Contributions to the Existence of Arrays With Two Levels

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Abstract. In this paper we consider some combinatorial structures called balanced arrays (B-arrays) with a finite number of elements, and we derive some necessary conditions in the form of inequalities for the existence of these arrays. The results obtained here make use of the Hölder Inequality.

1. Introduction and Preliminaries

Consider a set $S = \{0, 1, , \ldots, s-1\}$ with s elements. An array T with m constraints (rows), N runs (columns, treatment combinations) and with s levels is merely a matrix T of size $(m \times N)$ whose elements are $0, 1, 2, \ldots, s-1$. Here we restrict ourselves to arrays with two levels denoted by 0 and 1, and are called binary arrays. If $\underline{\alpha}$ is any column of T, we define weight of $\underline{\alpha}$, denoted by $\omega(\underline{\alpha})$, to be the number of 1's in $\underline{\alpha}$. Clearly $0 \leq \omega(\underline{\alpha}) \leq m$. The array T is said to be of strength $t (t \leq m)$ if in every sub-matrix $T^*(t \times N)$ of T, the vectors of weight $i (0 \leq i \leq t)$ occur with a frequency μ_i(say), and μ_i depends only on i. In this paper we confine ourselves to arrays with $t = 4$, but the results presented can be extended to arrays of strength t without much difficulty. Next we give the definition of balanced array (B-array) by imposing further combinatorial constraint on T.

Definition 1.1. An array T of size $(m \times N)$ and of strength four is said to be balanced if in every $(4 \times N)$ submatrix T_0 of T, every (4×1) vector of weight $i (i = 0, 1, 2, 3, 4)$ appears a constant number μ_i(say) of times. The vector $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is called the index set of T, and T is sometimes denoted by B-array $(m, N; \underline{\mu}', s = 2, t = 4)$. It is quite evident that

$$N = \sum_{i=0}^{4} \binom{4}{i} \mu_i = \mu_0 + 4\mu_1 + 6\mu_2 + 4\mu_3 + \mu_4 \tag{1}$$

It is quite clear that we obtain a different kind of array if we impose a different kind of combinatorial structure on T. These arrays have been found to be quite useful in Statistical Design of Experiments and Combinatorics. B-arrays, for example, are orthogonal arrays (O-arrays) when $\mu_i = \mu$ for each i, and are the incidence matrices of balanced incomplete block designs (BIBD) when $t = 2$, and each column of T has the same number of 1's in it. It is quite clear that the parameters (v, b, r, k, λ) of a BIBD and (μ_0, μ_1, μ_2) of B-array T are such that $\mu_2 = \lambda, \mu_1 = r - \lambda$, and $\mu_0 = b - 2r + \lambda$. Also B-arrays have been extensively used in the construction of fractional factorial designs of different resolutions. For those

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interested to gain further insight into the importance of B-arrays to combinatorics and statistical design of experiments may consult the list of references given at the end.

The existence and construction of B-arrays with $\underline{\mu}' = (\mu_0, \mu_1, \dots, \mu_t)$ and $m > t$ is a non-trivial problem since such arrays may not exist even though the parameter values satisfy all the known existence conditions. Furthermore, the construction of these arrays with the maximum possible number of constraints is an important problem both in combinatorics and design of experiments and its study may lead to a solution to the packing problem. Such problems for O-arrays for a given μ and m have been studied, among others, by Rao [11], Seiden [14], Seiden and Zemach [15], etc. etc. For B-arrays the corresponding problems have been investigated by Chopra [6], Longyear [8], Rafter and Seiden [10], and Saha et. al. [13]. In this paper we obtain similar results for arrays with $t = 4$, but these results can be easily generalized to B-arrays with $t = 2l$ resulting in a notation which is both messy and cumbersome. For B-arrays with $t = 2l + 1$, similar results can be obtained by considering them as arrays with $t = 2l$.

