# Pasch decompositions of lambda-fold triple systems

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**Abstract**

We find the set of integers $v$ for which $\lambda K_v$ may be decomposed into sets of four triples forming Pasch configurations, for all $\lambda$. We also remove the remaining exceptional values of $v$ for decomposing $K_v$ into sets of other four-triple configurations.

## 1 Introduction

A *λ-fold triple system*, or more simply, triple system, is a pair $(V, B)$ where $V$ is a $v$-set of elements (or points) and $B$ is a collection of 3-subsets of $V$ (called *triples* or *lines*) such that every unordered pair of elements chosen from $V$ belongs to precisely $\lambda$ triples. (The case $\lambda = 1$ is of course a Steiner triple system of order $v$, or $STS(v)$.) The cardinality $v$ of $V$ is called the *order* of the triple system.

We may also think of a triple system as a decomposition of the complete multigraph $\lambda K_v$ into a collection of triples, so that if $x$ and $y$ are any two vertices then the edge $\{x, y\}$ occurs in precisely $\lambda$ triples.

A *partial* triple system is a collection of triples chosen from a $v$-set $V$ so that each unordered pair of elements from $V$ belongs to *at most* one triple. We shall use the term *configuration* to describe any partial triple system with a small number of triples. One such configuration is known as the ***Pasch configuration*** that is defined by the four lines or triples $\{a,b,c\}, \{a,d,e\}, \{b,d,f\}$ and $\{c,e,f\}$ with vertex set $\{a,b,c,d,e,f\}$. We shall denote such a Pasch configuration by $P(a,f;b,e;c,d)$ (or $P(b,e;a,f;c,d)$, etc.); the only pairs of points from the six not contained in any of the four triples are $\{a,f\}, \{b,e\}$ and $\{c,d\}$.

If in a triple system $(V, B)$ we can partition $B$ into $|B|/4$ Pasch configurations, we call this a ***Pasch-decomposition*** of the triple system.

Several recent papers, such as [4], [2] and [3], have considered decompositions of $STS$'s into small configurations. Both the problem of taking *any* $STS(v)$ and determining whether it might be decomposable into certain configurations, and also the problem of *constructing* a $STS(v)$ which may be decomposed appropriately, have been considered. In this paper we restrict our attention to the Pasch configuration described above, and deal with the latter problem, of *existence* of a $\lambda$*-fold* triple system which may be decomposed into Pasch configurations.

Of the possible 16 four-line configurations (where lines are triples), only one, the Pasch configuration, is on six points. See [3] for details of these 16 configurations. In that paper the authors solve the problem of existence of an $STS(v)$ which may be decomposed into any four-line configuration, except for the value $v = 81$ for six of the configurations (in their notation, $C_6, C_{10}, C_{11}, C_{12}, C_{14}$ and $C_{16}$, the Pasch one). We give these in the Appendix (the five non-Pasch ones at the end).

It is clear that for a $\lambda$-fold triple system of order $v$ to be decomposable into Pasch configurations, $\lambda\binom{v}{2}$ must be divisible by 12 and $\lambda(v-1)$ must be divisible by 4. These translate into the necessary conditions given in Table 1.

| $\lambda$ | $v$ |
|---|---|
| 1, 5 (mod 6) | 1, 9 (mod 24) |
| 3 (mod 6) | 1 (mod 8) |
| 2, 10 (mod 12) | 1, 9 (mod 12) |
| 4, 8 (mod 12) | 0, 1 (mod 3) |
| 6 (mod 12) | 1 (mod 4) |
| 0 (mod 12) | all $v \geq 6$ |

**Table 1:** Necessary conditions for a Pasch-decomposition of $\lambda K_v$.

We shall now show existence of a triple system with a decomposition into Pasch configurations whenever the above necessary conditions hold, *except* that the unique $STS(9)$ cannot be so decomposed, as stated in [3].

For completeness we include the case $\lambda = 1$ ([3]). All our constructions use a decomposition of the complete tri-partite graph $K_{2,2,2}$ into a Pasch configuration; we give this now. Let $K_{2,2,2}$ have elements $\{a_1, a_2\} \cup \{b_1, b_2\} \cup \{c_1, c_2\}$. Then the triples

$$\{a_1, b_1, c_1\}, \{a_1, b_2, c_2\}, \{a_2, b_1, c_2\}, \{a_2, b_2, c_1\}$$

form a Pasch configuration; recall that we may also denote these four triples by $P(a_1, a_2; b_1, b_2; c_1, c_2)$.

