Ramsey Functions Associated with Second Order Recurrences

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Abstract. Numbers similar to the van der Waerden numbers $w(n)$ are studied, where the class of arithmetic progressions is replaced by certain larger classes. If $\mathcal{A}'$ is such a larger class, we define $w'(n)$ to be the least positive integer such that every 2-coloring of $\{1, 2, \ldots, w'(n)\}$ will contain a monochromatic member of $\mathcal{A}'$. We consider sequences of positive integers $\{x_1, \ldots, x_n\}$ which satisfy $x_i = a_i x_{i-1} + b_i x_{i-2}$ for $i = 3, \ldots, n$, with various restrictions placed on the a_i and b_i. Upper bounds are given for the corresponding functions $w'(n)$. Further, it is shown that the existence of somewhat stronger bounds on $w'(n)$ would imply certain bounds for $w(n)$.

Introduction

A well-known theorem of van der Waerden [7] states that for each positive integer n, there exists a least positive integer $w(n)$ such that no matter how $\{1, 2, \ldots, w(n)\}$ is partitioned into two sets, at least one of the two sets will contain an n-term arithmetic progression. A long-standing problem in the area of Ramsey Theory has been to determine the rate of growth of the van der Waerden numbers $w(n)$. The most significant progress on this problem was made only recently by Shelah [5] who showed that there exists a primitive recursive upper bound on $w(n)$. There is still a huge discrepancy, however, between the bound Shelah obtains and the best known lower bound of $w(p+1) \geq p2^p$ for p a prime (see [1]). For example, it is unknown whether $w(n) \leq n^{n^{\cdot^{\cdot^{n}}}}$ a tower of height n. The only known nontrivial values of $w(n)$ are $w(3) = 9$, $w(4) = 35$, and $w(5) = 178$ (see [2] and [6]). If $\mathcal{A}$ represents the class of arithmetic progressions, and if $\mathcal{A}' \supseteq \mathcal{A}$, then by van der Waerden's theorem we may define $w'(n)$ to be the least positive integer such that if $\{1, 2, \ldots, w'(n)\}$ is partitioned into two sets, then at least one set will contain a member of $\mathcal{A}'$. It is clear that $w'(n) \leq w(n)$. The motivating idea behind this article is to make $\mathcal{A}'$ large enough so that a reasonable upper bound for $w'(n)$ can be found, but to have $\mathcal{A}'$ close enough to $\mathcal{A}$ so that information about $w'(n)$ might lead to information about $w(n)$. This idea was used in [3] and [4] where $\mathcal{A}'$ was taken to be a collection of sequences obtained by iteration of certain polynomials (an arithmetic progression is obtained by iterating $f(x) = x + d$). In those articles, upper bounds were obtained for $w'(n)$. Further, it was shown that the existence of somewhat lower bounds on $w'(n)$ would imply the existence of similar bounds on $w(n)$. In this paper we consider a different

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type of generalization of the class of arithmetic progressions. Namely, we look at an arithmetic progression as being the solution to the recurrence relation $x_k = 2x_{k-1} - x_{k-2}, k \geq 3, x_1, x_2 \in \mathbf{Z}^+, x_1 < x_2$, and then allow the coefficients of x_{k-1} and x_{k-2} to be less restricted.

The following notation and terminology will be used throughout the paper. All variables will represent integers. We denote $\{a, a+1, \ldots, b\}$ and $\{a, a+1, \ldots\}$ by $[a, b]$ and $[a, \infty)$, respectively. A *2-coloring* of $[1, m]$ is a function $\chi : [1, m] \rightarrow \{0, 1\}$. A set S is *monochromatic* under a 2-coloring χ if χ is constant on S.

