Interpolation and Related Topics

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Abstract

The basic interpolation theorem states that if graph G contains spanning trees having m and n leaves, with $m < n$, then for each integer k, $m < k < n$, G contains a spanning tree having k leaves. Various generalizations and related topics will be discussed.

1 Introduction

All graphs we consider are finite, undirected graphs without loops or multiple edges. $V(G)$ and $E(G)$ denote the vertex set and the edge set of graph G. The order of G is the number of vertices and the size of G is the number of edges. If H is a subgraph of G then $n_k(H)$ is the number of vertices of degree k in H. If $\mathcal{Q}$ is a family of subgraphs of G then n_k interpolates on $\mathcal{Q}$ if given $g_1, g_2 \in \mathcal{Q}$ and integer j such that $n_k(g_1) < j < n_k(g_2)$ then there exists a subgraph $g \in \mathcal{Q}$ with $n_k(g) = j$.

Definition 1 The pendant polynomial of G is $P_1(G) = \sum_r^s b_k x^k$, where b_k is the number of spanning trees in G with k leaves, $b_r \neq 0$ and $b_s \neq 0$.

Definition 2 The cycle rank of G is $\rho(G) = |E(G)| - |V(G)| + 1$.

Theorem 1 (Schuster, 1983) *n_1 interpolates on the spanning trees of a connected graph.*

Theorem 2 (Heinrich & Liu, 1988) *Given a connected graph G with $P_1(G) = \sum_r^s b_k x^k$ then $b_j \geq 2\rho$, where $r < j < s$.*

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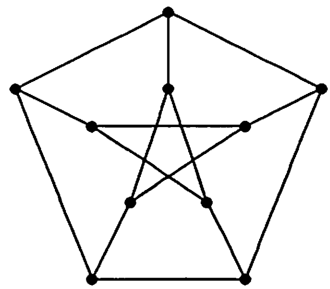
EXAMPLE 1



$$P_1 = 120x^2 + 820x^3 + 810x^4 + 240x^5 + 10x^6$$

EXAMPLE 2

The pendant polynomials of a few well known graphs

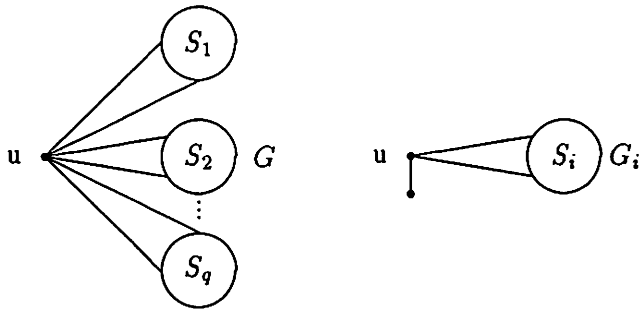
$P_1(K_{3,3}) = 36x^2 + 36x^3 + 9x^4$
$P_1(K_5) = 60x^2 + 60x^3 + 5x^4$
$P_1(K_6) = 360x^2 + 720x^3 + 210x^4 + 6x^5$

Lemma 1 $P_1(G) = \dfrac{\sum_e P_1(G-e)}{\rho}$

Proof: The sum counts each spanning tree T exactly ρ times because there are ρ edges that are not in T.

Lemma 2 *Let G and G_i be the graphs shown below, where u is a cut vertex in both graphs. Then*

$$P_1(G) = \frac{\prod_{i=1}^{q} P_1(G_i)}{x^q} \tag{1}$$



Proof: Let T be a spanning tree of G with k leaves. Since the leaves of T must be in $S_1, \cdots, S_q$ we have

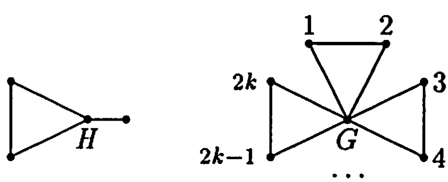
$$L_1 + L_2 + \cdots + L_q = k \tag{2}$$

where L_i is the number of leaves in T from S_i. Thus we see that the number of spanning trees with k leaves corresponds to the number of solutions of equation (2). This number can be found using the simple polynomial generating function given in equation (1). The factor x^q accounts for the extra leaf in each of the G_i.

EXAMPLE 3
An example using Lemma 1.

$$\begin{aligned}
\frac{\sum_e P_1(K_6 - e)}{\rho} &= \frac{15P_1(K_6 - e)}{10} = P_1(K_6) \\
&= 360x^2 + 720x^3 + 210x^4 + 6x^5 \\
P_1(K_6 - e) &= \frac{2P_1(K_6)}{3} = 240x^2 + 480x^3 + 140x^4 + 4x^5
\end{aligned}$$

EXAMPLE 4
An example using Lemma 2.

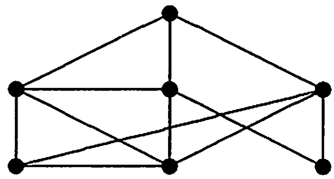


$P_1(H) = 2x^2 + x^3$

$P_1(G) = \frac{(2x^2 + x^3)^k}{x^k} = x^{2k} + 2kx^{2k-1} + \cdots + (2x)^k$

Notice that in $P_1(G)$ the coefficient of x^{2k-1} is $2k = 2\rho$. Therefore the lower bound given in Theorem 2 is sharp.

EXAMPLE 5
An approximation of $P_1(G)$



Instead of enumerating all 354 spanning trees of the above graph, generate 75 random spanning trees and find the pendant polynomial P_1' of this set of 75 trees. Thus we will have

$$\frac{P_1(G)}{\tau(G)} \approx \frac{P_1'}{75} = 0.1x^3 + 0.46x^4 + 0.36x^5 + 0.05x^6$$

where $\tau(G) = 354$.

2 Generalizations and Variations

The following definitions are due to Harary and Plantholt [6, 1989].

Definition 3 Let $f : \mathcal{F} \to \mathcal{N}$, where $\mathcal{F}$ is a family of subgraphs of a graph G and $\mathcal{N}$ is the set of nonnegative integers. Then f is a **positive invariant** if $H \in \mathcal{F}$ and $e \notin E(H)$ implies that

$$f(H) \leq f(H+e) \leq f(H) + 1. \tag{3}$$

On the other hand, f is a **negative invariant** if $H \in \mathcal{F}$ and $e \notin E(H)$ implies that

$$f(H) - 1 \leq f(H+e) \leq f(H). \tag{4}$$

EXAMPLE 6

The maximum degree $\Delta(G)$ is a positive invariant and the independence number $\alpha(G)$ is a negative invariant.

Definition 4 Let T be a spanning tree of G and $e \in E(G) - E(T)$. The subgraph $T + e$ contains a unique cycle C. Let f be an edge of C such that $f \neq e$. The operation $T + e - f$ is an **elementary exchange** and $T + e - f$ is a spanning tree distinct from T.

Theorem 3 (Harary & Plantholt, 1989) *Let $\mathcal{I}$ be a positive or negative invariant defined on a family $\mathcal{F}$ of graphs. Then $\mathcal{I}$ interpolates on $\mathcal{F}$ if $\mathcal{F}$ has one of the following properties:*

1. *For any two graphs $F, H \in \mathcal{F}$ there is a sequence of elementary exchanges from F to H consisting entirely of graphs from $\mathcal{F}$.*
2. *For any two graphs $F, H \in \mathcal{F}$, graph H can be obtained from F by a sequence of single edge deletions or additions, with each intermediate graph also in $\mathcal{F}$.*

Corollary 1 *Let $\mathcal{I}_1 = \kappa(G)$, $\mathcal{I}_2 = \kappa'(G)$, $\mathcal{I}_3 = \alpha(G)$, $\mathcal{I}_4 =$ the domination number of G and $\mathcal{I}_5 = \Delta(G)$. Define the following sets of subgraphs of G:*

- *F_1 = Spanning subgraphs of size at least s_1 and at most s_2.*
- *F_2 = Spanning subgraphs with maximum degree at most M.*
- *F_3 = Spanning subgraphs with hereditary property $\mathcal{P}$.*
- *F_4 = Connected spanning subgraphs of size s.*

Then $\mathcal{I}_j$ interpolates on F_m, where $1 \leq j \leq 5$; $1 \leq m \leq 4$.