Now our basic construction is as follows. If $v = 2s + h$ where $h \geq 0$, we require a group divisible design $GD(3, \lambda, M; s)$ where $M = \{g\}$ or $\{g, f^*\}$ (the asterisk means *one* group of size $f$), and decompositions into Pasch configurations of $\lambda K_{2g+h}, \lambda K_{2f+h}$ and (if $h > 1$) $\lambda K_{2g+h} \backslash \lambda K_h$, which is the complete multigraph on $2g + h$ vertices with a "hole" of size $h$, so that all the $\lambda \binom{h}{2}$ edges on some set of vertices of size $h$ are removed. The required group divisible designs ($GDD$s) are either well-known, or obtained from [1]. We also require decompositions of $\lambda K_{2s+h}$ for small values of $s$, if no $GD(3, \lambda, M; s)$ exists for small $s$. Then our decomposition of $K_v$ is composed as follows, on the element set $\{(i, j) | 0 \leq i \leq s, j = 1, 2\} \cup \{\infty_i \mid 1 \leq i \leq h\}$.

(1) On the set $\mathbb{Z}_s$ take a $GD(3, \lambda, M; s)$ where $M = \{g\}$ or $\{g, f^*\}$.

(2) If $h = 0$ or 1, for each group $\{i_1, i_2, \ldots, i_w\}$ of the $GDD$, where $w = g$ or $f$, place on the set

$$S = \{(i_1, j), (i_2, j), \ldots, (i_w, j) \mid j = 1, 2\}$$

(if $h = 0$) or $S \cup \{\infty\}$ if $h = 1$, a decomposition of $\lambda K_{2g+h}$ or $\lambda K_{2f+h}$ into Pasch configurations.
OR

(2)′ If $h \geq 2$, for *one* group of the $GDD$ (of size $f$ if $M = \{g, f^*\}$), place on the set $S \cup \{\infty_i\}_{i=1}^{h}$ a decomposition of $\lambda K_{2f+h}$ into Pasch configurations. For the remaining groups, all of size $g$, place on $S \cup \{\infty_i\}_{i=1}^{h}$ a decomposition of $(\lambda K_{2g+h} \backslash \lambda K_h)$ into Pasch configurations.

(3) For each block $\{i_1, i_2, i_3\}$ of the $GDD$, on the set $\{(i_1, 1), (i_1, 2)\} \cup \{(i_2, 1), (i_2, 2)\} \cup \{(i_3, 1), (i_3, 2)\}$, place a Pasch configuration of $K_{2,2,2}$, described earlier.

The collection of Pasch configurations in (2) and (3) (if $h \leq 1$) or in (2)′ (if $h > 1$) and (3) gives a suitable decomposition of $K_v$.

## 2 The cases $\lambda = 1$ and 2

For $\lambda = 1$ (done in [3]), $v = 24s+1$ or $24s+9$. In the former case we use a $GD(3,1,12;12s), s \geq 3$, and decompositions of $K_{25}$ and $K_{49}$. In the latter case we use a $GD(3,1,\{12,16^*\};12s+4), s \geq 4$ ([1]), together with decompositions of $K_{25}, K_{33}, K_{57}$ and $K_{81}$; these are all given below.

**Lemma 2.1** *For each $v \in \{25,33,49,57,81\}$, there exists a $STS(v)$ which can be decomposed into Pasch configurations.*

Proof: For $v = 25$, with point set $\mathbf{Z}_{25}$, take the following Pasch configurations (addition is modulo 25).

$$\{P(0,13;1,2;6,10)+i \mid 0 \leq i \leq 24\}.$$

For $v = 33$, with point set $\{(x,y) \mid x \in \mathbf{Z}_{11}, y = 1,2,3\}$, the following Pasch configurations decompose $K_{33}$:

$$\begin{array}{l} P((0,1),(0,3);(1,1),(0,2);(3,1),(5,1))+(i,0), \\ P((0,2),(10,3);(1,2),(6,1);(7,2),(9,2))+(i,0), \\ P((0,3),(8,2);(1,3),(9,1);(3,3),(5,3))+(i,0), \\ P((0,1),(9,1);(2,2),(7,2);(1,3),(9,3))+(i,0), \end{array}$$

where $0 \leq i \leq 10$ and addition is modulo 11.