An *f-sequence* is a sequence $\{x_1, \ldots, x_n\}$ satisfying $x_1 > 0, x_2 > x_1$, and $x_k = a_k x_{k-1} + b_k x_{k-2}$ for $k = 3, \ldots, n$, where $a_k \geq 2, b_k = 1 - a_k$. A *$g$-sequence* is the same as an f-sequence except that $b_k = -1$ for each k. An *h-sequence* is the same as an f-sequence except $a_k = 2$ for each k, while $b_k \geq -1$. It is easy to see that every f-, g-, or h-sequence is strictly increasing. In order to get more information about $w(n)$, we define the following, more restrictive, types of sequences. Let $\lambda : [3, \infty) \rightarrow [2, \infty)$, and let $\mu : [3, \infty) \rightarrow [-1, \infty)$. Then an *$f_\lambda$-sequence* is an f-sequence such that, for each k, either $a_k = 2$ or $a_k \geq \lambda(k)$. A *g_λ-sequence* is a g-sequence such that either $a_k = 2$ or $a_k \geq \lambda(k)$. An *h_μ-sequence* is an h-sequence such that either $b_k = -1$ or $b_k \geq \mu(k)$ for each k.

It is clear that every arithmetic progression is also an f_λ-, a g_λ-, and an h_μ-sequence. Thus, if λ is a given function, if $\mathcal{A}'$ represents the collection of f_λ-sequences of length n, and if $\mathcal{A}''$ represents the collection of f-sequences of length n, then we have $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}''$. Therefore the associated Ramsey functions satisfy $w(n) \geq w'(n) \geq w''(n)$. We will use $f_\lambda(n)$ and $f(n)$ to represent $w'(n)$ and $w''(n)$, respectively. Likewise, we will use $g_\lambda(n)$, $g(n)$, $h_\mu(n)$, and $h(n)$ to represent the Ramsey functions corresponding to the collection of g_λ-sequences, g-sequences, h_μ-sequences, and h-sequences; and again we have the relationships $w(n) \geq g_\lambda(n) \geq g(n)$ and $w(n) \geq h_\mu(n) \geq h(n)$. In §2, we obtain upper bounds for $f_\lambda(n)$, $g_\lambda(n)$, and $h_\mu(n)$; more significantly, we show in §3 that if these bounds could be improved, then we would have "nice" bounds on the van der Waerden numbers themselves.

2. Upper Bounds

From now on, whenever the notation $f_\lambda(n)$ or $g_\lambda(n)$ is used, it is understood that λ is a function mapping $[3, \infty)$ into $[2, \infty)$. Also, when $h_\lambda(n)$ is used, it is understood that λ maps $[3, \infty)$ into $[-1, \infty)$.

In this section we obtain upper bounds for $f_\lambda(n)$, $g_\lambda(n)$, and $h_\lambda(n)$, for a given function λ. We first mention the basic relationships among these functions and $w(n)$. All of the inequalities in the next theorem are clear from the definitions.

Theorem 2.1. *Let λ_1 and λ_2 be functions defined on $\{3, 4, \ldots\}$ with $\lambda_1(i) \geq$*

$\lambda_2(i)$ *for all* i. *Then, for all* n

(i) $w(n) \geq f_{\lambda_1}(n) \geq f_{\lambda_2}(n)$.

(ii) $w(n) \geq g_{\lambda_1}(n) \geq g_{\lambda_2}(n)$.

(iii) $w(n) \geq h_{\lambda_1}(n) \geq h_{\lambda_2}(n)$.

The next theorem provides upper bounds for f_λ, g_λ, and h_λ.

Theorem 2.2. *Let* λ *be a given function. Then for* $n \geq 3$

(i) $f_\lambda(n) \leq f_\lambda(3) \prod_{k=4}^{n}(\lambda(k) + k - 1)$ *(1)*

(ii) $g_\lambda(n) \leq g_\lambda(3) \prod_{k=4}^{n}(\lambda(k) + k - 1)$ *(2)*

(iii) $h_\lambda(n) \leq h_\lambda(3) \prod_{k=4}^{n}(\lambda(k) + k - 1)$ *(3)*

Proof. (i) This is obvious if $n = 3$. Using induction on n, assume that $n \geq 4$ and that the result is true for $n-1$. Let $M(n)$ represent the right side of (1), and let $\chi : [1, M(n)] \to \{0,1\}$ be a 2-coloring. By the induction hypothesis we know there exists a monochromatic f_λ-sequence of length $n-1$ contained in $[1, M(n-1)]$. Say $\mathrm{x} = \{x_1, \ldots, x_{n-1}\}$ is such a sequence and that $\chi(\mathrm{x}) = 0$. For $i = \lambda(n) - 1, \ldots, \lambda(n) + n - 2$ define