2.1 Spanning trees of maximum degree at least 4

First, a few definitions.

Definition 5 $Q_4(G)$ is the set of spanning trees of maximum degree at least 4.

Definition 6 Let g be a connected subgraph of G. A maximum intersecting spanning tree with respect to g, denoted $\mathcal{M}(g;G)$ or $\mathcal{M}(g)$, is a spanning tree T such that

$$|E(T) \cap E(g)| = |V(g)| - 1.$$

Theorem 4 (Barefoot) *n_1 interpolates on $\mathcal{M}(g)$.*

Proof: See the appendix.

Definition 7 Let Q be a set of spanning trees. Then $P_1(Q) = \sum_i \alpha_i x^i$ where $\alpha_i =$ the number of spanning trees in Q with i leaves. A polynomial $\sum_i a_i x^i$ is gapless if $a_{i-1}a_{i+1} \neq 0$ implies that $a_i \neq 0$.

Theorem 5 (Barefoot) *$P_1(Q_4)$ is gapless.*

Proof: By induction on the size of G. The result is easily verified if G is a tree or a unicyclic graph. Thus, let G be a smallest counterexample of order n and size $m \geq n+1$. Let $Q_4(G) = \{T_1, T_2, \cdots, T_\lambda\}$, where

$$n_1(T_i) \leq n_1(T_{i+1}), \tag{5}$$

$$n_1(T_r) \leq n_1(T_{r+1}) - 2. \tag{6}$$

Assume that there exists an elementary exchange with edges $e = u_1u_2$ and $f = v_1v_2$ such that

$$e \in E(T_{r+1}) - E(T_r), \tag{7}$$

$$f \in E(T_r) - E(T_{r+1}), \tag{8}$$

$$T_{r+1} - e + f \in Q_4(G). \tag{9}$$

Let $H = T_{r+1} - e + f$. Notice that $|n_1(T_{r+1}) - n_1(H)| \leq 2$. If $n_1(H) \geq n_1(T_{r+1})$ then H and T_r are in $G - e$. This would be a contradiction because G is a smallest counterexample. Therefore $n_1(H) \leq n_1(T_{r+1}) - 1$. Consequently $n_1(T_{r+1}) - n_1(H) = 2$. This implies that u_1 and u_2 have degree at least 3 in T_{r+1} while v_1 and v_2 are leaves of $T_{r+1} - e$. Thus, $u_i \neq v_j$ in T_{r+1} and we conclude that v_1 and v_2 have degree 2 in $T_{r+1} + f$.

Let $C = v_1, v_2, \cdots, v_s$ be the cycle of $T_{r+1} + f$ and let h be the smallest integer such that $d(v_1) = d(v_2) = \cdots = d(v_{h-1}) = 2$ and $d(v_h) \geq 3$. The integer h exists since u_1 and u_2 are on C. Thus, $n_1(T_{r+1} + f - v_{h-1}v_h) = n_1(T_{r+1}) - 1$. Moreover, $T_{r+1} + f - v_{h-1}v_h \in Q_4$ unless v_h is the only vertex of degree at least 4 in T_{r+1}. If this is the case let j be the largest integer such that $d(v_1) = d(v_s) = \cdots = d(v_j) = 2$ and $d(v_{j+1}) \geq 3$. Vertices u_1 and u_2 are on C therefore $j + 1 \neq h$. Thus, $T_{r+1} + f - v_j v_{j+1} \in Q_4$ and $n_1(T_{r+1} + f - v_j v_{j+1}) = n_1(T_{r+1}) - 1$. This contradicts equations (5-6).

The remainder of the proof concerns the more difficult case when $T_{r+1} - e + f \notin Q_4$ whenever $e \in E(T_{r+1}) - E(T_r)$ and $f \in E(T_r) - E(T_{r+1})$. Let u be a vertex of degree at least 4 in T_{r+1} and v a vertex of degree at least 4 in T_r.

Proposition 1 *Every edge of G is in either T_{r+1} or T_r.*

Proof: G is the smallest counterexample!

Proposition 2 *Vertices u and v have degree exactly 4 in G.*

Proof: If $d(u) \geq 5$ in T_{r+1} then $d(u) \geq 4$ after any exchange. Thus $d(u) = 4$ in T_{r+1}. Now, if $d(u) \geq 5$ in G then we can find an edge $f \notin E(T_{r+1})$ that is incident to u in G. Consequently $d(u) \geq 4$ after any exchange of the form $T_{r+1} + f - e$, where $e \in E(T_r) - e(T_{r+1})$. Therefore $d(u) = 4$ in G and the same argument applies to v.

Proposition 3 *If $e \in E(T_{r+1}) - E(T_r)$ then e is incident with u in T_{r+1} and if $f \in E(T_r) - E(T_{r+1})$ then f is incident with v in T_r.*

Proof: If e is not incident with u in T_{r+1} then $d(u) = 4$ in $T_{r+1} - e$. This implies that $T_{r+1} - e + f' \in Q_4$ for some $f' \in E(T_r) - E(T_{r+1})$. The same argument applies to v.

Proposition 4 $|E(G)| \leq |V(G)| + 2$.

Proof: From Proposition 1,

$$E(G) = E(T_{r+1}) \cup E(T_r) = E(T_{r+1}) \cup (E(T_r) - E(T_{r+1})).$$

From Proposition 3, we know that each edge of $E(T_r) - E(T_{r+1})$ is incident with v. Furthermore, v is incident with at least one edge of T_{r+1}. Thus, we have $|E(T_r) - E(T_{r+1})| \leq 3$. Therefore

$$|E(G)| = |E(T_{r+1})| + |E(T_r) - E(T_{r+1})| \leq |V(G)| - 1 + 3 = |V(G)| + 2.$$

Proposition 5 *Neither u nor v has degree 4 in both T_{r+1} and T_r.*

Proof: Assume u has degree 4 in T_{r+1} and T_r. Let g be the subgraph consisting of the 4 edges incident to u. By Theorem 4, n_1 interpolates on $\mathcal{M}(g)$, the set of maximum intersecting spanning trees with respect to g. Notice that both T_{r+1} and T_r are in $\mathcal{M}(g)$. Thus, there is a tree $T \in \mathcal{M}(g)$ with $n_1(T) = n_1(T_r) + 1$. Since $T \in Q_4$ this is a contradiction. Therefore u cannot have degree 4 in both T_{r+1} and T_r.

Definition 8 Let $\{u_1, u_2, u_3, u_4\}$ be the neighbors of u and $\{v_1, v_2, v_3, v_4\}$ the neighbors of v. $E(u)$ is the subgraph consisting of the 4 edges incident with u, $E(v)$ is the subgraph consisting of the 4 edges incident with v and $E(u, v)$ is the subgraph consisting of the edges incident with u or v. The **u_k-branch** of T_{r+1} is the component of $T_{r+1} - u$ that contains u_k. $E_r = E(T_r), E_{r+1} = E(T_{r+1}), E^r = E(T_r) - E(T_{r+1})$ and $E^{r+1} = E(T_{r+1}) - E(T_r)$. Now, if $e \in E^r$ let C_e be the cycle of $T_{r+1} + e$.