For $v = 49$, with point set $\mathbf{Z}_{49}$, we may take the Pasch configurations:

$$\{P(0,22;1,12;3,5)+i, P(0,42;6,8;22,31)+i \mid i \in \mathbf{Z}_{49}\}.$$

For $v = 57$, on the point set $\mathbf{Z}_{19} \times \{1,2,3\}$, and with $q_i$ as below, we may take the Pasch configurations $\bigcup_{\substack{i=1 \\ j \in \mathbf{Z}_{19}}}^{7} \{q_i + (j,0)\}$ (addition modulo 19).

$$\begin{array}{rcl} q_1 & = & P((0,2),(7,3);(6,2),(10,1);(6,3),(11,3)), \\ q_2 & = & P((0,3),(7,3);(10,2),(10,3);(12,1),(16,1)), \\ q_3 & = & P((0,3),(14,3);(2,3),(6,3);(4,2),(7,2)), \\ q_4 & = & P((0,1),(6,1);(1,1),(6,3);(2,2),(12,2)), \\ q_5 & = & P((0,1),(7,3);(8,1),(2,3);(17,1),(18,2)), \\ q_6 & = & P((0,1),(18,1);(3,1),(6,1);(3,2),(11,3)), \\ q_7 & = & P((0,2),(17,2);(1,2),(7,2);(12,1),(15,2)). \end{array}$$

For $v = 81$, on the point set $\mathbf{Z}_{27} \times \{1,2,3\}$, and with $q_i$ as below, the set $\bigcup_{\substack{i=1 \\ j \in \mathbf{Z}_{27}}}^{10} \{q_i + (j,0)\}$, addition modulo 27, is a Pasch-decomposition.

$$q_1 \;=\; P((0,1),(23,1);(10,2),(23,2);(17,2),(22,3)),$$

$$
\begin{aligned}
q_2 &= P((0,1),(8,3);(3,2),(11,1);(13,2),(7,3)),\\
q_3 &= P((0,1),(25,1);(7,2),(4,1);(4,3),(26,2)),\\
q_4 &= P((0,1),(12,1);(7,1),(5,2);(4,2),(18,2)),\\
q_5 &= P((0,1),(2,3);(8,2),(24,1);(10,1),(12,2)),\\
q_6 &= P((0,1),(24,3);(2,3),(3,3);(16,2),(15,1)),\\
q_7 &= P((0,1),(17,1);(8,3),(11,3);(18,1),(25,1)),\\
q_8 &= P((0,1),(10,2);(1,3),(12,3);(16,3),(25,3)),\\
q_9 &= P((0,2),(14,3);(3,2),(25,3);(18,2),(25,2)),\\
q_{10} &= P((0,2),(12,3);(3,3),(20,3);(11,2),(10,3)).
\end{aligned}
$$

A crucial decomposition required when $\lambda = 2$ is one of $2K_9$; recall that $K_9$ has no Pasch decomposition.

**Lemma 2.2** *$2K_9$ has a decomposition into Pasch configurations.*

Proof: Let the element set be $\mathbf{Z}_3 \times \mathbf{Z}_3$. A decomposition is:

$$
\begin{aligned}
\{&P((0,0),(2,2);(0,1),(1,0);(0,2),(2,1)) + (i,0),\\
&P((0,0),(2,2);(0,1),(2,0);(1,1),(1,2)) + (i,0) \mid 0 \le i \le 2\}.
\end{aligned}
$$

Now when $v = 12s+1$, we use two copies of a $GD(3,1,6;6s), s \ge 3$, and Pasch decompositions of $2K_{13}$ and $2K_{25}$. When $v = 12s+9$, we use two copies of a $GD(3,1,\{6,4^*\};6s+4), s \ge 3$, together with Pasch decompositions of $2K_9, 2K_{21}$ and $2K_{33}$. Obviously we may use our decompositions of $K_{25}$ and $K_{33}$ in Lemma 2.1; the other ones are given below.

**Lemma 2.3** *There are Pasch-decompositions of $2K_{13}$ and $2K_{21}$.*

Proof: For $2K_{13}$ we use the point set $\mathbf{Z}_{13}$, and the Pasch configurations $\{q+i \mid 0 \le i \le 12\}$ where addition is modulo 13, and

$$q = P(0,10;1,7;2,3).$$

For $2K_{21}$, we use the point set $\mathbf{Z}_7 \times \{1,2,3\}$. Then, with $q_i$ as below, $\bigcup_{\substack{i=1\\ j\in\mathbf{Z}_7}}^{5} \{q_i + (j,0)\}$ is a Pasch-decomposition of $2K_{21}$.

$$
\begin{aligned}
q_1 &= P((0,1),(6,3);(1,1),(5,1);(2,2),(6,2)),\\
q_2 &= P((0,1),(5,2);(3,1),(0,2);(2,1),(3,3)),\\
q_3 &= P((0,1),(5,3);(0,2),(6,2);(3,1),(0,3)),\\
q_4 &= P((0,2),(5,3);(3,2),(2,3);(1,2),(6,2)),\\
q_5 &= P((0,3),(3,3);(1,3),(6,3);(2,1),(4,1)).
\end{aligned}
$$

This completes the case $\lambda = 2$.

