Eulerian orientations and circuits of complete bipartite graphs

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Abstract. In a complete bipartite graph $K_{s,t}$, each vertex of one vertex set is joined to each vertex of the second vertex set by exactly one edge. An Eulerian orientation of $K_{s,t}$ assigns directions to the edges in such a way that the resulting digraph has an Eulerian dicircuit. Similarly, any Eulerian circuit of $K_{s,t}$ ascribes directions to the edges and results in an Eulerian orientation. This paper investigates Eulerian orientations and circuits of $K_{s,t}$. Exact solutions are presented for $s = 2$ and $s = 4$. Computer searches were used to obtain results for other small values of s and t. These results also lead to two conjectures which deal with upper and lower bounds on the numbers of Eulerian circuits.

1. Notation and preliminary concepts

Elements of the first vertex set will be denoted by $H_1, H_2, H_3, \ldots$ and elements of the second vertex set will be denoted by $A_1, A_2, A_3, \ldots$. Since any connected graph has an Eulerian circuit if and only if each vertex has even degree, the bipartite graphs will generally be denoted by $K_{2m,2n}$.

We do not distinguish between two Eulerian circuits if one of them is just a cyclic permutation of the other. However, if one circuit is the reverse of another, then the circuits are regarded as being different.

The number of Eulerian circuits of a graph may be found by firstly finding all Eulerian orientations and then adding the associated numbers of Eulerian dicircuits, e.g. Fleischner (1983). It should be noted that different Eulerian orientations may have different numbers of dicircuits. Good (1946) has effectively shown that an orientation is an Eulerian orientation if and only if the indegree of any vertex equals its outdegree. With this in mind, we define the orientation matrix, F, of an oriented complete bipartite graph $K_{2m,2n}$ to be the $2m \times 2n$ zero-one matrix with $F_{i,j} = 1$ if and only if there is an arc from H_i to A_j. Obviously there is a 1-to-1 relationship between the orientations and the orientation matrices. We note that F represents an Eulerian orientation if and only if each row sum is n and each column sum is m.

Later on we shall also let I_m denote the $m \times m$ identity matrix, $J_{m,n}$ an $m \times n$ matrix of ones, $0_{m,n}$ an $m \times n$ matrix of zeros, and $B_m(s,t)$ an $m \times m$ matrix with diagonal elements all set to s, and all off-diagonal elements equal to t. We shall make use of the well-known expansion

$$|B_m(s,t)| = (s-t)^{m-1}[s+(m-1)t]$$

and the special case

$$|B_2(s,t)| = (s+t)(s-t).$$

2. Orientations and circuits of $K_{2,t}$

Lemma 1 *$K_{2,2n}$ has $\binom{2n}{n}$ different Eulerian orientations.*

Proof: The n outgoing arcs from H_1 can be chosen in $\binom{2n}{n}$ ways. The directions of all other arcs are then fixed.

Lemma 2. *$K_{2,2n}$ has $2(2n-1)!$ Eulerian circuits.*

Proof: Every circuit must have either an arc from H_1 to A_1 or an arc from A_1 to H_1. Initially consider all circuits with arcs from H_1 to A_1 and let this be the first arc of a circuit. The second arc must lead to H_2. There are now $(2n-1)$ choices for the third arc. The fourth arc must lead to H_1. There are $(2n-2)$ choices for the fifth arc. The sixth arc must lead to H_2, and the process continues. This leads to $(2n-1)!$ distinct circuits. Since there must be an equal number of circuits with arcs from A_1 to H_1, the result follows.

Similarly, the number of Eulerian trails in $K_{2,2n+1}$ which begin at H_1 and finish at H_2 must be $2(2n)!$.

3. Computer searches

Most of the theory for the case $K_{4,2n}$, and the conjectures, was inspired by close examination of the output from the computer searches. It is therefore worthwhile to summarize the results from these searches before proceeding with the theory. This section also introduces the "mini-minor" which is merely a useful determinant in calculating numbers of dicircuits.

3.1 Search methodology

The searches proceeded by initially finding all possible orientation matrices with appropriate row and column sums, and then calculating corresponding numbers of dicircuits. Without loss of generality we can relabel the vertices so that the orientation matrices have the form in Table 1, where the x's form a set of $(n-1)*(2m-1)$ unknowns, the y's form a different set of $(n-1)*(2m-1)$ unknowns, and the z's form a set of $(2m-1)$ unknowns. Since row sums and column sums are known, it is possible to search through whole rows or whole columns at a time. Whenever $m < n$, calculation, and computer time trials, show that it is more efficient to search for whole columns. Thus for each column of x's we try a column vector with $(m-1)$ 1's and m 0's. Candidate y column vectors have m 1's. For given x's and y's, the last column vector (z's) has either 0 or 1 solution and can be solved explicitly. The actual search space can now be truncated by noting that interchanging two x vectors, or two y vectors, has the effect of relabelling the A's but not changing the number of circuits. A further saving is gained by noting that if given x's and y's yield a solution, then interchanging all 0's and 1's in columns A_2 to A_{2n-1} leads to an equivalent solution with the same number of circuits.