Proposition 6 *$E(u, v)$ contains a cycle.*

Proof: If $E(u, v)$ is acyclic then there is a spanning tree T that contains $E(u, v)$. Let $Q = \mathcal{M}(E(u)) \cup \mathcal{M}(E(v))$. By Theorem 4, $P_1(\mathcal{M}(E(u))$ and $P_1(\mathcal{M}(E(v))$ are gapless. Furthermore, $T \in \mathcal{M}(E(u)) \cap \mathcal{M}(E(v))$. This implies that the coefficient of $x^{n_1(T)}$ is nonzero in $P_1(\mathcal{M}(E(u)))$ and $P_1(\mathcal{M}(E(v)))$. Consequently $P_1' = P_1(\mathcal{M}(E(u))) + P(\mathcal{M}(E(v)))$ is gapless. Since each coefficient of $P_1(Q)$ is nonzero iff the corresponding coefficient of P_1' is nonzero, we conclude that $P_1(Q)$ is gapless. By definition T_{r+1} and T_r are in Q. Therefore, there is a tree $T' \in Q$ such that $n_1(T') = n_1(T_r) + 1$. Since this is a contradiction we conclude that $E(u, v)$ contains a cycle.

From Proposition 6 we conclude that u and v have at least one common neighbor. There are six cases to examine according to the degree of v in T_{r+1} and whether uv is an edge of G. In Cases 1-3 we will assume that uv is not an edge and u, u_1, v, u_2 is a cycle in $E(u,v)$. In Cases 4-6, uv is an edge and u, u_1, v is a cycle in $E(u,v)$, where $v = u_2$.

The basic proof technique is outlined below:

Proof technique using a non-elementary exchange method

1. From Propositions 3 and 4: $d(v)$ in T_{r+1} = $d(u)$ in T_r. T_r is obtained from T_{r+1} by the equation

$$T_r = T_{r+1} - E^{r+1} + E^r, |E^{r+1}| = |E^r| \leq 3. \tag{10}$$

   All edges of E^{r+1} are incident to u and all edges of E^r are incident to v. The edges of E^r must be incident with the components of $T_{r+1} - E^{r+1}$ so that T_r is connected. Define S_0 by the equation

$$S_0 = T_{r+1} - E^{r+1} + E', \text{where} \, E' \subseteq E^r. \tag{11}$$

   Count the components, edges and leaves of S_0 and keep in mind that

$$n_1(T_r) \leq n_1(T_{r+1}) - 2. \tag{12}$$

   In some cases the configuration of S_0 forces $n_1(T_r) \geq n_1(T_{r+1}) - 1$ so that T_r has too many leaves. In other cases, S_0 forces certain neighbors of v to be leaves, or forces certain neighbors of u to have degree at least two; otherwise equation (12) is violated. (see Figure 1)

2. Use the configuration of T_r to find an elementary exchange that produces the spanning tree T with $n_1(T) = n_1(T_r) + 1$ and $T \in Q_4$!

Case 1 *Assume that uv is not an edge of G and $d(v) = 3$ in T_{r+1}.*

Consider $T_{r+1} + vv_i$, where $vv_i \in E^r$. Since this graph has at least two vertices of degree ≥ 4 and uv is not an edge, we conclude that there exists an exchange of the form $T_{r+1} + vv_i - e$ satisfying equations (7-9).

Case 2 *Assume that uv is not an edge of G and $d(v) = 2$ in T_{r+1}.*

Subcase 1 *u is nonadjacent to u_1 and u_2 in T_r, and v is nonadjacent to u_1 and u_2 in T_{r+1}.*

Since u has degree 2 in T_r, we see that

$$T_r = T_{r+1} - uu_1 - uu_2 + vu_1 + vu_2. \tag{13}$$

Notice that this transformation preserves the degree of every vertex except u and v. Therefore $n_1(T_r) = n_1(T_{r+1})$. Since this is a contradiction we conclude that v is adjacent to u_1 or u_2 in T_{r+1}. Assume that v is adjacent to u_1 in T_{r+1}.

Subcase 2 *u is nonadjacent to u_1 and u_2 in T_r, and v is adjacent to u_1 in T_{r+1}.*

Let $S_0 = T_{r+1} - uu_1 - uu_2 + vu_2$. This transformation preserves the degree of every vertex except u, v and u_1. Since S_0 has two components and $T_r = S_0 + vv_i$, we conclude that $n_1(T_r) \geq n_1(T_{r+1}) - 1$. Since this is a contradiction we conclude that u is adjacent to u_1 or u_2 in T_r but not both. Assume that u is adjacent to u_1 in T_r.

Subcase 3 *u is adjacent to u_1 in T_r, and v is nonadjacent to u_1 and u_2 in T_{r+1}.*

Let v be on the u_3-branch of T_{r+1} and consider $S_0 = T_{r+1} - uu_2 - uu_3 + vu_2$. Counting components and leaves of S_0, we see that $n_1(T_r) \geq n_1(T_{r+1}) - 1$.

Subcase 4 *u is adjacent to u_1 in T_r, and v is adjacent to u_1 in T_{r+1}.*

Assume that u is adjacent to u_3 in T_r and let $S_0 = T_{r+1} - uu_2 - uu_4 + vu_2$. Counting components, leaves and edges of S_0 we see that T_r has too many leaves. In other words, $n_1(T_r) \geq n_1(T_{r+1}) - 1$.

Subcase 5 *u is adjacent to u_1 in T_r and v is adjacent to u_2 in T_{r+1}.*

Assume that u is also adjacent to u_3 in T_{r+1} and let $S_0 = T_{r+1} - uu_2 - uu_4 + vu_1$. Notice that if u_1 is not a leaf of T_{r+1} then $n_1(T_r) \geq n_1(T_{r+1}) - 1$. Also $d(u_4) \geq 2$ in T_{r+1}, otherwise T_r has too many leaves. Let v_4 be the leaf in the u_4-branch of T_{r+1}. This implies that $d(v_4) = 2$ in T_r. Let C_{uu_4}[†] $= (z_1 = v_4), \cdots, z_x = u_4, z_{x+1} = u, z_{x+2} = v$ and h the smallest integer such that $d(z_1) = \cdots = d(z_{h-1}) = 2$ and $d(z_h) \geq 3$. Since u_4 is on this cycle $h \leq x$. The spanning tree $T_r + uu_4 - z_{h-1}z_h$ has $n_1(T_r) + 1$ leaves and is a member of Q_4.

Case 3 *Assume that uv is not an edge of G and $d(v) = 1$ in T_{r+1}.*

Since v is a leaf of T_{r+1}, u is a leaf of T_r.

Subcase 1 *u is nonadjacent to u_1 and u_2 in T_r,*
and v is nonadjacent to u_1 and u_2 in T_{r+1}.

This case is similar to Case 1, subcase 1. Assume that $uu_4 \in E(T_r)$ and let $S_0 = T_{r+1} - uu_1 - uu_2 - uu_3 + vu_1 + vu_2$. Regardless of whether v is in the u_3-branch or u_4-branch of T_{r+1} we conclude that $n_1(T_r) \geq n_1(T_{r+1})$.

Subcase 2 *u is nonadjacent to u_1 and u_2 in T_r*
and v is adjacent u_1 in T_{r+1}.

Assuming that $uu_4 \in E(T_r)$, let $S_0 = T_{r+1} - uu_1 - uu_2 - uu_3 + vu_2$. If u_3 is a leaf of T_{r+1} then we conclude that T_r has too many leaves. Actually neither u_3 nor u_4 are leaves of S_0 for the same reason. Let v_3 be on the u_3-branch of T_{r+1} and v_4 on the u_4-branch of T_{r+1}. Since $n_1(S_0) \geq n_1(T_{r+1})$, v_3 and v_4 must be leaves of S_0. Thus, $d(v_3) = d(v_4) = 2$ in T_r. Let $C_{uu_3} = (z_1 = v_4), z_2, \cdots, z_x = u_3, \cdots, v$ and let h be the smallest integer such that $d(z_1) = \cdots = d(z_{h-1}) = 2$ and $d(z_h) > 2$. With u_3 on C_{uu_3}, $h \leq x$. The spanning tree $T = T_r + uu_4 - z_{h-1}z_h$ provides a contradiction. Hence, we will assume that uu_1 is an edge of T_r.