# 3 The cases $\lambda = 3, 4, 6$ and 12.

From now on we shall list necessary Pasch-decompositions in the Appendix, as the reader has probably seen enough to get the picture!

For $\lambda = 3$ we have $v = 8s + 1$, and use three copies of a $GD(3, 1, 4; 4s)$ if $s \geq 3$ and $s \equiv 0$ or 1 (mod 3), or three copies of a $GD(3, 1, \{4, 8^*\}; 4s)$ if $s \geq 5$ and $s \equiv 2$ (mod 3). Then Pasch-decompositions of $3K_9$ and $3K_{17}$ are also required; see the Appendix. This completes the case $\lambda = 3$.

When $\lambda = 4, v \equiv 0$ or 1 (mod 3): we write $v = 6s + h$ where $h = 0, 1, 3$ or 4. Then we use two copies of a $GD(3, 2, 3; 3m), m \geq 3$, together with Pasch-decompositions of $4K_v$ as follows:

| $h$ | $v$ |
|---|---|
| 0 | 6, 12 |
| 1 | 7, 13 |
| 3 | 9, 9[3], 15 |
| 4 | 10, 10[4], 16 |

(Here 9[3] means $4(K_9 \backslash K_3)$, and 10[4] likewise.) For $v = 9$ and 13, see Lemmas 2.2, 2.3; for the rest, see the Appendix.

When $\lambda = 6, v \equiv 1$ (mod 4): if $v \equiv 1$ (mod 8), we may take two copies of a Pasch-decomposition of $3K_v$, so assume that $v \equiv 5$ (mod 8) and let $v = 8s + 5$. If $s \equiv 1$ or 2 (mod 3) then $v \equiv 1$ or 9 (mod 12) and we may take three copies of a decomposition of $2K_v$. So let $s \equiv 0$ (mod 3) and write $v = 24S + 5$. We use a $GD(3, 1, \{6, 8^*\}; 12S + 2)$ when $S \geq 2$, and a Pasch decomposition of $6K_{29}$, given in the Appendix.

When $\lambda = 12$, the only cases left to consider are $v \equiv 2, 8$ or 11 (mod 12). When $v = 12s + 2$ we use twelve copies of a $GD(3, 1, \{3, 7^*\}; 6s + 1)$ for $s \geq 3$, and Pasch-decompositions of $12K_{14}$ and $12K_{26}$. When $v = 12s + 8$, we use twelve copies of a $GD(3, 1, \{6, 4^*\}; 6s + 4)$ for $s \geq 3$, and Pasch-decompositions of $12K_8, 12K_{20}$ and $12K_{32}$. Finally when $v = 12s + 11$ we use a twelve-fold $GD(3, 1, \{3, 5^*\}; 6s + 5)$ for $s \geq 2$, and Pasch-decompositions of $12K_{11}$ and $12K_{23}$. See the Appendix for all these Pasch-decompositions.

# 4 Summary

Having dealt with $\lambda K_v$ for $\lambda = 1, 2, 3, 4, 6$ and 12, it can easily be checked from the necessary conditions in Table 1 that all other values of $\lambda$ may be dealt with by combining Pasch-decompositions for smaller values of $\lambda$. Thus we have proved:

**Theorem 4.1** *There exists a $\lambda$-fold triple system of order $v$ which can be decomposed into Pasch-configurations if and only if $\lambda$ and $v$ satisfy the necessary conditions in Table 1, and $(\lambda, v) \neq (1,9)$.*


# Acknowledgements

CAR thanks the University of Queensland for its hospitality and support from the Ethel Raybould Fellowship during the work on this paper, while EJB thanks Auburn University for its hospitality during the writing of this paper. Thanks also to Alex Rosa for supplying recent preprints.

Research supported by the Australian Research Council (EJB) and NSF grant DMS-9024645 (CAR).


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# Appendix

$3K_9$: $(\mathbf{Z}_9, \mathrm{B})$ where $B = \{P(0,6;1,8;2,3) + i \mid i \in \mathbf{Z}_9\}$.
$3K_{17}$: $(\mathbf{Z}_{17}, \mathrm{B})$ where

$$B = \{P(0,9;1,5;2,3) + i, P(0,12;1,3;7,8) + i \mid i \in \mathbf{Z}_{17}\}.$$

$4K_6$: $(\mathbf{Z}_6, \mathrm{B})$ where $B$ is the complement of a 1-factorization of $K_6$ (for example, the 1-factor $\{0,1\},\{2,3\},\{4,5\}$ yields the Pasch configuration $P(0,1;2,3;4,5)$, and similarly for the other four 1-factors).
$4K_7$: $(\mathbf{Z}_7, \mathrm{B})$ where $B = \{P(1,6;2,5;3,4) + i \mid i \in \mathbf{Z}_7\}$.
$4K_{12}$: $(\mathbf{Z}_{11} \cup \{\infty\}, \mathrm{B})$ where

$$B = \{P(0,6;1,4;2,3) + i, P(\infty,9;1,10;4,5) + i \mid i \in \mathbf{Z}_{11}\}.$$

