GRAPH PARTITIONS

T.D. PORTER[1]
SOUTHERN ILLINOIS UNIVERSITY
CARBONDALE, ILLINOIS 62901

ABSTRACT. For any graph G, and all $s = 2^k$, we show there is a partition of the vertex set of G into s sets $U_1, \ldots, U_s$, so that both, $e(G[U_i]) \leq \frac{e(G)}{s^2} + \sqrt{\frac{e(G)}{s}}$, for $i = 1, \ldots, s$ and $\sum_{i=1}^{s} e(G[U_i]) \leq \frac{e(G)}{s}$.

Introduction.

The graphs under consideration are without loops and multiple edges. For any undefined terms please consult [1]. Take a graph G, and a partition $(U_1, \ldots, U_s)$ of $V(G)$ into s classes with $\bigcup U_i = V(G)$, $U_i \cap U_j = \emptyset$ for $i \neq j$. We also refer to $(U_1, \ldots, U_s)$ as a coloring of the vertex set using s colors where U_j denotes the set of vertices colored j. Let $\gamma_s(U_1, \ldots, U_s) = \max\{e(U_1), \ldots, e(U_s)\}$, where $e(U_i)$ denotes the number of edges in the induced subgraph $G[U_i]$, and $\gamma_s(G) = \min_{(U_1,\ldots,U_s)} \gamma_s(U_1, \ldots, U_s)$. Entringer [3], introduced the function $\gamma_2(G)$, and Paul Erdös [4] conjectured $\gamma_2(G)/e(G) \leq \frac{1}{4} + O(1/\sqrt{e(G)})$, $e(G)$ denotes the number of edges in G. In [5], the present author verified Erdös' conjecture and showed it was best possible. In [2] and [7], Clark,

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Shahroki, and Székely, show the computation of $\gamma_2(G)$ is NP-hard, they relate it to the known complexity of the max cut problem. In [6], Porter and Székely, solve a matrix discrepancy problem that approximates $\gamma_2(G)$.

The present paper extends these results to $s > 2$. Moreover, let $M_s(U_1, \ldots, U_s) = e(G) - \sum_{i=1}^{s} e(U_i)$, and $M_s(G) = \max_{(U_1,\ldots,U_s)} M_s(U_1, \ldots, U_s)$, note $M_2(G)$ is the well known max cut of G, and the usual chromatic number $\chi(G)$ can be defined as

$$\chi(G) = \min\{s | \gamma_s(G) = 0\} = \min\{s | M_s(G) = e(G)\}.$$

In this paper we are primarily concerned with fixing s, and finding partitions that give $\gamma_s(G)$ and $M_s(G)$. The main result is that for any G, and all $s = 2^k$, there exists a partition of the vertex set of G into s sets, $U_1, \ldots, U_s$, so that simultaneously $\gamma_s(U_1, \ldots, U_s) \leq \frac{e(G)}{s^2} + \sqrt{\frac{e(G)}{s}}$ and $M_s(U_1, \ldots, U_s) \geq \frac{s-1}{s} e(G)$. The lower bound for M_s holds for all s.

Lemma 1. *If $e(G) < s$ then $\gamma_s(G) = 0$.*

Proof. First, if G is connected then with $e(G) < s$ we have $|V(G)| \leq s$, i.e., notice with $e(G)$ fixed the largest $|V(G)|$ is attained when G is a tree. So we may assign different colors to each $u \in V(G)$, hence $\gamma_s(G) = 0$. If G is disconnected, then write $G = G_1 \bigcup G_2 \bigcup \cdots \bigcup G_w$ as a union of its components, then $\gamma_s(G) = \max\{\gamma_s(G_1), \ldots, \gamma_s(G_w)\}$ and with $e(G_i) < s$ we have as above $|V(G_i)| \leq s$ for each i, hence $\gamma_s(G_i) = 0$. □

Theorem 1.

For any graph G, there exists a bipartition $(A, \bar{A})$ of the vertex set $V(G)$ so that $\frac{\gamma_2(A, \bar{A})}{e(G)} \leq \frac{1}{4}\left(1 + \sqrt{\frac{2}{e(G)}}\right)$ *and* $M_2(A, \bar{A}) \geq \frac{e(G)}{2}$.

Proof. See [5].

Theorem 2.
For any graph G, and all $s = 2^p$, there exists a partition $(U_1, \ldots, U_s)$ of $V(G)$ so that

$$\gamma_s(U_1, \ldots, U_s) \leq \frac{e(G)}{s^2} + \sqrt{\frac{e(G)}{s}} \qquad (a)$$

$$M_s(U_1, \ldots, U_s) \geq \frac{s-1}{s} e(G). \qquad (b)$$

Proof. The proof is by induction on p. The ground case $p = 1$ is given by Theorem 1. Now take any $p \geq 2$. First, if $e(G) < 2^p$ then by Lemma 1, $\gamma 2^p(G) = 0$, and we are done. So assume $e(G) \geq 2^p$.