Subcase 3 *$uu_1 \in E(T_r)$ and v is nonadjacent to u_1 and u_2 in T_{r+1}.*

Assume that v is on the u_3-branch of T_{r+1}. If u_1 is not a leaf of T_{r+1} then T_r has too many leaves. Thus, $d(u_1) = 2$ in T_r. This implies that u_3 is not a leaf of T_r. Therefore, $n_1(T_r + uu_3 - uu_1) = n_1(T_r) + 1$.

[†]We are employing step 2 of the proof technique.

Subcase 4 *$uu_1 \epsilon E(T_r)$ and v is adjacent to u_1 in T_{r+1}.*

Looking at $S_0 = T_{r+1} - uu_2 - uu_3 - uu_4 + vu_2$ we see that v_3 and v_4 are leaves, where v_i is on the u_i-branch of T_{r+1}. Moreover, $d(u_3) \geq 2$ and $d(u_4) \geq 2$ in S_0. Let $C_{uu_3} = (v_3 = z_1), z_2, \cdots, z_x = u_3, \cdots, z_{x+1} = u, z_{x+2} = u_1, z_{x+3} = v$ and h the smallest integer such that $d(z_1) = d(z_2) = \cdots = d(z_{h-1}) = 2$ and $d(z_h) \geq 3$. The spanning tree $T_r + uu_3 - z_{h-1}z_h$ provides the contradiction.

Subcase 5 *$uu_1 \epsilon E(T_r)$ and v is adjacent to u_2 in T_{r+1}.*

Let $S_0 = T_{r+1} - uu_2 - uu_3 - uu_4 + vu_1$. If u_1 is not a leaf of T_{r+1} then apply the agrument of the previous case. If u_1 is a leaf of T_{r+1} then $n_1(T_r) = n_1(T_{r+1}) - 1$. Therefore, $d(u_3) \geq 2$ and $d(u_4) \geq 2$ in S_0. There must be a leaf in the u_3-branch or the u_4-branch of T_{r+1}. If this leaf is in the u_3-branch then use $T_r + uu_3$ to find the spanning tree $T = T_r + uu_3 - z_{h-1}z_h$ as in the previous case and if the leaf is in the u_4-branch use $T_r + uu_4$.

Since every case leads to a contradiction we now know that $uv \epsilon E(G)$.
Assume that u, u_1, v is a cycle in $E(u, v)$ and let $v = u_2$.

Case 4 *$uv \epsilon E(G)$ and $d(v) = 3$ in T_{r+1}.*

Let $f \epsilon E^r$. Some edge e of the cycle C_f is not in T_r. Since uv is in T_r and T_{r+1}, $e \neq uv$. Thus, $T_{r+1} + f - e$ is an exchange satisfying (7-9) with $d(v) = 4$.

Case 5 *$uv \epsilon E(G)$ and $d(v) = 2$ in T_{r+1}.*

Vertex u has degree 2 in T_r. Edge uu_1 is not in T_r because uv and vu_1 are in T_r. Wlog, let $uu_3 \epsilon E_r$. Set $S_0 = T_{r+1} + vu_1 - uu_1 - uu_4$. If u_4 is a leaf of T_{r+1} then we conclude that $n_1(T_r) = n_1(T_{r+1})$. On the other hand, if u_4 is not a leaf then we conclude that $n_1(T_r) \geq n_1(T_{r+1}) - 1$.

Case 6 *$uv \epsilon E(G)$ and $d(v) = 1$ in T_{r+1}.*

Vertex u has degree 1 in T_r and $uu_1 \not\epsilon E_r$. Set $S_0 = T_{r+1} + vu_1 - uu_1 - uu_3 - uu_4$. If u_3 or u_4 is a leaf of T_{r+1} then we conclude that T_r has too many leaves. In fact, u_3 and u_4 have degree at least in 3 in T_{r+1}. Let v_3 be the neighbor of v in the u_3-branch of

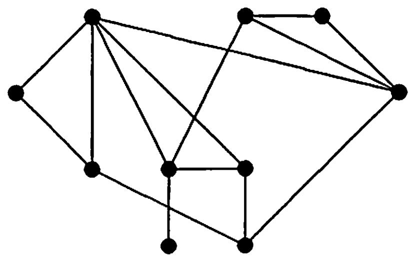
T_{r+1}. Notice that v_3 must be a leaf of T_{r+1}. Let $C_{uu_3} = (z_1 = v_3), \cdots, z_x = u_3, z_{x+1} = u, z_{x+2} = v$ and h the smallest integer such that $d(z_1 = v_3) = \cdots = d(z_{h-1}) = 2$ and $d(z_h) > 2$, where $h \leq x$. $T = T_r + uu_3 - z_{h-1}z_h$ provides the contradiction.

Since every case leads to a contradiction we conclude that there is an exchange that satisfies (7-9). (Phew!). Therefore, $P_1(Q_4)$ is gapless.

Conjecture: Let $q_3(G)$ be the spanning trees with maximum degree 3. Then n_1 interpolates on q_3.

EXAMPLE 7

An example of $P_1(Q_4)$ and $P_1(q_3)$.



$$P_1(G) = 9x^2 + 107x^3 + 292x^4 + 247x^5 + 60x^6 + 4x^7$$

$$P_1(Q_4) = 47x^4 + 124x^5 + 55x^6 + 4x^7$$

$$P_1(q_3) = P_1(G) - P_1(Q_4) - 9x^2 = 107x^3 + 245x^4 + 123x^5 + 5x^6$$

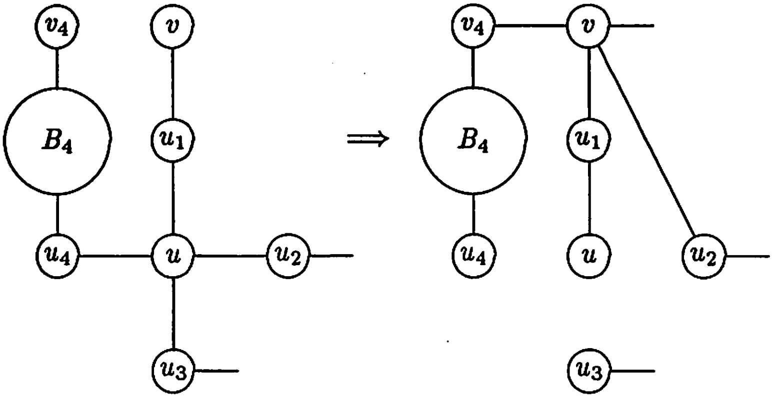
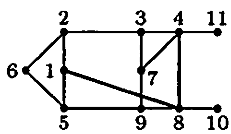


Figure 1. Non-elementary exchange with $d(v) = 1$ in T_{r+1} and $uv \notin E(G)$.

2.2 q-Filters

Consider the following question for the graph shown below: Is there an edge set Q such that $G_1 - Q$ is connected and contains no spanning tree with 5 leaves?



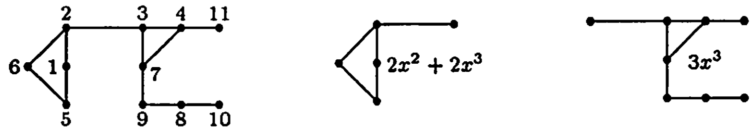
G_1

$$P_1 = x^2 + 46x^3 + 180x^4 + 194x^5 + 62x^6 + 4x^7$$

Find a spanning T with $k \neq 5$ leaves and let $Q = E(G) - E(T)$. Then $G - Q$ will have k leaves! Thus, as long as $P_1(G)$ has at least two nonzero coefficients it is always possible to find a connected spanning subgraph without any spanning trees with k leaves.

Definition 9 Let $P_1(G) = \sum_r^s b_k x^k$. Then a **q-filter** is a minimum edge set Q such that $G - Q$ is connected and does not contain a spanning tree with q leaves. $|Q|$ is denoted $\mu_q(G)$.