$4K_{15}$: $((\mathbf{Z}_7 \times \{1,2\}) \cup \{\infty\}, B)$ where $B = \bigcup_{i=1}^{5} \{P_i + (j,0) \mid j \in \mathbf{Z}_7\}$;

$$\begin{aligned}
P_1 &= P((0,1),(4,2);(3,1),(3,2);(4,1),(5,2)),\\
P_2 &= P((0,1),(6,1);(1,1),(4,2);(1,2),(4,1)),\\
P_3 &= P((0,1),(5,2);(1,1),(1,2);(0,2),(2,1)),\\
P_4 &= P((\infty,(4,2);(0,1),(0,2);(2,1),(2,2)),\\
P_5 &= P((\infty,(4,2);(0,1),(5,1);(1,2),(3,2)).
\end{aligned}$$

$4K_{16}$: $((\mathbf{Z}_8 \times \{1,2\}, B)$ where $B = \bigcup_{i=1}^{5} \{P_i + (j,0) \mid j \in \mathbf{Z}_8\}$;

$$\begin{aligned}
P_1 &= P((0,1),(7,1);(1,1),(6,2);(2,1),(7,2)),\\
P_2 &= P((0,1),(3,1);(3,2),(6,1);(5,1),(7,2)),\\
P_3 &= P((0,1),(6,2);(1,1),(1,2);(0,2),(4,1)),\\
P_4 &= P((0,1),(6,2);(0,2),(1,2);(4,1),(3,2)),\\
P_5 &= P((0,2),(3,2);(1,2),(7,2);(5,1),(6,1)).
\end{aligned}$$

$4(K_9 \backslash K_3)$: Element set $\{1,2,\ldots,9\}$ with hole $\{1,2,3\}$. A Pasch-decomposition is:

$$\begin{aligned}
\{&P(1,2;4,5;6,7), P(1,2;6,7;8,9), P(1,2;4,5;8,9),\\
&P(1,3;4,6;5,7), P(1,3;6,8;7,9), P(1,3;4,8;5,9),\\
&P(2,3;4,7;5,6), P(2,3;6,9;7,8), P(2,3;4,9;5,8),\\
&P(4,5;6,7;8,9), P(4,5;6,7;8,9)\}.
\end{aligned}$$

$4(K_{10} \backslash K_4)$: Element set $\mathbf{Z}_6 \cup H$ where the hole $H$ is $\{\infty_1,\infty_2,\infty_3,\infty_4\}$. A Pasch-decomposition is:

$$\begin{aligned}
\{&P(\infty_1,\infty_2;0,4;1,3)+i, P(\infty_1,\infty_2;0,3;1,4)+i,\\
&P(\infty_3,\infty_4;0,3;1,4)+i, P(\infty_3,\infty_4;0,1;3,4)+i,\\
&P(0,3;1,4;2,5) \mid 0 \le i \le 2\}.
\end{aligned}$$

$6K_{29}$: $(\mathbf{Z}_{29}, B)$ where

$$\begin{aligned}
B = \{&P(0,20;1,2;10,12)+i, P(0,17;1,16;21,24)+i,\\
&P(0,16;1,6;22,25)+i, P(0,7;1,5;2,3)+i, P(0,20;2,17;5,6)+i,\\
&P(0,19;3,26;12,15)+i, P(0,26;4,6;13,18)+i \mid i \in \mathbf{Z}_{29}\}.
\end{aligned}$$

$12K_8$: $(\mathbf{Z}_7 \cup \{\infty\}, B)$ where $B = \bigcup_{i=1}^{4} \{P_i + j \mid j \in \mathbf{Z}_7\}$;

$$\begin{aligned}
P_1 &= P(0,5;1,6;3,4), P_2 = P(\infty,4;1,3;0,2),\\
P_3 &= P(\infty,5;1,2;0,3), P_4 = P(\infty,5;1,2;0,3).
\end{aligned}$$

$12K_{11}$: $(\mathbf{Z}_{11}, B)$ where $B = \bigcup_{i=1}^{5} \{P_i + j \mid j \in \mathbf{Z}_{11}\}$;

$$\begin{aligned}
P_1 &= P(0,5;1,4;2,3), P_2 = P(0,5;1,4;2,3), P_3 = P(0,9;1,3;4,6),\\
P_4 &= P(0,10;1,4;5,6), P_5 = P(0,9;2,3;6,7).
\end{aligned}$$