$$y_i = x_{n-1} + i(x_{n-1} - x_{n-2}).$$

It is clear that $\{x_1, x_2, \ldots, x_{n-1}, y_i\}$ is an n-term f_λ-sequence for each i. Note also that the largest of the y_i is $x_{n-1} + (\lambda(n)+n-2)(x_{n-1} - x_{n-2}) \leq x_{n-1}(\lambda(n) + n - 1) \leq M(n)$. Hence, in case any of the y_i has color 0, $[1, M(n)]$ will contain a monochromatic n-term f_λ-sequence. Otherwise, $\chi(y_i) = 1$ for each i, so that $[1, M(n)]$ contains a monochromatic arithmetic progression, and hence, an f_λ-sequence.

(ii) This is proved in the same way as (i), except we let $y_i = ix_{n-1} - x_{n-2}$ for $i = \lambda(n), \ldots, \lambda(n) + n - 1$, and then note that the largest of the y_i cannot exceed the right side of (2).

(iii) Using the same idea as in (i) and (ii), we let $M(n)$ denote the right side of (3), and assume $\{x_1, \ldots, x_{n-1}\}$ has color 0 and is contained in $[1, M(n-1)]$. For $i = \lambda(n), \ldots, \lambda(n) + n - 1$ let $y_i = 2x_{n-1} + ix_{n-2} \leq x_{n-1}(\lambda(n) + n + 1) \leq M(n)$. Then either $\{x_1, \ldots, x_{n-1}, y_i\}$ is colored 0 for some i, in which case we have a monochromatic h_λ-sequence, or else $\{y_i\}$ forms an n-term monochromatic arithmetic progression. ∎

By using $\lambda(n) \equiv 2$, $\lambda(n) \equiv 2$, and $\lambda(n) \equiv -1$, in (1), (2) and (3), respectively, we obtain the following upper bounds for $f(n)$, $g(n)$, and $h(n)$. The constant coefficients are obtained by noting that $f(3) = 7$, $g(3) = 8$, and $h(3) = 7$ (see §4).

Corollary 2.3. *For* $n \geq 3$,

(i) $f(n) \leq \frac{7}{24}(n+1)!$

(ii) $g(n) \leq \frac{1}{3}(n+1)!$

(iii) $h(n) \leq \frac{7}{6}n!$.

Remarks. (i) It is easy to see that given $m \in \mathbf{Z}^+$, the greater the magnitude of the function λ, the fewer the number of f_λ-, g_λ-, or h_λ-sequences there are in $[1, m]$. For example, letting $\lambda(i) = i^3$, we have that, except for arithmetic progressions, there are no f_λ-sequences of length 4 in $[1,34]$ (the non-arithmetic f_{i^3}-sequence of length 4 with the least value of x_4 is $\{1,2,3,66\}$ with $x_3 = 2x_2 - x_1$ and $x_4 = 64x_3 - 63x_2$). Thus $f_{i^3}(4) = w(4) = 35$, while $f_2(4) = f(4) = 13$ (see §4).

(ii) Each of the upper bounds given by Theorem 2.2 actually holds for somewhat smaller classes of sequences. Let us define an $\widehat{f}_\lambda$-sequence to be an f_λ-sequence $\{x_i\}$ such that $x_i = a_i x_{i-1} - (a_i - 1)x_{i-2}$ with $a_i \in [\lambda(i), \lambda(i) + i - 1] \cup \{2\}$ for each i. Likewise, define a $\widehat{g}_\lambda$-sequence to be a g_λ-sequence with the added restriction that a_i cannot exceed $\lambda(i) + i - 1$, and an $\widehat{h}_\lambda$-sequcence to be an h_λ-sequence such that b_i cannot exceed $\lambda(i) + i - 1$. Then the same proof that we gave of Theorem 2.2 shows that the bounds we obtained for f_λ, g_λ, and h_λ also are bounds for $\widehat{f}_\lambda$, $\widehat{g}_\lambda$, and $\widehat{h}_\lambda$, respectively. The less restricted types of sequences f_λ, g_λ, and h_λ are used here because they yield stronger results when finding sufficient conditions for bounds on $w(n)$ in §3.

