *and* $k \neq 2, 3,$ *or* 6.

Proof. We begin with the ODLS(n), and fill the k disjoint common transversals with $(4, 1)$-IOILS and the others with OLS(3), but the back diagonal with modified $(4, 1)$-IOILS, that is, by permuting the first 3 columns so that the main diagonal of the upper left part in the $(4, 1)$-IOILS becomes its back diagonal. Note that there exist ODLS(k) from Theorem 1.1, we obtain the required design by permuting rows and columns, in which the size n hole consists of the central cells of filling 3×3 arrays in ODLS(n).

Combining Lemmas 3.1 and 3.3 we have

Lemma 3.4. *If even $n \notin E_2$, then there exists a $(3n+k,n)$-IODLS, for $2 \leq k \leq n$ and $k \neq 2,3$, or 6.*

Lemma 3.5. *If even $n \geq 70$, then there exists a $(3n+2,n)$-IODLS.*

Proof. We begin with the $(n,2)$-IODLS for which existence comes from Lemma 5.6 below. We fill the diagonals in the upper left part with $(4,1)$-IOILS but the back diagonal with modified $(4,1)$-IOILS, and the others with OLS(3). We then have a $(3n+2,8)$-IODLS. Note that there exists a $(8,2)$-IODLS, so the result follows. The size n hole consists of size 2 hole in $(8,2)$-IODLS and the central cells of filling 3×3 arrays in $(n,2)$-IODLS.

For the case $k=3$ we need

Lemma 3.6. *If even $n \geq 72$, then there exist 4 PODLS(k) such that $n = 4k+t$ or $n = 8k+t$, $t \in E_1$.*

Proof. From [9], it is not difficult to check that the assertion is true.

Lemma 3.7. *If there exist 4 PODLS(k), then there exist $(4k+t,2)$-IODLS and $(8k+t,2)$-IODLS, $t \in E_1$, and $t \leq k$, each of which possesses a common transversal which meets each of the diagonals of the subarray in the upper left part precisely once.*

Proof. We begin with 4 PODLS(k). Then there exists an ODLS(k) which possesses one common T transversal which meet each of the diagonals precisely once, and k disjoint common transversals each of which meets the above transversal precisely once. From this ODLS(k), we fill t disjoint transversals with $(5,1)$-IODLS or $(9,1)$-IODLS from ODLS(5) or ODLS(9) respectively, and the others with ODLS(4) or ODLS(8) respectively. In particular, two cells which be contained one common transversal T and one transversal of k disjoint common transversals and diagonal respectively, we fill $(5,1)$-IOILS or $(9,1)$-IOILS from ODLS(5) or ODLS(9), or modified $(5,1)$-IOILS or modified $(9,1)$-IOILS. Then we obtain the required design by filling the size t hole with $(t,2)$-IODLS and permuting rows and columns. The required common transversal consists of the back diagonals (about ODLS(5) or ODLS(9)) of filling IOILS and the main diagonals of filling ODLS or IODLS in T.

Combining Lemmas 3.6 and 3.7 we have

Lemma 3.8. *If even $n \geq 72$, then there exists an $(n,2)$-IODLS with a common transversal wich meets each of the diagonals of the subarray on the upper left part precisely once.*

Lemma 3.9. *If even $n \geq 72$, then there exists a $(3n+3,n)$-IODLS.*

Proof. We begin with $(n,2)$-IODLS as in Lemma 3.8, and fill the diagonals of the subarray on the upper left part with $(4,1)$-IOILS or modified $(4,1)$-IOILS, and the others with OLS(3) but the common transversal with $(4,1)$-IOILS or $(5,1,1)$-IOILS (two holes of size 1, or modified $(5,1,1)$-IOILS. Then we obtain the required design by filling the size 8 hole with $(8,2)$-IODLS and pemuting rows and columns.

For the case $k = 6$, we need the following result from Lemma 3.1.

Lemma 3.10. *If even $n \geq 56$, then there exists an USOLSSOM(k) such that $n = 3k + t, t \in E_1$.*

Lemma 3.11. *If there exists an USOLSSOM(k), then there exists a $(3k+t, t)$-IODLS with t disjoint common transversals including the main diagonal and the back diagonal which consist of the elements which is not in the subarray.*

Proof. We begin with the USOLSSOM(k), then we have an ODLS(k) with k disjoint common transversals including the main diagonal and the back diagonal. From this ODLS(k), we fill t disjoint transversals including the main diagonal and the back diagonal with $(4,1)$-IOILS or modified $(4,1)$-IOILS, and the others with OLS(3). The result follows.