2. Main Results with Discussion

Lemma 2.1. *A B-array τ of strength four and with $m = 4$ always exists.*

Lemma 2.2. *A B-array $\tau(m, N; \underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4))$ is also of strength t' where $0 < t' \leq 4$. The index sets of τ when considered as an array of strength 3, 2, and 1 are respectively $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $(\beta_0, \beta_1, \beta_3)$, and (γ_0, γ_1) where $\alpha_i = \mu_i + \mu_{i+1} (i = 0, 1, 2, 3)$, $\beta_j = \alpha_j + \alpha_{j+1} (j = 0, 1, 2)$, and $\gamma_k = \beta_k + \beta_{k+1} (k = 0, 1)$.*

Remark. It is quite evident that α_i, β_j, and γ_k are all linear functions of the μ_i's.

Definition 2.1. A B-array $\tau(m \times N)$ is said to be trim if $x_0 = x_m = 0$ where x_j denotes the number of columns of weight j in τ, and τ is called non-trim if at least one of x_0 or $x_m \neq 0$.

Remark. It is obvious that the existence a trim B-array τ with $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ implies the existence of τ^* (τ^* being non-trim) with index set $(\mu_0^*, \mu_1, \mu_2, \mu_3, \mu_4^*)$ satisfying strict inequality in at least one of $\mu_i^* \geq \mu_i (i = 0, 4)$. If we delete from an array τ all vectors of weight o and m, then we obtain an array τ^* which is trim.

Definition 2.2. Two columns of an $(m \times N)$ B-array τ with elements 0 and 1 are said to have $i (0 \leq i \leq m)$ coincides if these columns have 1 occurring in i of the corresponding positions.

Theorem 2.1. *Consider a B-array $\tau(m, N; s = 2, t = 4, \underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4))$. If $\omega(\underline{\alpha}) = l$, $\underline{\alpha}$ being some column, say the first, of τ, then the following results are true;*

$$\sum_{j=0}^{m} x_j = N - 1 = A_0(\text{say}) \tag{2.1}$$

$$\sum_{j=0}^{m} j x_j = \sum_{j=0}^{1} \binom{l}{i}\binom{m-l}{l-i}(\gamma_i - 1) = A_1(\text{say}) \tag{2.2}$$

$$\sum j^2 x_j = 2\sum_{i=0}^{2} \binom{l}{i}\binom{m-l}{2-i}(\beta_i - 1) + \sum_{i=0}^{1} \binom{l}{i}\binom{m-l}{1-i}(\gamma_i - 1) = A_2(\text{say}) \tag{2.3}$$

$$\sum j^3 x_j = 6\sum_{i=0}^{3} \binom{l}{i}\binom{m-l}{3-i}(d_i - 1) + 6\sum_{i=0}^{2} \binom{l}{i}\binom{m-l}{2-i}(\beta_i - 1) + \sum \binom{l}{i}\binom{m-l}{1-i}(\gamma_i - 1) = A_3(\text{say}) \tag{2.4}$$

$$\sum j^4 x_j = 24\sum_{i=0}^{4} \binom{l}{i}\binom{m-l}{4-i}(\mu_i - 1) + 36\sum_{i=0}^{3} \binom{l}{i}\binom{m-l}{3-i}(\alpha_i - 1) + 14\sum_{i=0}^{2} \binom{l}{i}\binom{m-l}{2-i}(\beta_i - 1) + \sum_{i=0}^{1} \binom{l}{i}\binom{m-l}{1-i}(\gamma_i - 1) = A_4(\text{say}) \tag{2.5}$$

where x_j = number of columns of τ other than the first having exactly j coincidences with the first column.

Proof Outline: We can obtain (2.1) through (2.5) from the results given below obtained by counting in two ways the total number of $(4-i)$-tuples appearing in columns other than the first which are identical with the corresponding $(4-i)$-tuples in the first column.

$$\sum_{j=0}^{m} \binom{j}{4} x_j = \sum_{i=0}^{4} \binom{l}{i}\binom{m-l}{4-i}(\mu_i - 1)$$

$$\sum \binom{j}{3} x_j = \sum \binom{l}{i}\binom{m-l}{3-i}(\alpha_i - 1)$$

$$\sum \binom{j}{2} x_j = \sum \binom{l}{i}\binom{m-l}{2-i}(\beta_i - 1)$$

$$\sum \binom{j}{1} x_j = \sum \binom{l}{i}\binom{m-l}{1-i}(\gamma_i - 1)$$

$$\sum x_j = N - 1$$

Next, we state Hölder Inequality for later use.