Since G_1 has only one Hamiltonian path $\mu_2 = 1$. An exhaustive search shows that $\mu_5 = 3$. To show that $\mu_5 \leq 3$, let $Q = \{18, 59, 48\}$. To calculate $P_1(G_1 - Q)$ use the fact that 23 is a cut edge of $G_1 - Q$. Using Lemma 2 we see that $P_1(G_1 - Q) = 6x^3 + 6x^4$.



$$P_1(G_1 - Q) = \frac{3x^3(2x^2 + 2x^3)}{x^2} = 6x^3 + 6x^4$$

Given q-filter Q, we know that $P_1(G - Q)$ is gapless. Therefore, all trees of $G - Q$ have at least $q + 1$ leaves or all trees of $G - Q$ have at most $q - 1$ leaves.

Definition 10 If $P_1(G) = \sum_r^s b_k x^k$ then the **filter polynomial** of G is $\mu(G) = \sum_r^s \mu_k x^k$.
The q-filter Q is low if

$$P_1(G - Q) = \sum_{k=L}^{H} \alpha_k x^k \text{, where } q + 1 \leq L \leq H \leq s$$

and **high** if

$$P_1(G - Q) = \sum_{k=L}^{H} \alpha_k x^k \text{, where } r \leq L \leq H \leq q - 1$$

A high q-filter is denoted q^+ and a low q-filter is denoted q^-. Returning to graph G_1, we find that

$$\mu(G_1) = x^2 + 2x^3 + 3x^4 + 3x^5 + 2x^6 + x^7$$

According to the definition a 2-filter of G_1 is low and a 7-filter of G_1 is high. What can we say about the q-filters when $3 \leq q \leq 6$?

Definition 11 To indictate whether q-filters are low or high an underline or overline is used with the coefficient of x^q in $\mu(G) = \sum_r^s \mu_k x^k$. If there are q^+-filters and q^--filters then no symbol is used with the coefficient of x^q.
For example,

$$\mu(G_1) = \underline{x^2 + 2x^3 + 3x^4} + \overline{3x^5 + 2x^6 + x^7}$$

Thus, if $q \leq 4$ then all q-filters are low; and if $q \geq 5$ then all q-filters are high.

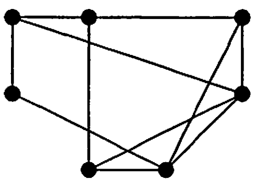
Proposition 7 *If Q is a q^+-filter and $p \geq q$ then $\mu_q \geq \mu_p$. If Q is a q^--filter and $p \leq q$ then $\mu_q \geq \mu_p$.*

Proof: If Q is a q^+-filter then all spanning trees of $G - Q$ have at most $q - 1$ leaves. Thus, deleting Q also removes all spanning trees with at least q leaves.Therefore $|Q|$ is an upper bound for μ_p, where $p \geq q$. The same idea can be applied to q^--filters.

Another possibility is shown in the following two examples.

EXAMPLE 8

An example with q^+-filters and q^--filters.

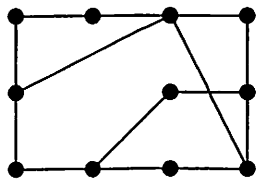


G_2

$$P_1(G_2) = 46x^2 + 109x^3 + 48x^4 + x^5$$
$$\mu(G_2) = \underline{3x^2} + 4x^3 + \overline{3x^4 + x^5}$$

Notice that there are 3^+-filters and 3^--filters.

EXAMPLE 9



G_3

$$P_1(G_3) = 10x^2 + 90x^3 + 147x^4 + 77x^5 + 13x^6$$
$$\mu(G_3) = \underline{x^2 + 2x^3} + 3x^4 + \overline{2x^5 + x^6}$$

Theorem 5 (q-filter theorem) *Let $P_1(G) = \sum_r^s b_k x^k$, where $r \leq s-1$. Then there exists integers L and H such that $1 \leq H - L \leq 2$ and*

$q \leq L$ implies that all q-filters are low.

$q \geq H$ implies that all q-filters are high.

Proof: The proof will be by induction on the size of G. The theorem is obvious if G is unicyclic because $\mu(G)$ has at most three terms. Therefore assume that G is a smallest counterexample of order n and size $m \geq n+1$. The following proposition will help determine $\mu(G)$.

Proposition 8 *Suppose that F_1 is a j^+-filter and F_2 is an h^--filter, where $j \leq h-1$ then*

- $\mu_j = \mu_h$
- *F_1 is an h^+-filter and F_2 is a j^--filter.*

Proof: By Proposition 7, $\mu_j \leq \mu_h$ and $\mu_j \geq \mu_h$. Therefore $\mu_j = \mu_h$. Since $|F_1| = \mu_j = \mu_h$ and $G - F_1$ has no spanning trees with h leaves we conclude that F_1 is an h-filter. Therefore F_1 is a h^+-filter. Similarly, F_2 is j^--filter.

Proposition 9 *If all k-filters are high then $q \geq k$ implies that all q-filters are high and if all k-filters are low then $q \leq k$ implies that all q-filters low.*

Proof: Assume that all k-filters are high and that Q is a q^--filter, $q \geq k$. According to Proposition 8, Q is a k^--filter. Since this is a contradiction, Q does not exist. The same idea applies for low filters.

We can use Propositions (7-9) to conclude that

1. There must be at least two integers h and j such there are h^+-filters, h^--filters, j^+-filters and j^--filters.

2. $\mu_j = \mu_h$.

3. If $P_1(G) = \sum_r^s b_k x^k$ then all r-filters are low and all s-filters are high.

4. If $\mu_0 = \mu_j = \mu_h$ then for every integer i, $\mu_0 \geq \mu_i$.

Thus, we can assume that

$$\mu(G) = \underline{\sum_{r}^{p-1} \mu_k x^k} + \mu_0 \sum_{p}^{h} x^k + \overline{\sum_{h+1}^{s} \mu_k x^k} \qquad (14)$$

where $\mu_0 \geq \mu_i$ and $p < h$. Equation (14) implies that $\mu_p = \mu_{p+1}$ and there is a p^+-filter Q_1 and a $(p+1)^-$-filter Q_2. Furthermore, Q_1 is a $(p+1)^+$-filter and Q_2 is a p^--filter.

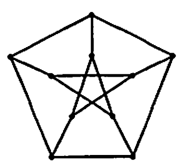
Let e be an edge of G that is not a cut edge. Notice that if G has a tree with at least p leaves then $Q_1 - e$ is a p^+-filter of $G - e$. Suppose that all spanning trees of $G - e$ have at most $p - 1$ leaves. This means that e is a p^+-filter. Consequently $\mu_p = \mu_{p+1} = \mu_0 = 1$. Therefore let f be a $(p+1)^-$-filter. Since e is a p^+-filter and f is a $(p+1)^-$-filter we see that every spanning tree of G contains e or f. Thus, $\{e, f\}$ is an edge cut. Let T_0 be a spanning tree of G with at least $p + 2$ leaves. Clearly e is an edge of T_0. Furthermore we can assume that f is not an edge of T_0 because if f is in every spanning tree with at least $p + 2$ leaves then f must be in every spanning tree and this would mean that f is a cut edge.

Given the configuration that we now have for G we see that $T_0 - e + f$ is a spanning tree of G. Since an exchange can decrease the number of leaves in T_0 by at most two we see that $T_0 - e + f$ has at least p leaves. This is a contradiction because e is a p-filter! Thus, we conclude that every spanning tree of G has at most $p + 1$ leaves. Let T_1 be a spanning with at most $p - 1$ leaves. Using an argument similar to the one in the previous paragraph we can assume that f is in T_1 but e is not. Similarly we see that $T_1 - f + e$ is a spanning tree with at most $p+1$ leaves that does not contain f. This contradicts the fact that f is a $(p+1)^-$-filter. Therefore every spanning tree of G has at least p leaves. This means that $P_1(G) = b_p x^p + b_{p+1} x^{p+1}$. This is an obvious contradiction. Thus, we must conclude that $G - e$ has a spanning tree with at least p leaves. Therefore $Q_1 - e$ is a p^+-filter of $G - e$. The same type of argument shows that $Q_2 - e$ is a $(p+1)^-$-filter of $G - e$.