3. Comparisons with $w(n)$

In this section we consider the magnitude of the terms of f_λ-, g_λ-, and h_λ-sequences which do not contain any arithmetic progressions of a fixed length. We then show that if certain upper bounds were to hold for the Ramsey functions associated with these types of sequences, then this would imply the existence of similar upper bounds for $w(n)$. We will need the following lemmas.

Lemma 3.1. *If $\{x_1, \ldots, x_n\}$ is an f_λ-, g_λ-, or h_λ-sequence, and if $1 \leq i \leq k \leq n-1$, then $x_{k-1} - x_k \geq x_{i+1} - x_i$.*

Proof. It is obvious from the definitions that the sequence of increments $\{x_{i+1} - x_i : i = 1, \ldots, n-1\}$ in any f_λ-, g_λ-, or h_λ-sequence must be non-decreasing. ∎

Lemma 3.2. *Let $\lambda(r+s) \geq (r+1)\lambda(s)$ for all $r \geq 0$ and $s \geq 3$. Let $k \geq 3$, $i \geq 1$, $t \geq 2$, and $i + t \leq k$. Then*

(i) *If $\{y_1, \ldots, y_k\}$ is an f_λ-sequence and $\{y_i, \ldots, y_{i+t}\}$ is not an arithmetic progression, then*

 (a) $y_{i+t} - y_{i+t-1} \geq (y_{i+1} - y_i)[\lambda(i+2) - 1]$ *and*

 (b) $y_{i+t} \geq (t-1)(y_{i+1} - y_i)[\lambda(i+2) - 1]$.

(ii) *If $\{y_1, \ldots, y_k\}$ is a g_λ-sequence and $\{y_i, \ldots, y_{i+t}\}$ is not an arithmetic progression, then $y_{i+t} \geq (t-1)y_{i+1}[\lambda(i+2) - 2]$.*

(iii) *If $\{y_1, \ldots, y_k\}$ is a h_λ-sequence and $\{y_i, \ldots, y_{i+t}\}$ is not an arithmetic progression, then $y_{i+t} \geq (t-1)y_i\lambda(i+2)$.*

Proof. (i) (a) Obviously, for some j such that $i \le j \le i+t-2$, $\{y_j, y_{j+1}, y_{j+2}\}$ must fail to be an arithmetic progression. By Lemma 3.1, $y_{i+t} - y_{i+t-1} \ge y_{j+2} - y_{j+1} \ge [\lambda(j+2)-1](y_{j+1}-y_j) \ge [\lambda(i+2)-1](y_{i+1}-y_i)$.

(b) Given y_j and y_{j+1} with $i \le j \le i+t-2$, it is clear that y_{j+2} will be minimized when $\{y_j, y_{j+1}, y_{j+2}\}$ is an arithmetic progression. Thus, since $\{y_i, \dots, y_{i+t}\}$ is not an arithmetic progression, we may assume that there is exactly one k, $0 \le k \le t-2$, such that $\{y_{i+k}, y_{i+k+1}, y_{i+k+2}\}$ is not an arithmetic progression. Then

$$y_{i+k+2} \ge y_{i+k+1}\lambda(i+k+2) - y_{i+k}(\lambda(i+k+2)-1)$$

and

$$y_{i+t} = y_{i+k+1} + (t-k-1)(y_{i+k+2} - y_{i+k+1}).$$

Therefore,

$$\begin{aligned} y_{i+t} &\ge y_{i+k+1} + (t-k-1)(y_{i+k+1} - y_{i+k})[\lambda(i+k+2)-1] \\ &\ge (t-k-1)(y_{i+k+1} - y_{i+k})[(k+1)\lambda(i+2)-1] \end{aligned}$$