Combining Lemmas 3.10 and 3.11 we have

Lemma 3.12. *If even $n \geq 56$, then there exists an (n,t)-IODLS, $t = 8, 10, 12, 14, 16$, or 18, with t disjoint common transversals including the main diagonal and the back diagonal which consist of the elements which are not in the subarray.*

Lemma 3.13. *If even $n \geq 56$, then there exists a $(3n+6, n)$-IODLS.*

Proof. We begin with the (n,t)-IODLS as in Lemma 3.12, and fill 6 disjoint common transversals including the main and back diagonals with $(4,1)$-IOILS or modified $(4,1)$-IOILS, and the others with OLS(3). Then we obtain the required design by filling the size $3t+6$ hole with $(3t+6, t)$-IODLS and permuting rows and columns.

Up to now, we have obtained

Theorem A. *If even $n \geq 72$, then there exists a $(3n+k, n)$-IODLS, $2 \leq k \leq n$.*

4. The case n odd

We need the following results about t PODLS(n).

Lemma 4.1. *([7,21]) If odd $n \notin E_3$, then there exist 3 PODLS(n), where*

$$E_3 = \{3, 5, 15, 21, 33\}$$

Lemma 4.2. *([9,17]) If odd $n \notin E_4$, then there exist 4 PODLS(n), where*

$$E_4 = E_3 \cup \{39, 55, 69\}$$

Lemma 4.3. *([10,17]) If odd $n \notin E_5$, then there exist 5 PODLS(n), where*

$$E_5 = E_4 \cup \{51\}$$

Lemma 4.4. *Suppose n odd, and there exist t PODLS(n). Then the following exist:*

1. *$t-1$ PODLS(n) ($t-3$ PODLS(n)) with n disjoint common transversals, one of which contains the central cell, and such that other $n-1$ meet both the main and back diagonals (4 common transversals meeting in the central and including the main and back diagonals) in one cell each.*
2. *$t-2$ PODLS(n) with 4 common transversals, meeting including the main and back diagonals, which contain the central cell but are otherwise disjoint.*

Proof. Note that if we begin with the t PODLS(n), from t-th DLS we may determine in $t-1$ DLS n disjoint common transversals, in which one pass the central cell and the others meet each of the diagonals precisely once, or one common transversal which meet each of the diagonals precisely once. The result follows.

Lemma 4.5. *Suppose n odd and there exists an ODLS(n) with n disjoint common transversals, in which one contains the central cell and the others meet each of the diagonals precisely once. Then there exists a $(3n+2+k, n)$-IODLS, $0 \le k < n$ and $k \neq 2, 3$ or 6.*

Proof. Begin with the ODLS(n), and fill the diagonals and the k disjoint transversals which do not contain the central cell with $(4,1)$-IOILS, $(5,1)$-IOILS, modified $(4,1)$-IOILS or modified $(5,1)$-IOILS, but leave the central cell comply. Fill all other cells with OLS(3). Finally, fill the size 5 hole with ODLS(5) and fill the size k hole with ODLS(k), and permute rows and columns.

Lemma 4.6. *Suppose n is odd and there exists an ODLS(n) with 4 common transversals meeting in the central cell and including the main diagonal and the back diagonal. Then there exists a $(3n+4, n)$-IODLS.*

Proof. We begin with the ODLS(n) and fill the 4 transversals with $(4,1)$-IOILS or modified $(4,1)$-IOILS and let the central cell be empty, the others with OLS(3). Then we obtain the required result by filling the size 7 hole with modified ODLS(7), that is, by moving the first two rows and columns to the last such that the back diagonal of the ODLS(7) possesses the cells $(2,0)$, $(1,1)$, $(0,2)$, $(6,3)$, $(5,4)$, $(4,5)$ and $(3,6)$, and permuting rows and columns.

Lemma 4.7. *Suppose n is odd and there exists an ODLS(n) with 4 common transversals meeting in the central cell and including the main and back diagonals, and n further disjoint common transversals, in which one contains the central cell* *and the others meet each of the above 4 transversals precisely once. Then there exists a* $(3n+k, n)$*-IODLS,* $k = 5$ *and* 8.