We know that there are integers α and β such that $q \leq \alpha$ implies that all q-filters are low in $G - e$ and $q \geq \beta$ implies that all q-filters are high in $G - e$. Apparently $\alpha \leq p - 1$ and $\beta \geq p + 2$. Thus, $G - e$ is also a counterexample. Since this is impossible equation (14) must be false. Therefore we see that $|h - p| = 0$ or $r \leq k \leq s$ implies that all k-filters are high or all k-filters are low.

Definition 12 A graph is **slow** if $H - L = 2$ and **fast** if $H - L = 1$.

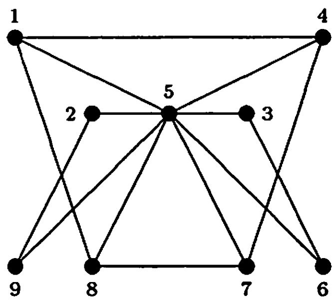
EXAMPLE 10



$$P_1 = 120x^2 + 820x^3 + 810x^4 + 240x^5 + 10x^6$$

$$\mu = \underline{3x^2 + 5x^3} + 5x^4 + \overline{3x^5 + 3x^6}$$

EXAMPLE 11



$$P_1 = 32x^3 + 128x^4 + 152x^5 + 76x^6 + 16x^7 + x^8$$
$$\mu = \underline{x^3 + 2x^4} + 4x^5 + \overline{3x^6 + 2x^7 + x^8}$$

$$P_1(G - \{29, 36, 47, 78\}) = 2x^6 + 5x^7 + x^8$$
$\Rightarrow$ $\{29, 36, 47, 78\}$ is a 5^-–filter.

$$P_1(G - \{15, 25, 35, 45\}) = 4x^3 + 7x^4$$
$\Rightarrow$ $\{15, 25, 35, 45\}$ is a 5^+–filter.

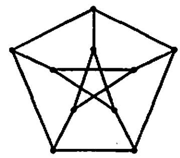
3 Counting vertices of degree 2 - $P_2(G)$

Instead of counting leaves we will consider the more difficult (and more interesting) problem of counting vertices of degree two.

Definition 13 $P_2(G) = \sum_r^s c_k x^k$, where c_k is the number of spanning trees in G with k vertices of degree two. The minimum size of an edge set Q such that $G - Q$ is connected and has no spanning trees with q vertices of degree two is denoted $\mu_q^{(2)}$ and $\mu^{(2)}(G) = \sum_r^s \mu_k^{(2)} x^k$.

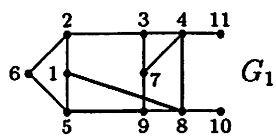
The following examples show that $P_2(G)$ may not be gapless.

EXAMPLE 12



$$P_2 = 10 + 240x^2 + 810x^4 + 820x^6 + 120x^8$$

EXAMPLE 13

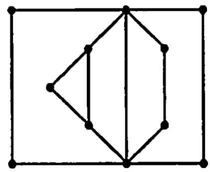


$$P_1 = x^2 + 46x^3 + 180x^4 + 194x^5 + 62x^6 + 4x^7$$

$$P_2 = 2 + 21x + 33x^2 + 138x^3 + 66x^4 + 165x^5 + 15x^6 + 46x^7 + x^9$$

$P_2(G)$ usually has more terms than $P_1(G)$ because it is possible for a spanning tree to have no vertices of degree two. Consequently, the spanning trees of G are more dispersed. The next example shows that it is possible for P_2 to be gapless although this appears to be the exception.

EXAMPLE 14



$$P_1 = 42x^3 + 156x^4 + 182x^5 + 99x^6 + 30x^7 + 4x^8$$
$$P_2 = 2 + 16x + 52x^2 + 93x^3 + 132x^4 + 104x^5 + 72x^6 + 42x^7$$

Lemma 3 *If T is a tree of order $n \geq 3$ then $n_1 = 2 + \sum_k (k-2)n_k$.*

Proof:

$$\sum_v d(v) = \sum_k kn_k = 2|E(T)| = 2(|V(T)| - 1)$$
$$= 2(-1 + \sum_k n_k) = -2 + 2\sum_k n_k.$$

Solving for n_1 gives $n_1 = 2 + \sum_k (k-2)n_k$.

Lemma 4 *Let G be a connected graph of order $n \geq 3$ and $\Delta(G) \leq 3$. If T is a spanning tree then $n_2(T) \equiv n \pmod 2$.*

Proof: From Lemma 3, we have $n_1 = 2 + n_3$. Also, $n = n_1 + n_2 + n_3$. Therefore, $n - n_2 = 2n_1 - 2$.

Lemma 5 *If T is a tree of order $n \geq 3$ then $n_2 \neq n - 3$.*

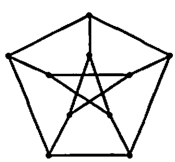
Proof: By Lemma 3, $n_1 = 2 + n_3 + 2n_4 + \cdots$. Assume that $n_2 = n - 3$. This implies that $\Delta(T) \geq 3$, otherwise T is a path with $n_2(T) = n - 2$. Thus, for some $j \geq 3$, we have $n_j \geq 1$. Now, $n_1 \geq 2 + (j-2)n_j \geq 3$ so that $n = |V(T)| \geq n_1 + n_2 + n_j = 3 + (n-3) + 1 = n + 1$. Therefore $n_2 \neq n - 3$.

The previous Lemmas tell us that if G has order n then

- if $\Delta(G) = 3$ then
  - n even $\Rightarrow P_2(G)$ is an even polynomial and n odd $\Rightarrow P_2(G)$ is an odd polynomial.
  - if T is a spanning tree then $n_2(T) = n + 2 - 2n_1(T)$.
  - $\mu_k = \mu^{(2)}_{n+2-2k}$ and $b_k = c_{n+2-2k}$.
- If G is Hamiltonian and $\Delta(G) \geq 3$ then $P_2(G)$ has at least one gap. This means that $P_2(G)$ may have gaps even if G has lots of edges.

Thus, if $\Delta(G) = 3$, there is no significant difference between n_1, P_1, μ and $n_2, P_2, \mu^{(2)}$ as far as the spanning trees are concerned. This implies that the concepts of $n_2, P_2(G)$ and $\mu^{(2)}$ are only significant when $\Delta(G) \geq 4$.

EXAMPLE 15



$$P_1 = 120x^2 + 820x^3 + 810x^4 + 240x^5 + 10x^6$$
$$P_2 = 10 + 240x^2 + 810x^4 + 820x^6 + 120x^8$$
$$\mu = 3x^2 + 5x^3 + 5x^4 + 3x^5 + 3x^6$$
$$\mu^{(2)} = 3 + 3x^2 + 5x^4 + 5x^6 + 3x^8$$

Next, we will consider the size of the gaps in $P_2(G)$.

Definition 14 Let $P_2(G) = \sum_r^s c_k x^k$. If $c_{k-1}c_{k+1} \neq 0$ and $c_k = 0$ then $P_2(G)$ has a k-gap or a gap of length 1.

Theorem 6 *If G is connected, $P_2(G)$ is gapless or has gaps of length 1.*

Proof: By induction on the size of G. The assertion is true if G is a tree or a unicyclic graph. Let G be a smallest counterexample of order n and size $m \geq n+1$. Let T_1 and T_2 be spanning trees of G such that

$$n_2(T_2) \geq n_2(T_1) + 3 = u + 3, \tag{15}$$

$$u < j < n_2(T_2) \Rightarrow c_j = 0. \tag{16}$$

Let $e \in E(T_2) - E(T_1)$, $f \in E(T_1) - E(T_2)$ such that $T_3 = T_2 - e + f$ is a spanning tree of G. If $n_2(T_3) \geq n_2(T_2)$ then both T_1 and T_3 are in $G - e$. This is impossible since G is a smallest counterexample. Thus, $n_2(T_3) \leq n_2(T_2) - 1$ so that $n_2(T_3) \leq u$. Since an exchange can increase the number of vertices of degree two by at most four we conclude that $u + 3 \leq n_2(T_2) \leq u + 4$.