Also, $k(t-k-2) \ge 0$, which implies $(t-k-1)(k+1) \ge t-1$. Hence,

$$\begin{aligned} y_{i+t} &\ge [(t-1)\lambda(i+2) - (t-1)](y_{i+k+1} - y_{i+k}) \\ &\ge (t-1)(y_{i+1} - y_i)[\lambda(i+2)-1]. \end{aligned}$$

(ii) As in the proof of (i) we may assume there is exactly one k, $0 \le k \le t-2$, such that $\{y_{i+k}, y_{i+k+1}, y_{i+k+2}\}$ is not an arithmetic progression. Thus,

$y_{i+k+2} \ge \lambda(i+k+2)y_{i+k+1} - y_{i+k}$ and

$y_{i+t} = y_{i+k+1} + (t-k-1)(y_{i+k+2} - y_{i+k+1})$. It follows that

$$\begin{aligned} y_{i+t} &\ge (t-k-1)y_{i+k+1}(\lambda(i+k+2)-2) \\ &\ge (t-k-1)y_{i+k+1}[(k+1)\lambda(i+2)-2] \\ &\ge y_{i+k+1}[(t-1)\lambda(i+2) - 2(t-k-1)] \\ &\ge y_{i+1}(t-1)[\lambda(i+2)-2]. \end{aligned}$$

(iii) Assuming that $\{y_{i+k}, y_{i+k+1}, y_{i+k+2}\}$ is not an arithmetic progression, we have $y_{i+k+2} \ge 2y_{i+k+2} + y_{i+k}\lambda(i+k+2)$ and

$$\begin{aligned} y_{i+t} &= y_{i+k+1} + (t-k-1)(y_{i+k+2} - y_{i+k+1}) \\ &\ge (t-k-1)[y_{i+k+1} + y_{i+k}\lambda(i+k+2)] \\ &\ge (t-k-1)[(k+1)y_{i+k}\lambda(i+2)] \\ &\ge (t-1)y_i\lambda(i+2). \end{aligned}$$

∎

In the next theorem we give lower bounds for terms of sequences which contain no arithmetic progressions of length n.

Theorem 3.3. *Let $k \geq n$ and assume that* $\mathbf{x} = \{x_1, \ldots, x_k\}$ *contains no* n*-term arithmetic progression. Let* $\lambda(r+s) \geq (r+1)\lambda(s)$ *for all* $r \geq 0$ *and* $s \geq 3$*. Then for* $j \geq 1$*,*

(i) *If* $\mathbf{x}$ *is an* f_λ*-sequence then*

(a) $x_{j(n-1)+2} - x_{j(n-1)+1} \geq \prod_{i=0}^{j-1} [\lambda(3+i(n-1)) - 1]$*, and*

(b) $x_{j(n-1)+1} \geq (n-2) \prod_{i=0}^{j-1} [\lambda(3+i(n-1)) - 1]$.

(ii) *If* $\mathbf{x}$ *is an* g_λ*-sequence then*

$$x_{j(n-2)+2} \geq 2(n-2)^j \prod_{i=0}^{j-1} [\lambda(3+i(n-2)) - 2] .$$

(iii) *If* $\mathbf{x}$ *is an* h_λ*-sequence then*

$$x_{j(n-1)+1} \geq (n-2)^j \prod_{i=0}^{j-1} \lambda(3+i(n-1)).$$

Proof. (i) (a) We use induction on j. Applying Lemma 3.2(i)(a) to $\{x_1, \ldots, x_{n+1}\}$ we have $x_{n+1} - x_n \geq (x_2 - x_1)(\lambda(3) - 1) \geq \lambda(3) - 1$, so that the result is true for $j = 1$. Now assume it holds for j, and apply Lemma 3.2(i)(a) to $\{x_{j(n-1)+1}, \ldots, x_{j(n-1)+n}\}$. This yields

$$x_{j(n-1)+n} - x_{j(n-1)+n-1} \geq \left(x_{j(n-1)+2} - x_{j(n-1)+1}\right) [\lambda(3+j(n-1)) - 1] .$$

Thus, by the induction hypothesis,

$$x_{j(n-1)+n} - x_{j(n-1)+n-1} \geq \prod_{i=0}^{j} [\lambda(3+i(n-1)) - 1] ,$$

and the result follows by Lemma 3.1.