$12K_{14}$: $(\mathbf{Z}_{13} \cup \{\infty\}, B); B = \bigcup_{i=1}^{7} \{P_i + j \mid j \in \mathbf{Z}_{13}\}$;

$$\begin{aligned} P_1 &= P(0,6;1,2;3,9), P_2 = P(0,2;1,9;7,12), \\ P_3 &= P(0,5;1,4;2,3), P_4 = P(0,9;1,4;2,3), \\ P_5 &= P(\infty,8;0,1;5,6), P_6 = P(\infty,9;0,1;5,6), \\ P_7 &= (\infty,10;0,1;6,7). \end{aligned}$$

$12K_{20}$: $(\mathbf{Z}_{19} \cup \{\infty\}, B); B = \bigcup_{i=1}^{10} \{P_i + j \mid j \in \mathbf{Z}_{19}\}$;

$$\begin{aligned} &P_1 = P(0,17;1,12;5,11), P_2 = P(0,3;1,16;5,12), \\ &P_3 = P(0,9;1,14;3,12), P_4 = P(0,17;1,13;6,11), \\ &P_5 = P(0,8;1,17;5,18), P_6 = P(0,5;1,4;2,3), \\ &P_7 = P(0,12;2,9;5,6), P_8 = P(\infty,10;1,14;4,5), \\ &P_9 = P(\infty,13;0,1;8,9), P_{10} = P(\infty,14;0,1;8,9). \end{aligned}$$

$12K_{23}$: $(\mathbf{Z}_{23}, B); B = \bigcup_{i=1}^{11} \{P_i + j \mid j \in \mathbf{Z}_{23}\}$;

$$\begin{aligned} &P_1 = P(0,11;1,19;7,12), P_2 = P(0,16;1,21;3,8), \\ &P_3 = P(0,2;1,22;12,15), P_4 = P(0,8;1,11;13,14), \\ &P_5 = P(0,7;1,14;5,13), P_6 = P(0,6;1,4;2,3), \\ &P_7 = P(0,9;2,7;4,5), P_8 = P(0,13;3,10;6,7), \\ &P_9 = P(0,20;4,14;8,9), P_{10} = P(0,20;4,6;12,15), \\ &P_{11} = P(0,22;5,8;13,14). \end{aligned}$$

$12K_{26}$: $(\mathbf{Z}_{25} \cup \{\infty\}, B); B = \bigcup_{i=1}^{13} \{P_i + j \mid j \in \mathbf{Z}_{25}\}$;

$$\begin{aligned} &P_1 = P(0,4;1,3;13,15), P_2 = P(0,8;1,4;5,24), \\ &P_3 = P(0,5;1,8;16,17), P_4 = P(0,12;1,23;18,21), \\ &P_5 = P(0,10;1,7;11,16), P_6 = P(0,3;1,20;14,17), \\ &P_7 = P(0,14;1,22;5,24), P_8 = P(0,9;2,7;4,5), \\ &P_9 = P(0,11;2,8;4,5), P_{10} = P(0,23;4,17;10,11), \\ &P_{11} = P(\infty,13;1,19;6,7), P_{12} = P(\infty,16;1,22;8,11), \\ &P_{13} = P(\infty,16;1,23;8,11). \end{aligned}$$

$12K_{32}$: $(\mathbf{Z}_{31} \cup \{\infty\}, B); B = \bigcup_{i=1}^{16} \{P_i + j \mid j \in \mathbf{Z}_{31}\}$

$$\begin{aligned} &P_1 = P(0,4;1,26;18,28), P_2 = P(0,17;1,14;5,16), \\ &P_3 = P(0,13;1,18;24,30), P_4 = P(0,23;1,11;16,27), \\ &P_5 = P(0,13;1,20;8,9), P_6 = P(0,23;1,18;5,6), \\ &P_7 = P(0,8;1,23;7,16), P_8 = P(0,22;1,6;8,30), \\ &P_9 = P(0,30;2,8;4,16), P_{10} = P(0,9;2,7;4,5), \\ &P_{11} = P(0,20;2,12;6,8), P_{12} = P(0,16;3,25;6,13), \\ &P_{13} = P(0,23;3,10;13,20), P_{14} = P(\infty,21;1,27;7,10), \\ &P_{15} = P(\infty,19;1,28;7,10), P_{16} = P(\infty,21;0,1;10,11). \end{aligned}$$

**Decompositions of $K_{81}$ into configurations $C_6, C_{10}, C_{11}, C_{12}$ and $C_{14}$** (see [3]). All use the element set $\mathbf{Z}_{27} \times \{1,2,3\}$, and are starter configurations mod $(27,-)$.