Proof. Begin with the ODLS(n), and fill the 4 transversals meeting in the central cell and including the main diagonal and the back diagonal and the one or four in n transversals which do not contain the central cell with $(4,1)$-IOILS, $(5,1,1)$-IOILS, modified $(4,1)$-IOILS or modified $(5,1,1)$-IOILS, but leave the central cell comply. Fill all other cells with OLS(3). Finally, fill the size 1 or 4 hole with ODLS(1) or ODLS(4) and fill the size 7 hole with modified ODLS(7) as in Lemma 4.6, and permute rows and columns.

Combining above lemmas we have

Lemma 4.8.

(1) *If odd* $n \notin E_5$*, then there exists a* $(3n+k, n)$*-IODLS,* $2 \leq k < n$
(2) *If odd* $n \notin E_4$*, then there exists a* $(3n+k, n)$*-IODLS,* $2 \leq k < n$ *and* $k \neq 5$.
(3) *If odd* $n \notin E_3$*, then there exists a* $(3n+k, n)$*-IODLS,* $2 \leq k < n$ *and* $k \neq 4$ *or* 5.

Up to now we have obtained

Theorem B. *If odd* $n \geq 69$*, then there exists a* $(3n+k, n)$*-IODLS,* $2 \leq k < n$.

5. The main result

Let $P = \{S_1, S_2, \ldots, S_n\}$ be a partition set S, where $n \geq 2$. A *partitioned incomplete Latin square* (or PILS) having partition P is an $|p| \times |p|$ array L, indexed by S, which satisfies the following properties:

(1) a cell of L either contains a symbol from S or is empty
(2) the subarray indexed by $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are called *holes*)
(3) the elements occurring in row (or column) s of L are precisely these in $S \backslash S_i$, where $s \in S_i$.

The *type* of L is the multiset $\{|S_1|, |S_2|, \ldots, |S_n|\}$. We use the notation $1^{u_1} 2^{u_2} \ldots$ to describe a type, where there are precisely u_i occurrences of i, for $i = 1, 2, \ldots$.

Suppose L and M are PILS having the same partition P. We say that L and M are *orthogonal* if their superposition yields every ordered pair in $S^2 \setminus (\cup S_i^2)$. The term "orthogonal PILS" is abbreviated to OPILS.

We shall assume that the reader is familiar with the standard terminology of group-divisible designs (GDDs) and Wilson's "Fundamental Construction" (see, for example, [19]). Of course, a GD[$k, 1, n; kn$] is equivalent to $k-2$ POLS(n).

Lemma 5.1. *([13]) If* $v \geq 3n+1$ *and* $v \neq 6$*, then there exists a* (v, n)*-ISOLS except possible for* $(v, n) \in (6m+i, 2m) : \quad i = 2$ *or* $6\}$.

Lemma 5.2. *If positive integer* $n \neq 2, 3$ *or* 6, *then there exists an* $OPILS(1^n)$ *with transversal back diagonal.*

Proof. By Theorem 1.1, there exists an ODLS(n). By applying a suitable permutation to the rows, and permuting symbols, we can ensure that cell (i, i) contains the pair (i, i), for all i.

We need the following recursive construction for OPILS

Lemma 5.3. *([4]) Suppose that* $(X, \mathbf{G}, \mathbf{A})$ *is a GDD,* w *is a weighting, and let* $k \geq 1$. *Further, suppose that, for every block* $A \in \mathbf{A}$, *there are* k *OPILS of type* $w(A)$. *Then there are* k *OPILS of type* $\{\sum_{x \in G} w(x) : \quad G \in \mathbf{G}\}$.

We now state the main construction.