Let $(U_1, \ldots, U_{2^{p-1}})$ be a partition realizing the inductive hypothesis, i.e.

$$\gamma_{2^{p-1}}(U_1, \ldots, U_{2^{p-1}}) \leq \frac{e(G)}{2^{2p-2}} + \sqrt{\frac{e(G)}{2^{p-1}}},$$

and $M_{2^{p-1}}(U_1, \ldots, U_{2^{p-1}}) \geq \frac{2^{p-1}-1}{2^{p-1}} e(G)$, to each U_i, let $(A_i, \bar{A}_i)$ be the bipartition of U_i given by Theorem 1, i.e.,

$$\gamma_2(A_i, \bar{A}_i) \leq \frac{1}{4}\left(e(U_i) + \sqrt{2e(U_i)}\right)$$

and $M_2(A_i, \bar{A}_i) \geq \frac{e(U_i)}{2}$. We show $(A_1, \bar{A}_1, \ldots, A_{2^{p-1}}, \bar{A}_{2^{p-1}})$ satisfies the theorem.

We have,

$$\begin{aligned}
&\gamma_{2^p}(A_1, \bar{A}_1, \ldots, A_{2^{p-1}}, \bar{A}_{2^{p-1}}) \\
&= \max_{1 \leq i \leq 2^{p-1}} \gamma_2(A_i, \bar{A}_i) \\
&\leq \max \frac{1}{4}\left(e(U_i) + \sqrt{2e(U_i)}\right) \quad \text{(by Thm. 1)} \\
&\leq \frac{1}{4}\left(\frac{e(G)}{2^{2p-2}} + \sqrt{\frac{e(G)}{2^{p-1}}} + \sqrt{2\left(\frac{e(G)}{2^{2p-2}} + \sqrt{\frac{e(G)}{2^{p-1}}}\right)}\right)
\end{aligned}$$

the last inequality by the inductive hypothesis, we show

$$\leq \frac{1}{4}\left(\frac{e(G)}{2^{2p-2}}+\sqrt{\frac{e(G)}{2^{p-1}}}+\sqrt{2\left(\frac{e(G)}{2^{2p-2}}+\sqrt{\frac{e(G)}{2^{p-1}}}\right)}\right) \leq \frac{e(G)}{2^{2p}}+\sqrt{\frac{e(G)}{2^{p}}},$$

to establish the inductive step.

Note sufficient to show,

$$\sqrt{\frac{e(G)}{2^{2p-2}}+\sqrt{\frac{e(G)}{2^{p-1}}}} \leq \left(\frac{4-\sqrt{2}}{2}\right)\sqrt{\frac{e(G)}{2^{p-1}}}. \tag{1}$$

We have for all $p \geq 2$,

$$\frac{1}{2^{2p-2}}+\sqrt{\frac{1}{2^{p}2^{p-1}}} \leq \frac{(4-\sqrt{2})^2}{2^{p+1}},$$ and with $e(G) \geq 2^p$, that

$$\frac{1}{2^{2p-2}}+\sqrt{\frac{1}{e(G)2^{p-1}}} \leq \frac{(4-\sqrt{2})^2}{2^{p+1}},$$ hence

$$\frac{e(G)}{2^{2p-2}}+\sqrt{\frac{e(G)}{2^{p-1}}} \leq \frac{(4-\sqrt{2})^2}{4}\left(\frac{e(G)}{2^{p-1}}\right).$$

This establishes (1) and completes the proof of Theorem 2(a).

To show (b), we have by the inductive hypothesis, that for $(U_1,\ldots,U_{2^{p-1}})$,

$$e(G)-\sum_{i=1}^{2^{p-1}} e(U_i) = M_{2^{p-1}}(U_1,\ldots,U_{2^{p-1}}) \geq \frac{2^{p-1}-1}{2^{p-1}}e(G),$$

i.e., $\sum_{i=1}^{2^{p-1}} e(U_i) \leq \frac{e(G)}{2^{p-1}}$. Also, by our choice of $(A_i,\bar{A}_i)$ we have for each i, $e(A,\bar{A}_i) \geq \frac{e(U_i)}{2}$, hence $e(A_i)+e(\bar{A}_i) \leq \frac{e(U_i)}{2}$, then

$$\begin{aligned} M_{2^p}(A_1,\bar{A}_1,\ldots,A_{2^{p-1}},\bar{A}_{2^{p-1}}) &= e(G) - \sum_{i=1}^{2^{p-1}}(e(A_i)+e(\bar{A}_i)) \\ &\geq e(G) - \tfrac{1}{2}\sum_{i=1}^{2^{p-1}} e(U_i) \\ &\geq e(G) - \tfrac{1}{2}\cdot\frac{e(G)}{2^{p-1}} \\ &= \frac{2^p-1}{2^p}e(G). \end{aligned}$$

This completes the inductive step and the proof of Theorem 2(b). □

We state the next lemmas to conclude with a lower bound on the maximum s-cut of G. For $u \in V(G)$, $H \subset V(G)$, define $d_H(u)$ be the number of vertices in H adjacent to u.