$C_6$: {{(0,1),(11,2),(5,3)},{(5,3),(2,2),(3,2)},{(0,1),(3,2),(18,3)},
{(15,3),(25,3),(9,2)}},{{(0,1),(4,2),(17,2)},{(17,2),(3,3),(25,3)},
{(0,1),(25,3),(16,3)},{(26,3),(12,1),(14,1)}},{{(0,1),(6,2),(13,2)},
{(13,2),(23,3),(16,2)},{(0,1),(16,2),(25,2)},{(3,1),(2,3),(5,3)}},
{{(0,1),(5,2),(9,1)},{(9,1),(1,2),(6,1)},{(0,1),(6,1),(15,3)},
{(20,3),(12,3),(24,3)}},{{(0,1),(7,2),(13,1)},{(13,1),(9,3),(20,3)},
{(0,1),(20,3),(9,2)},{(19,1),(0,3),(25,3)}},{{(0,1),(14,2),(17,1)},
{17,1),(25,1),(8,2)},{(0,1),(8,2),(20,1)},{(1,1),(2,1),(12,3)}},
{{(0,1),(0,2),(17,3)},{(17,3),(16,2),(23,1)},{(0,1),(23,1),(0,3)},
{(1,2),(5,3),(2,1)}},{{(0,1),(1,2),(16,1)},{(16,1),(21,1),(13,3)},
{(0,1),(13,3),(12,1)},{(1,1),(3,2),(23,3)}},{{(0,2)(2,2)(6,2)},
{(6,2),(1,2),(16,2)},{(0,2),(16,2),(25,3)},{(3,2),(11,2),(8,3)}},
{{(0,3),(6,3),(15,2)},{(15,2),(7,3),(14,3)},{(0,3),(14,3),(0,2)},
{(1,2),(23,3),(24,3)}}.

$C_{10}$: {{(0,1),(17,2),(10,2)},{(10,2),(23,3),(22,3)},{(22,3),(23,2),(13,3)},
{(0,1),(13,3),(3,3)}},{{(0,1),(11,2),(7,2)},{(7,2),(8,1),(7,3)},
{(7,3),(26,1),(4,3)},{(0,1),(4,3),(25,1)}},{{(0,1),(14,2),(3,2)},
{(3,2),(0,3),(5,1)},{(5,1),(5,2),(20,2)},{(0,1),(20,2),(16,3)}},
{{(0,1),(23,2),(10,1)},{(10,1),(7,2),(5,1)},{(5,1),(22,3),(21,2)},
{(0,1),(21,2),(18,2)}},{{(0,1),(1,2),(16,1)},{(16,1),(9,1),(24,1)},
{(24,1),(7,3),(21,3)},{(0,1),(21,3),(1,1)}},{{(0,1),(5,2),(19,2)},
{(19,2),(15,1),(11,1)},{(11,1),(20,1),(14,1)},{(0,1),(14,1),(9,2)}},
{{(0,1),(6,2),(0,3)},{(0,3),(2,2),(7,2)},{(7,2),(1,2),(9,3)},
{(0,1),(9,3),(1,3)}},{{(0,1),(2,3),(7,3)},{(7,3),(9,1),(0,3)},
{(0,3),(8,2),(11,3)},{(0,1),(11,3),(23,3)}},{{(0,1),(12,3),(14,3)},
{(14,3),(0,2),(26,2)},{(26,2),(9,2),(15,3)},{(0,1),(15,3),(19,3)}},
{{(0,2),(2,2),(9,3)},{(9,3),(5,2),(14,2)},{(14,2),(22,2),(5,3)},
{(0,2),(5,3),(11,3)}}.