Case 1 $n_2(T_2) = u + 4$ *and* $n_2(T_3) = u$.

Let $e = u_1v_1$ and $f = u_2v_2$. Adding e to T_3 creates two vertices of degree two and deleting f from $T_3 + e$ creates two vertices of degree two. Thus, u_1 and v_1 are leaves of T_3 while u_2 and v_2 are vertices of degree 3. Let $P = (x_1 = u_2), x_2, \cdots, x_h = u_1$ be the (u_2, u_1)-path in T_2 and let q be the smallest integer such that $d(x_q) = 3$ and $d(x_{q+1}) \neq 3$. Since edge e is on P, q must exist. If $d(x_{q+1}) = 2$ then $n_2(T_3 + e - x_qx_{q+1}) = u + 2$ and if $d(x_{q+1}) \geq 4$ then $n_2(T_3 + e - x_qx_{q+1}) = u + 3$. In either case we obtain a contradiction.

Case 2 $n_2(T_2) = u + 3$ *and* $n_2(T_3) = u - 1$.

The argument of the previous case shows that there is a spanning tree T such that $n_2(T) = u + 1$ or $u + 2$.

Case 3 $n_2(T_2) = u + 3$ *and* $n_2(T_3) = u$.

In this case there are two ways to create three vertices of degree two.

Subcase 1 $n_2(T_3 + e) = u + 1$ *and* $n_2(T_3 + e - f) = u + 3$.

One vertex of degree two is added when e is added to T_3. Therefore, either u_1 or v_1 is a leaf of T_3 but not both. Assume that in T_3 we have $d(v_1) \geq 3$ and $d(u_2) = d(v_2) = 3$. Let $P = (x_1 = u_2), \cdots, x_h = u_1$ be the (u_2, u_1)-path in T_3 and let q be the smallest integer such that $d(x_q) = 3$ and $d(x_{q+1}) \neq 3$. Therefore, $n_2(T_3 + e - x_qx_{q+1}) = u + 1$ or $u + 2$ depending on whether $d(x_{q+1}) = 2$ or $d(x_{q+1}) \geq 4$.

Subcase 2 $n_2(T_3 + e) = u + 2$ *and* $n_2(T_3 + e - f) = u + 3$.

Vertices u_1 and v_1 must be leaves of T_3 and assume that $d(u_2) = 3$ and $d(v_2) \geq 4$. Since $T_2 = T_3 + e - f$, f must be on the cycle of $T_3 + e$. Thus, there are two paths between u_1 and u_2 in $T_3 + e$. Let $P = (x_1 = u_2), \cdots, x_h = u_1$ be the path that does not contain v_2 and let q be the largest integer such that $d(x_q) = 3$ and $d(x_{q+1}) \neq 3$. If $d(x_{q+1}) = 2$ then $n_2(T_3 + e - x_q x_{q+1}) = u + 2$. If $d(x_{q+1}) = 4$ consider the (x_{q+1}, u_1)-section of P, $P(x_{q+1}, u_1) = x_{q+1}, x_{q+2}, \cdots, x_h = u_1$ and let t be the largest integer such that $d(x_t) \geq 4$ and $d(x_{t+1}) = 2$. Notice that the definition of q implies that no vertex on $P(x_{q+1}, u_1)$ can have degree 3. Thus, we have $n_2(T_3 + e - x_t x_{t+1}) = u + 1$. Therefore, the smallest counterexample G does not exist and the theorem is established.

Now we will consider q-filters.

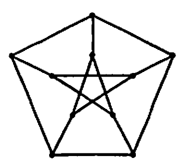
Definition 15 $P_2(G) = \sum_r^s c_k x^k$, where $r \leq s - 1$ and let Q be a q-filter of G. Then Q is sharp, denoted $q^{\#}$, if

$$P_2(G - Q) = \sum_c^d \alpha_k x^k$$

where $c < q < d$.

The use of the underline and overline in $\mu^{(2)}(G)$ is the same as before.

EXAMPLE 16



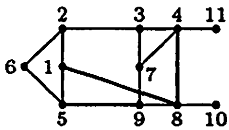
$$P_2 = 10 + 240x^2 + 810x^4 + 820x^6 + 120x^8$$

$$\mu^{(2)} = \underline{3 + 3x^2} + 5x^4 + \overline{5x^6 + 3x^8}$$

The concept of fast and slow graphs is more complicated for $\mu^{(2)}(G)$. It is clear that there exists integers L and H such that $q \leq L$ implies that all q-filters are low and $q \geq H$ implies that all q-filters are high. However, the presence of sharp filters can cause $H - L$ to be greater than two! Consequently, graphs have more than two "speeds" with respect to $\mu^{(2)}$.

Definition 16 $\mu^{(2)}(G) = \sum_r^s \mu_k^{(2)} x^k$. $L^{(2)}$ and $H^{(2)}$ are the integers such that $q \leq L^{(2)}$ implies that all q-filters are low and $q \geq H^{(2)}$ implies that all q-filters are high. The **speed** of G is $1/(H^{(2)} - L^{(2)})$.

EXAMPLE 17



G_1

$$P_2(G_1) = 2 + 21x + 33x^2 + 138x^3 + 66x^4 + 165x^5 + 15x^6 + 46x^7 + x^9$$

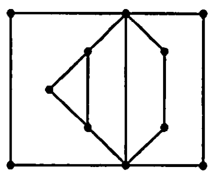
$$P_2(G_1 - \{48\}) = 2x + 46x^3 + 96x^5 + 38x^7 + x^9$$

$$\mu_0^{(2)} = \mu_2^{(2)} = \mu_4^{(2)} = \mu_6^{(2)} = 1$$

$$\mu^{(2)}(G_1) = \underline{1 + 2x} + x^2 + \underline{3x^3} + x^4 + \overline{3x^5} + x^6 + \overline{2x^7 + x^9}$$

$$\text{speed}(G_1) = 1 \div (7 - 1) = 1/6 \Rightarrow \text{very slow}$$

EXAMPLE 18



$$P_2(G) = 2 + 16x + 52x^2 + 93x^3 + 132x^4 + 104x^5 + 72x^6 + 42x^7$$

$$\mu^{(2)}(G) = \underline{1 + 2x + 2x^2 + 3x^3} + 3x^4 + \overline{2x^5 + 2x^6 + x^7}$$

$$\text{speed}(G) = 1/2 \Rightarrow \text{fast}$$

Notice that the speed of the Petersen graph is 1/4.

4 Conclusion

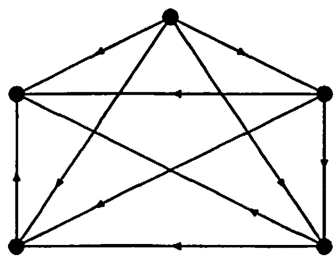
4.1 Interpolation and digraphs ?

Given the results on interpolation in undirected graphs, interpolation in digraphs is the next logical step. The question is which generalization should be used for digraphs ?

- count in-leaves or out-leaves in each spanning tree of the digraph D.
- count vertices of in-degree 1 or out-degree 1 in each spanning tree.
- count in-leaves and out-leaves in each spanning tree.
- count vertices of in-degree 1 and vertices of out-degree 1 in each spanning tree.

In the following example $P_1^{(1)}$ counts the number of vertices of in-degree 1 in each spanning tree, $P_1^{(2)}$ counts the number of in-leaves in each spanning tree and the $(k+1, j+1)$-entry of M is the the number of spanning trees with k in-leaves and j out-leaves.