Lemma 2. *For a graph G, and a partition $(U_1,\ldots,U_s)$ of $V(G)$ that gives the max s-cut, $M_s(G)$, for any U_i, $U_j \in \{U_1,\ldots,U_s\}$ we have $d_{U_j}(x) \geq d_{U_i}(x)$ for all $x \in U_i$.*

Proof. Take a partition $(U_1,\ldots,U_s)$ that gives the max s-cut, i.e., $M_s(G) = M_s(U_1,\ldots,U_s)$. Take any two classes U_i, U_j, *W.L.O.G* say $U_i = U_1$, $U_j = U_2$. We have $d_{U_2}(x) \geq d_{U_1}(x)$ for all $x \in U_1$, since suppose the contrary, i.e., there exist some $x \in U_1$ with $d_{U_2}(x) < d_{U_1}(x)$, let $\varepsilon = d_{U_1}(x) - d_{U_2}(x) > 0$. Then, $M_s(U_1 - x, U_2 + x, U_3,\ldots,U_s) = M_s(U_1,U_2,\ldots U_s) + \varepsilon > M_s(U_1,\ldots,U_s) = M_s(G)$, contradicting the definition of $M_s(G)$. □

Corollary 1. *For a graph G, and a partition $(U_1,\ldots,U_s)$ of $V(G)$ that gives the max s-cut, $M_s(G)$, for any $U_i, U_j \in \{U_1,\ldots,U_s\}$, $i \neq j$, we have $M_2(U_i,U_j) \geq 2\max\{e(U_i),e(U_j)\}$.*

Proof. Take any $(U_1, \ldots, U_s)$ with $M_s(U_1, \ldots, U_s) = M_s(G)$, and U_i, $U_j \in \{U_1, \ldots, U_s\}$, $i \neq j$. Let $e(U_i) = \max\{e(U_i), e(U_j)\}$. We have, by Lemma 2, that $d_{U_j}(x) \geq d_{U_i}(x)$ for all $x \in U_i$, hence,

$$M_2(U_i, U_j) = \sum_{x \in U_i} d_{U_j}(x) \geq \sum_{x \in U_i} d_{U_i}(x) = 2e(U_i)$$

$$= 2\max\{e(U_i), e(U_j)\}.$$

□

Theorem 3. *For any graph G, and all s, $M_s(G) \geq \frac{s-1}{s} e(G)$.*

Proof. Let $(U_1, \ldots, U_s)$ be a partition that gives $M_s(G)$, i.e., $M_s(U_1, \ldots, U_s) = M_s(G)$. We have the following two cases on $\sum e(U_i)$. First, if $\sum_{i=1}^{s} e(U_i) \leq \frac{e(G)}{s}$, then

$$M_s(G) = e(G) - \sum_{i=1}^{s} e(U_i) \geq e(G) - \frac{e(G)}{s} = \frac{s-1}{s} e(G).$$

Otherwise, $\sum_{i=1}^{s} e(U_i) \geq \frac{e(G)}{s}$, note that

$$M_s(U_1, \ldots, U_s) = \tfrac{1}{2} \sum_{i=1}^{s} M_2(U_i, V(G) \backslash U_i) \geq \tfrac{1}{2} \sum_{i=1}^{s} 2(s-1) e(U_i),$$

the last inequality from Corollary 1. Hence

$$M_s(U_1, \ldots, U_s) \geq (s-1) \sum_{i=1}^{s} e(U_i) \geq (s-1) \frac{e(G)}{s}.$$

□

Conclusions.

We end with some open problems. Say a graph G is in a 'state' of maximum s-cut when we have a partition $(U_1, \ldots, U_s)$ of $V(G)$ with $M_s(U_1, \ldots, U_s) = M_s(G)$. We have if G is in a state of maximum s-cut, then clearly, $\sum e(U_i)$ is minimized, i.e., $M_s(G) = e(G) - \min_{(U_1,\ldots,U_s)} \sum e(U_i)$. We ask what can we say about other norms? For example, is $(\sum e(U_i)^p)^{1/p}$ minimized for all $p \geq 1$ when G is in this state.

We have for the complete graph on $sn+1$ vertices, i.e., K_{sn+1}, that

$$\gamma_s(K_{sn+1}) = \frac{e(K_{sn+1})}{s^2} + O\left(\sqrt{\frac{e(K_{sn+1})}{s^2}}\right),$$

and

$$M_s(K_{sn+1}) = \frac{s-1}{s} e(K_{sn+1}) + O\left(\sqrt{e(K_{sn+1})}\right),$$

hence, if bounds in Thm. 2, Thm. 3, can be improved, they can only be modified in the error term. We would also like to extend the bounds on $\gamma_s(G)$ to all s.

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