$C_{11}$: {{(0,1),(12,2),(16,3)},{(16,3),(8,2),(17,3)},{(17,3),(20,3),(20,2)},
{(16,3),(20,2),(2,1)}},{{(0,1),(6,2),(26,2)},{(26,2),(8,2),(10,1)},
{(10,1),(13,2),(3,2)},{(26,2),(3,2),(25,1)}},{{(0,1),(9,2),(12,3)},
{(12,3),(20,1),(25,3)},{(25,3),(7,2),(0,3)},{(12,3),(0,3),(9,1)}},
{(0,1),(17,2),(14,2)},{(14,2),(8,2),(20,3)},{(20,3),(1,3),(9,2)},
{(14,2),(9,2),(22,2)}},{{(0,1),(13,2),(21,1)},{(21,1),(9,1),(4,2)},
{(4,2),(26,3),(24,1)},{(21,1),(24,1),(10,1)}},{{(0,1),(8,2),(21,3)},
{(21,3),(20,2),(22,2)},{(22,2),(10,2),(11,2)},{(21,3),(11,2),(0,3)}},
{{(0,1),(0,2),(7,3)},{(7,3),(1,1),(3,1)},{(3,1),(0,3),(7,1)},
{(7,3),(7,1),(8,1)}},{{(0,1),(2,2),(5,1)},{(5,1),(13,1),(22,1)},
{(22,1),(15,1),(26,2)},{(5,1),(26,2),(16,3)}},{{(0,1),(15,2),(17,3)},
{(17,3),(7,1),(3,2)},{(3,2),(18,3),(8,3)},{(17,3),(8,3),(22,1)}},
{{(0,1),(1,3),(23,3)},{(23,3),(8,1),(16,3)},{(16,3),(18,1),(0,3)},
{(23,3),(0,3),(2,2)}}.

$C_{12}$: {{(0,1),(22,2),(11,1)},{(11,1),(21,2),(5,3)},{(0,1),(5,3),(6,1)}, {(22,2),(26,3),(7,1)}},{{(0,1),(25,2),(14,2)},{(14,2),(11,1),(2,1)}, {(0,1),(2,1),(18,3)},{(25,2),(0,3),(3,3)}},{{(0,1),(21,2),(6,2)}, {(6,2),(15,1),(23,2)},{(0,1),(23,2),(5,2)},{(21,2),(12,1),(10,3)}}, {{(0,1),(24,2),(5,1)},{(5,1),(18,2),(17,1)},{(0,1),(17,1),(7,2)}, {(24,2),(8,1),(4,1)}},{{(0,1),(0,2),(22,3)},{(22,3),(13,1),(26,1)}, {(0,1),(26,1),(7,1)},{(0,2),(20,2),(22,2)}},{{(0,1),(26,2),(24,1)}, {(24,1),(1,2),(0,3)},{(0,1),(0,3),(7,3)},{(26,2),(7,2),(22,3)}}, {{(0,1),(1,3),(2,3)},{(2,3),(1,2),(4,3)},{(0,1),(4,3),(10,3)}, {(1,3),(4,1),(17,3)}},{{(0,1),(8,3),(20,3)},{(20,3),(3,1),(15,3)}, {(0,1),(15,3),(11,3)},{(8,3),(2,1),(16,3)}},{{(0,2),(1,2),(7,3)}, {(7,3),(10,2),(14,2)},{(0,2),(14,2),(8,3)},{(1,2),(4,2),(18,3)}}, {{(0,2),(6,2),(19,3)},{(19,3),(1,2),(10,3)},{(0,2),(10,3),(0,3)}, {(6,2),(4,3),(18,3)}}.

$C_{14}$: {{(0,1),(1,2),(17,3)},{(17,3),(22,2),(0,2)},{(0,1),(0,2),(20,3)}, {(22,2),(20,3),(1,1)}},{{(0,1),(26,2),(6,3)},{(6,3),(18,3),(7,2)}, {(0,1),(7,2),(16,2)},{(18,3),(16,2),(2,1)}},{{(0,1),(17,2),(13,3)}, {(13,3),(19,3),(12,3)},{(0,1),(12,3),(3,3)},{(19,3),(3,3),(1,1)}}, {{(0,1),(9,2),(24,1)},{(24,1),(18,1),(22,2)},{(0,1),(22,2),(25,3)}, {(18,1),(25,3),(1,1)}},{{(0,1),(20,2),(8,2)},{(8,2),(11,2),(25,1)}, {(0,1),(25,1),(8,3)},{(11,2),(8,3),(8,1)}},{(0,1),(24,2),(9,3)}, {(9,3),(10,1),(15,2)},{(0,1),(15,2),(23,1)},{(10,1),(23,1),(1,1)}}, {{(0,1),(6,2),(21,3)},{(21,3),(19,3),(11,3)},{(0,1),(11,3),(14,3)}, {(19,3),(14,3),(1,2)}},{{(0,1),(2,2),(11,1)},{(11,1),(18,1),(26,1)}, {(0,1),(26,1),(22,3)},{(18,1),(22,3),(14,2)}},{{(0,1),(11,2),(15,3)}, {(15,3),(23,2),(1,3)},{(0,1),(1,3),(5,3)},{(23,2),(5,3),(22,2)}}, {{(0,2),(4,2),(11,2)},{(11,2),(13,2),(21,2)},{(0,2),(21,2),(0,3)}, {(13,2),(0,3),(26,2)}}.