(b) By Lemma 3.2(i)(b), $x_n \geq (n-2)(\lambda(3) - 1)$, so the theorem is true for $j = 1$. Now consider $\{x_{(j-1)(n-1)+1}, \ldots, x_{j(n-1)+1}\}$. By Lemma 3.2(i)(b),

$$x_{j(n-1)+1} \geq (n-2)\left(x_{(j-1)(n-1)+2} - x_{(j-1)(n-1)+1}\right) [\lambda(3+(j-1)(n-1)) - 1] ,$$

and hence, by Theorem 3.3(i)(a),

$$x_{j(n-1)+1} \geq (n-2) \prod_{i=0}^{j-1} [\lambda(3+i(n-1)) - 1] \quad \text{for } j \geq 2 .$$

(ii) Applying Lemma 3.2(ii) to $\{x_1, \ldots, x_n\}$, we have $x_n \geq (n-2)x_2(\lambda(3) - 2) \geq 2(n-2)(\lambda(3) - 2)$, so the theorem holds if $j = 1$. Assume the inequality is true for j. Applying Lemma 3.2(ii) to $\{x_{j(n-2)+1}, \ldots, x_{j(n-2)+n}\}$, we have $x_{j(n-2)+n} \geq (n-2)x_{j(n-2)+2}[\lambda(3+j(n-2)) - 2]$, and, by the assumption,

$$x_{j(n-2)+n} \geq 2(n-2)^{j+1} \prod_{i=0}^{j} [\lambda(3+i(n-2)) - 2]$$

so that the theorem is true by induction.

(iii) Again using induction, by Lemma 3.2(iii), $x_n \geq (n-2)\lambda(3)$, and if the theorem is true for j, then

$$\begin{aligned} x_{j(n-1)+n} &\geq (n-2)x_{j(n-1)+1}\,\lambda(3+j(n-1)) \\ &\geq (n-2)^{j+1} \prod_{i=0}^{j} \lambda(3+j(n-1)). \end{aligned}$$

∎

As a result of Theorem 3.3, we are able to find sufficient conditions, in terms of bounds on $f_\lambda(n)$, $g_\lambda(n)$, and $h_\lambda(n)$, for the existence of a reasonable upper bound on $w(n)$. These are summarized in the following corollary, where $\lfloor\ \rfloor$ represents the greatest integer function.

Corollary 3.4. *Let* $\lambda(r+s) \geq (r+1)\lambda(s)$ *for all* $r \geq 0$ *and* $s \geq 3$. *Let* $n \geq 4$ *and* $u \geq n$.

(i) *Let* $v = \lfloor \frac{u-1}{n-1} \rfloor$. *If* $f_\lambda(u) < (n-2)\prod_{i=0}^{v-1}(\lambda(3+i(n-1)) - 1) = \alpha$, *then* $w(n) < \alpha$.

(ii) *Let* $v = \lfloor \frac{u-2}{n-2} \rfloor$. *If* $g_\lambda(u) < 2(n-2)^v \prod_{i=0}^{v-1}(\lambda(3+i(n-2)) - 2) = \beta$, *then* $w(n) < \beta$.

(iii) *Let* $v = \lfloor \frac{u-1}{n-1} \rfloor$. *If* $h_\lambda(u) < (n-2)^v \prod_{i=0}^{v-1}\lambda(3+i(n-1)) = \gamma$, *then* $w(n) < \gamma$.

Proof. (i) Since $(n-1)v+1 \leq u$, we have $f_\lambda((n-1)v+1) < \alpha$. Thus, if $[1,\alpha]$ is 2-colored there exists a monochromatic $((n-1)v+1)$-term f_λ-sequence. Then by Theorem 3.3(i)(b), there must be an n-term arithmetic progression of length n within this f_λ-sequence. Hence, $w(n) < \alpha$.

(ii) By hypothesis $g_\lambda((n-2)v+2) < \beta$, but then by Theorem 3.3(ii) any 2-coloring of $[1,\beta]$ must contain an n-term monochromatic arithmetic progression.