Lemma 5.4. *If* n, m, k *be positive integers,* m *odd,* $2 \leq n \leq 3m - 3$, $1 \leq k \leq 2m$ *and* $k \neq 3, 4$, *such that there exsts a* $GD[10, 1, m; 10m]$. *Then there exists a* $(7m + n + k, n)$*-IODLS. Further, for* $7m + n + 5 \leq v \leq 9m + n$, *there exists a* (v, n)*-IODLS.*

Proof. In all but three groups of the GD$[10, 1, m; 10m]$, we give the points weight 1. In the third last group, we give s (s odd and $s \geq 1$) points weight 1 and give the remaining points weight 0. In the second last group, we give t points weight 1 and give the remaining points weight 0. We observe that if $s + t = k \geq 1$ and $k \neq 3, 4$, then we can choose s and t such that both ODLS(s) and ODLS(t) exist. In the last group, we give weight 0, 2, or 3, such that the total weight be n. We can apply Lemma 5.3 with the necessary input designs from Lemma 5.1 in which one size 8 block input is an OPILS(1^8) from Lemma 5.2 (or when $s + t = 2m$, one size 9 block input is an OPILS(1^9) from Lemma 5.2), to obtain an OPILS($m^7 s^1 t^1 n^1$). We then fill the size m holes with ODLS(m, 1), the size s hole with ODLS(s, 1) and the size t hole with ODLS(t), and obtain the required design be permuting rows and columns as Wallis did in [16].

Lemma 5.5. *There is a series of positive inegers*

$$M = (m_i : i = 1, 2, 3, \ldots) = (17, 19, 23, 25, 27, 29, 31, 37, 41, \ldots),$$

such that $m_{i+1} - m_i \leq 8$, $7m_{i+1} + 4 \leq 9m_i$, *and there exist* $GD[10, 1, m_i; 8m_i]$ *for all* $i \geq 1$.

Proof. From existing tables on the number of POLS (see, for example, [1]), it is not difficult to check that such a series M exists with $m_{i+1} - m_i \leq 8$ and there exist GD$[10, 1, m_i; 10m_i]$. Since $m_{i+1} - m_i \leq 8$, it is also easy to see that $7m_{i+1} + 4 \leq 9m_i$ if $m_i \geq 31$. Moreover, for $17 \leq m_i \leq 31$, simple calculation shows that we have $7m_{i+1} + 4 \leq 9m_i$.

We are now in a position to prove

Theorem C. *For any positive integer $n > 48$, if $v \geq 10n/3 + 66$, then there exists a (v, n)-IODLS.*

Proof. Our proof relies heavily on Lemmas 5.4 and 5.5. First of all, for any fixed $n > 48$, there exists an $i \geq 1$ such that $3m_i - 3 < n \leq 3m_{i+1} - 3$. Thus we have $3m_{i+1} - n < 3(m_{i+1} - m_i) + 3 \leq 27$ and $m_{i+1} \leq (n + 26)/3$. Applying Lemmas 5.4 and 5.5 recursively, we know that there exist (v, n)-IODLS whenever $v \geq 7m_{i+1} + n + 5$. Therefore there exist (v, n)-IODLS whenever $v \geq 7(n + 26)/3 + n + 5$, that is, whenever $v \geq 10n/3 + 66$.

Lemma 5.6. *For any positive integer $v \geq 70$, there exists a $(v, 2)$-IODLS.*

Proof. First we apply Lemma 5.4 with $n = 2$ and $m \in M \cup \{9, 11, 14\}$. We then have the result is true except for $120 \leq v \leq 125$. Then we also apply Lemma 5.4 with $n = 8$ and $m = 13$, so we have $(v, 8)$-IODLS for these values. Note that there exists a $(8, 2)$-IODLS, so the result follows.

Proof of Theorem 1.3

Since $10n/3 + 66 \leq 4n$ whenever $n \geq 100$, the result is an immediate consequence of Theorems A, B, and C.

Remark

The starter-adder method devised by Wu [20] to construct ISOLS$(3t + 6, t)$ has been introduced in [12, Theorem 2.1], which also gave the required IOILSs in Lemma 2.2 except for $t = 18$. For the IOILS$(3 \times 18 + 6, 18)$ we may choose $m, a, \mathbf{a}, \mathbf{b}$ in [12, Theorem 2.1] as follows:

$$m = 9 \qquad a = 3$$

$$\mathbf{a} = (16, 38, 32, 36, 18, 22, 19, 39, 20, 40, 21, 8, 13, 6, 23, 24, 1)$$

$$\mathbf{b} = (14, 35, 28, 31, 12, 15, 11, 30, 10, 29, 9, 37, 41, 33, 7, 17, 25).$$

Acknowledgement

I would like to thank Professor L. Zhu for introducing me to this work. Also, I would like to thank the referees for their valuable suggestions for a previous version of this paper.

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