EXAMPLE 19



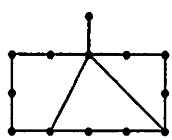
$$P_1^{(1)} = 13 + 16x + 72x^2 + 24x^4$$
$$P_1^{(2)} = 26 + 51x + 36x^2 + 11x^3 + x^4$$

$$M = \left\{ \begin{array}{ccccc} 0 & 0 & 15 & 10 & 1 \\ 0 & 30 & 20 & 1 & 0 \\ 15 & 20 & 1 & 0 & 0 \\ 10 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right\}$$

Notice that $P_1^{(1)}$ has a 3-gap. In any case there will be the accompanying filter theory and P_1-morphology to study.

4.2 P_2-morphology and gaps

In the graph shown below $\rho = 3$ and the coefficient of x^4 is 5. This implies that when $P_2(G) = \sum_r^s c_k x^k$ then for $r < k < s$, c_k can be less that 2ρ. What is a lower bound for c_k in this case?



$$P_2 = 6x^3 + 5x^4 + 23x^5 + 41x^6 + 26x^7 + 20x^8 + 16x^9 + x^{11}$$

Another question concerns the gaps in P_2.

Conjecture: In $P_2(G)$ all of the coefficients are nonzero or P_2 has one gap or P_2 is odd or P_2 is even.

Problem: Suppose that $P_2(G)$ has a k-gap. Find a minimal spanning subgraph gap(k, G) such that $P_2(\text{gap}(k, G))$ has a k-gap, where

$$P_2(\text{gap}(k, G)) = \sum_j d_j x^j$$

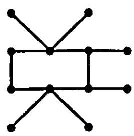
Studying the gap-subgraphs would be interesting because gaps occur in P_2 for different reasons:

- $\Delta(G) \leq 3$.
- G is Hamiltonian.

- G has a specific configuration like the graph shown in the next example.

EXAMPLE 20

A nonhamiltonian graph with $\Delta(G) > 3$ and a 2-gap in P_2.



$$P_2 = 1 + 2x + 2x^3 + x^4$$

Appendix

A. Maximum intersecting spanning trees

The maximum intersecting spanning tree is an important tool in the proof that n_1 interpolates on $Q_4(G)$.

Definition 6 Let g be a connected subgraph of the connected graph G. A maximum intersecting spanning tree with respect to g, denoted $\mathcal{M}(g;G)$ or $\mathcal{M}(g)$, is a spanning tree T such that

$$|E(T) \cap E(g)| = |V(g)| - 1.$$

Theorem 4 *Let G be a connected graph and g a connected subgraph. Then n_1 interpolates on $\mathcal{M}(g)$.*

Proof: By induction on the size of G. The assertion is clear if G is unicyclic. Hence, G will be a smallest counterexample of order n and size $m > n$, where $n \geq 4$. Let $\mathcal{M}(g) = \{T_1, T_2, \cdots, T_\lambda\}$, where

$$n_1(T_j) \leq n_1(T_{j+1}), \tag{17}$$

$$n_1(T_r) \leq n_1(T_{r+1}) - 2. \tag{18}$$

Suppose that every edge of g is in T_{r+1}. This means that g is a connected acyclic subgraph of G. Therefore every spanning tree of $\mathcal{M}(g)$ contains g. If there is an edge e that is not in T_r or T_{r+1} then $G - e$ would contain T_r and T_{r+1}. Since G is a smallest counterexample this is impossible. Therefore $E(G) = E(T_{r+1}) \cup E(T_r)$.

Let $e = u_1u_2 \in E(T_{r+1}) - E(T_r)$ and $f = v_1v_2 \in E(T_r) - E(T_{r+1})$ such that $J = T_{r+1} - e + f$ is a spanning tree of G. Notice that J contains g. If $n_1(J) \geq n_1(T_{r+1})$ then J and T_r are in $\mathcal{M}(g; G - e)$. This is a contradiction. Therefore, $n_1(J) \leq n_1(T_{r+1})$. Moreover, $n_1(T_{r+1}) - n_1(J) \leq 2$. Thus,

$$n_1(T_{r+1}) = n_1(J) + 2. \tag{19}$$

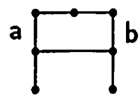
Equation (19) implies that u_1 and u_2 have degree at least 3 in T_{r+1} while v_1 and v_2 are leaves of T_{r+1}. Thus, $u_i \neq v_j$ and v_1 and v_2 have degree 2 in $T_{r+1} + f$. Let T_0 be the component of $T_{r+1} - e$ that does not contain g and let $C_f = w_1, \cdots, w_c, w_{c+1}, \cdots, w_d$ be the cycle of $T_{r+1} + f$, where v_2 and u_2 are in T_0, $w_1 = v_2, w_c = u_2, w_{c+1} = u_1$ and $w_d = v_1$. Let h be the smallest integer such that $d(w_h) \geq 3$ and $d(w_{h-1}) = 2$, where $h \leq c$. Let $H = T_{r+1} + f - w_hw_{h-1}$. Since edge w_hw_{h-1} is in T_0 we conclude that H contains g. Furthermore, $n_1(H) = n_1(T_{r+1}) - 1$. Since this is a contradiction $E(g) - E(T_{r+1}) \neq \emptyset$.

Let $f = v_1v_2 \in E(g) - E(T_{r+1})$. Consequently, $T_{r+1} + f$ contains a unique cycle C_f. Edge f must be in $E(T_r)$, otherwise both T_r and T_{r+1} are in $\mathcal{M}(g - f; G - f)$. Thus, some edge $e = u_1u_2$ of C_f is not in T_r. This means that $H = T_{r+1} - e + f \in \mathcal{M}(g)$ because $|E(H) \cap E(g)| \geq |E(T_{r+1}) \cap E(g)|$. Also, $|n_1(T_{r+1}) - n_1(H)| \leq 2$. Now, if $n_1(H) \geq n_1(T_{r+1})$ then T_r and H would be in $\mathcal{M}(g - e; G - e)$. Therefore, $n_1(H) \leq n_1(T_{r+1})$ which implies that $n_1(H) + 2 = n_1(T_{r+1})$. Thus, $d(u_1) > 2$ and $d(u_2) > 2$ in T_{r+1} while v_1 and v_2 are leaves of T_{r+1}.

Let $C_f = v_1, v_2, \cdots, v_s$, where $f = v_1v_2$. Let q be the smallest integer such that $d(v_{q-1}) = 2$ and $d(v_q) \geq 3$. Since u_1 and u_2 are on C_f, q does exist. Let $J = T_{r+1} + f - v_{q-1}v_q$. Since $|E(g) \cap E(J)| \geq |E(g) \cap E(T_{r+1})|$, we conclude that $J \in \mathcal{M}(g)$. Moreover, $n_1(J) = n_1(T_{r+1}) - 1$. Since this contradicts equations (17-18) we conclude that the counterexample G does not exist.

If g is a forest then n_1 may not interpolate on $\mathcal{M}(g)$. Consider the graph shown below and notice that if g consists of edges a and b then $P_1(\mathcal{M}(g))$ has a 3-gap.

EXAMPLE 21
An example in which $P_1(\mathcal{M}(g))$ has a 3-gap, where $g = \{a, b\}$.



$$P_1(\mathcal{M}(g)) = x^2 + 2x^4$$

B. Spanning tree enumeration

Enumeration of spanning trees is an important part of filter theory and P_2-morphology. There are hundreds of papers written on this topic and just as many algorithms! As demonstrated in the introduction, P_1 can be calculated by hand if the graph is fairly small and of low connectivity. For large graphs a simple recursive procedure can be used or even a backtracking method (See [9]).

As far as finding an approximate pendant polynomial, two methods were used:

1. A "random" version of Kruskal's algorithm.
2. Applying random elementary exchanges to a series of spanning trees to take a "random walk" through the tree graph.

No attempt was made to apply statistical methods to pendant polynomial approximation but this may be an interesting area of future research.

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