	A_1	A_2	...	A_n	A_{n+1}	A_{n+2}	...	A_{2n-1}	A_{2n}
H_1	1	1	...	1	0	0	...	0	0
H_2	1	x	...	x	y	y	...	y	z
...	...	...	...	...	...	...	...	...	...
H_m	1	x	...	x	y	y	...	y	z
H_{m+1}	0	x	...	x	y	y	...	y	z
H_{m+2}	0	x	...	x	y	y	...	y	z
...	...	...	...	...	...	...	...	...	...
H_{2m}	0	x	...	x	y	y	...	y	z

Table 1. General structure of an orientation matrix prior to solution.

When an orientation matrix has been found, it is possible to count the corresponding number of Eulerian dicircuits by evaluating the BEST formula (van Aardenne-Ehrenfest and de Bruijn 1951, Smith and Tutte 1941)

$$\Delta \prod_{v=1}^{2m+2m} (d_i(V_v) - 1)!$$

where Δ is the minor of the (incoming) degree matrix after eliminating the row and column associated with one vertex, and $d_i(V_v)$ is the indegree of vertex V_v. For bipartite graphs, the minor has a very simple structure and evaluation of the BEST formula is modified by substituting a $2m \times 2m$ determinant for the $(2m + 2n - 1) \times (2m + 2n - 1)$ determinant, viz.

$$m^{(2n-1)} |G| \prod_{v=1}^{2m+2m} (d_i(V_v) - 1)!$$

where $G_{ii} = n$ and $G_{ij} = -p_{ij}/m$ with p_{ij} being the number of directed paths of length 2 from H_i to H_j, such that the paths do not pass through A_{2n}. We call $|G|$ the mini-minor of the degree matrix. To clarify, consider the orientation matrix in Table 2. Its degree matrix is given in Table 3. The minor is obtained by deleting the row and column corresponding to A_6. Now rows A_1 to A_5 are each divided by 2, i.e. m. Eliminating the -1's in columns A_1 to A_5 of rows H_1 to H_4, then eliminating the $-\frac{1}{2}$'s in columns H_1 to H_4 of rows A_1 to A_5 yields a simple matrix with G as the upper left $2m \times 2m$ square submatrix corresponding to the H vertices. See Table 4.

	A_1	A_2	A_3	A_4	A_5	A_6
H_1	1	1	1	0	0	0
H_2	1	0	0	0	1	1
H_3	0	0	0	1	1	1
H_4	0	1	1	1	0	0

Table 2. An orientation matrix for $K_{4,6}$.

	H_1	H_2	H_3	H_4	A_1	A_2	A_3	A_4	A_5	A_6
H_1	3	0	0	0	−1	−1	−1	0	0	0
H_2	0	3	0	0	−1	0	0	0	−1	−1
H_3	0	0	3	0	0	0	0	−1	−1	−1
H_4	0	0	0	3	0	−1	−1	−1	0	0
A_1	0	0	−1	−1	2	0	0	0	0	0
A_2	0	−1	−1	0	0	2	0	0	0	0
A_3	0	−1	−1	0	0	0	2	0	0	0
A_4	−1	−1	0	0	0	0	0	2	0	0
A_5	−1	0	0	−1	0	0	0	0	2	0
A_6	−1	0	0	−1	0	0	0	0	0	2

Table 3. The degree matrix for the orientation matrix displayed in Table 2.

	H_1	H_2	H_3	H_4	A_1	A_2	A_3	A_4	A_5
H_1	3	$-\frac{2}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	0	0	0	0	0
H_2	$-\frac{1}{2}$	3	$-\frac{1}{2}$	$-\frac{2}{2}$	0	0	0	0	0
H_3	$-\frac{2}{2}$	$-\frac{1}{2}$	3	$-\frac{1}{2}$	0	0	0	0	0
H_4	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{2}{2}$	3	0	0	0	0	0
A_1	0	0	0	0	1	0	0	0	0
A_2	0	0	0	0	0	1	0	0	0
A_3	0	0	0	0	0	0	1	0	0
A_4	0	0	0	0	0	0	0	1	0
A_5	0	0	0	0	0	0	0	0	1

Table 4. A determinant, after manipulation, derived from the degree matrix displayed in Table 3.