(iii) We have $h_\lambda((n-1)v+1) < \gamma$. Hence, by Theorem 3.3(iii) any 2-coloring of $[1,\gamma]$ must contain an n-term monochromatic arithmetic progression. ∎

We now mention a few examples to illustrate how narrow the gaps are between the known upper bounds for $f_\lambda(n)$, $g_\lambda(n)$, $h_\lambda(n)$ given in §2, and the "desired" bounds of Corollary 3.4.

1. Letting $\lambda(i) = i!$, Theorem 2.2 tells us that $f_\lambda(n^2) < 9\prod_{i=4}^{n^2}(i! + i - 1)$. On the other hand, by Corollary 3.4, if $u = n^2$, then $v = n+1$, so that if

$f_\lambda(n^2) < (n-2)\prod_{i=0}^{n}(3+i(n-1))!$, then this would also serve as an upper bound for $w(n)$.

2. By Theorem 2.2, if $\lambda(i) = 2^i$, then $g_\lambda(n^2) \le 2^{n^4/2}(1+o(1))$. Applying Corollary 3.4 with $u = n^2 - 2$ (so that $v = n+2$), we see that if this bound could be improved to $n^{n+2}2^{n^3/2}(1+o(1))$, then this would also bound $w(n)$.
3. Let $\lambda(i) = 2^i$. By Theorem 2.2,

$$h_\lambda(n^2) \le h_\lambda(3)\prod_{k=4}^{n^2}(2^k+k+1)$$
$$\le h_\lambda(3)\prod_{k=4}^{n^2}2^{k+1} \le 2^{\sum_{i=1}^{n^2+1} i} = 2^{n^4/2}(1+o(1)).$$

By Corollary 3.4, if

$$h_\lambda(n^2) < (n-2)^{n+1}\prod_{i=0}^{n}2^{3+i(n-1)}$$
$$= (n-2)^{n+1}2^{3(n+1)} + \frac{1}{2}(n^3-n)$$
$$= n^{n+1}2^{n^3/2}(1+o(1)),$$

then this would also be a bound for $w(n)$.

4. Exact Values

In the table below we give the values of $w(n)$, $f(n)$, $g(n)$, $h(n)$, and $f_\lambda(n)$, $g_\lambda(n)$, $h_\lambda(n)$, for $n = 3$ and 4, for selected functions λ.

$n\backslash$	f	f_{2^i}	$f_{i!}$	g	g_{2^i}	$g_{i!}$	h	h_{2^i}	$h_{i!}$	w
3	7	9	9	8	9	9	7	9	9	9
4	13	28	34	21	35	35	17	35	35	35

Known values of $w(n)$ can be found in [2] and [6]. The fact that $g_{2^i}(4) = g_{i!}(4) = h_{2^i}(4) = h_{i!}(4) = w(4)$ follows from the same argument given in Remark (i) of §2 for $f_{i^3}(4)$. All other values in the table were found on an IBM-PC using the algorithm of [6].

References

[1] E.R. Berlekamp, *A construction for partitions which avoid long arithmetic progressions*, Canadian Math. Bull **11** (1968), 409–414.

[2] V. Chvatal, *Some unknown van der Waerden numbers*, in "R. K. Guy et al., eds., Combinatorial Structures and Their Applications", Gordeon and Breach, New York, 1970, pp. 31–33.

[3] R. N. Greenwell and B.M. Landman, *On the existence of a reasonable upper bound for the van der Waerden numbers*, J. Comb. Theory (A) **50** (1989), 82–86.

[4] B. M. Landman, *Ramsey functions related to the van der Waerden numbers*, Discrete Math **102** (1992), 265–278.

[5] S. Shelah, *Primitive recursive bounds for van der Waerden numbers*, J. Amer. Math. Soc **1** (1988).

[6] R. S. Stevens and R. Shantaram, *Computer generated van der Waerden partitions*, Math. Comp **17** (1978), 635–636.

[7] B. L. van der Waerden, *Beweis einer Baudetschen Vermutang*, Nieuw Arch. Wisk **15** (1927), 212–216.