Hölder Inequality. *If* $x_i, y_i \geq 0, p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum x_i^p\right)^{1/p} \left(\sum y_i^q\right)^{1/q} \quad \text{or} \quad \sum x_i^{1/p} y_i^{1/q} \leq \left(\sum x_i^{1/p} \sum y_i^{1/q}\right)$$

Remark: The above inequality is reversed if $p < 1 (p \neq 0)$. The equality holds if and only if the sets $\{x\}$ and $\{y\}$ are proportional.

Theorem 2.2. *Consider a* B*-array* $\tau(m, N; s = 2, t = 4, \underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4))$. *If* l *is the weight of some* $(m \times 1)$ *column of* τ, *then the following are true:*

(a) $A_3^3 \leq A_1 A_4^2$

(b) $A_2^3 \leq A_1^2 A_4$

(c) $A_2^3 \leq A_0 A_3^2$ *where* A_i*'s are as defined in theorem 2.1, and are functions of* l, m, *and* $\underline{\mu}'$.

Proof: In Hölder Inequality, choose $p = 3$, then $q = \frac{3}{2}$. To obtain the result in (a), we pick x_i and jx_j and $y_i = j^4 x_j$ and use Hölder Inequality,

$$\sum (jx_j)^{1/3} (j^4 x_j)^{2/3} \leq \left(\sum jx_j\right)^{1/3} \left(\sum j^4 x_j\right)^{2/3}$$

$$\sum j^3 x_j \leq \sqrt[3]{\sum jx_j} \sqrt[3]{\left(\sum j^4 x_j\right)^2}$$

Raising both sides to the power 3, we obtain

$$\left(\sum j^3 x_j\right)^3 \leq \sum jx_j \left(\sum j^4 x_j\right)^2 \quad \text{i.e. } A_3^3 \leq A_1 A_4^2$$

(b) If we choose $x_i = j^4 x_j, y_i = jx_j$, we obtain $\sum j^2 x_j \leq \left(\sum j^4 x_j\right)^{\frac{1}{3}} \left(\sum jx_j\right)^{\frac{2}{3}}$ and the result follows.

(c) Here we let $x_i = x_j, y_i = j^3 x_j$, and we have $\sum j^2 x_j \leq \left(\sum x_j\right)^{\frac{1}{3}} \sqrt[3]{\left(\sum j^3 x_j\right)^2}$ and we obtain the result.

Remark: From (b) and (c), it is clear that $A_2^3 \leq \min(A_1^2 A_4, A_0 A_3^2)$.

Theorem 2.3. *If there exists a B-array τ of size $(m \times N)$ with $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ and having a column vector with l 1's in it, then we must have:*

(a) $A_3^4 \leq A_0 A_4^3$

(b) $A_1^4 \leq A_0^3 A_4$ *where A_i's are functions of l, m, and $\underline{\mu}'$.*

Proof: In Hölder Inequality, take $p = 4$, and then $q = \frac{4}{3}$. To prove (a), we choose $x_i = x_j$, and $y_i = j^4 x_j$ and using Hölder Inequality, one obtains

$$\sum j^3 x_j \leq \left(\sum x_j\right)^{1/4} \left(\sum j^4 x_j\right)^{3/4}$$

$$\text{i.e. } \left(\sum j^3 x_j\right)^4 \leq \sum x_j \left(\sum j^4 x_j\right)^3 \quad \text{i.e. } A_3^4 \leq A_0 A_4^3$$

(b) Similarly, here we take $x_i = j^4 x_j$, and $j_i = x_j$ which will prove this part.

Remark. The results of theorems 2.2 and 2.3 are quite useful in discussing the existence of B-arrays τ for given $\underline{\mu}'$ provided some information on l (the number of 1's in some column of τ) is available (in the absence of information on l, one can always attach a vector of weight 0 or of m to τ, and deal with the existence of this new array. For a given $\underline{\mu}'$ we can prepare a computer program to check the results of theorems 2.2 and 2.3 for values of $m \geq 5$. If any one of these results is contradicted for $m = m^*$(say), then τ may exist for values of m satisfying $5 \leq m < m^*$. Consequently the maximum number of constraints for such an array τ is $m^* - 1$.

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