The elements of G are easily calculated by noting that any column of the orientation matrix provides a path of length 2 from H_i to H_j if there is a one in row H_i and a zero in row H_j.

3.2 Checks on results

The number of Eulerian orientations of $K_{2m,2n}$ equals the coefficient of $x_1^n \dots x_{2m}^n$ $y_1^m \dots y_{2n}^m$ in

$$\prod_{i=1}^{2m}\prod_{j=1}^{2n}(1 + x_i y_j).$$

McKay (1983) employed exact arithmetic and the method of Liskovec (1971) to evaluate, inter alia, some of these coefficients for $m = n$. We implemented Liskovec's method using complex double precision arithmetic and were able to confirm that the numbers of Eulerian orientations in Table 5 are correct to at least seven significant digits.

Unfortunately no alternative procedure for counting Eulerian circuits seems to be available. The following lemma provides some assistance.

Lemma 3. *The total number of Eulerian circuits of* $K_{2m,2n}$ *must be divisible by* $2(2m-1)!(2n-1)!$.

Proof: Half of the circuits must contain the sequence $\dots H_1 A_1 \dots$. With regard to these circuits, the remaining arcs involving H_1 can occur in $(2n-1)!$ ways. Independently of these, the remaining arcs involving A_1 can occur in $(2m-1)!$ ways. Thus half the number of circuits is divisible by $(2m-1)!(2n-1)!$. The other half of the circuits must contain the sequence $\dots A_1 H_1 \dots$, and this leads to the required result.

Lemma 3 provides a simple divisibility criterion as a partial check on the computed number of Eulerian circuits. However the method of searching guarantees that the obtained number of circuits is divisible by

$$2[(m-1)!]^{2n}[(n-1)!]^{2m}\binom{2m-1}{m}\binom{2n-1}{n}$$

and this expression is in fact divisible by $2(2m-1)!(2n-1)!$ for $K_{6,8}$ and $K_{8,8}$. Even though the divisibility criterion cannot usefully be applied in these cases, it was successfully applied in all other cases.

3.3 Results

The results of the computer searches are summarized in Table 5. The numbers of Eulerian circuits may seem large, but are not inconsistent with the numbers of circuits around complete graphs K_n as determined by McKay (1983).

Two Eulerian orientations must be equivalent, i.e. have the same minor and the same number of Eulerian dicircuits, if one can be obtained from the other by relabelling H's or A's or by reversing the directions of all arcs. We do not know if other criteria also lead to equivalence. In any case, the numbers of different minors in Table 5 represents numbers of different equivalence classes of Eulerian orientations. Appendices 1 and 2 present some more information on these classes for small m, n. The relative sizes of the classes in Appendix 1 have been obtained by fixing the directions of all edges incident with H_1 and A_1.

Size	Number of Eulerian orientations	Number of Eulerian circuits	# of different minors
4×4	90	6336	2
4×6	1 860	292 14720	3
4×8	44 730	54 67933 90080	4
4×10	1 172 556	28 86587 52012 28800	5
6×6	297 200	8469 00707 32800	6
6×8	60 871 300	19 48197 70542 91820 54400	16
6×10	14 367 744 720	20871 27887 97176 47095 18999 55200	35
8×8	116 963 796 250	107 81029 55089 45463 93993 19896 06400	80
8×10	273 957 842 462 220	4589 66046 86554 48806 47168 96257 52190 32064 00000	1147

Table 5. Numbers of Eulerian orientations and circuits of complete bipartite graphs.

4. Orientations and circuits of $K_{4,2n}$

The case $K_{4,2n}$ is easily dealt with once we realize that for any Eulerian orientation matrix, each column sum is 2. This leads to the apparently trivial, but important, observation that if an arbitrary column has a 1 in its first row then there is exactly one other row with a 1 in the same column, and a similar result holds for a zero in the first row. We have

Lemma 4. *Each Eulerian orientation of $K_{4,2n}$ generates a 3-partition of n, with zeros permitted.*

Proof: Consider only those columns of the orientation matrix which have a one in the first row. Suppose row 2 has α_1 ones in these columns. Similarly suppose rows 3, 4 have α_2, α_3 ones respectively in these same columns. Since each column total is exactly two, we must have $\alpha_1 + \alpha_2 + \alpha_3 = n$.

For example, in Table 2 we have $n = 3$ and $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 2$.

Lemma 5. *The 3-partition of n, with zeros permitted, generated by considering columns with a zero in the first row, is identical to the 3-partition obtained by considering ones in the first row.*

Proof: Consider any row except the first. If it has α ones in common with the first row, then it has $n - \alpha$ zeros in the columns where the first row has a one. Since the total number of zeros in any row must be n, it has α zeros in columns common with zeros of the first row.

Lemmas 4 and 5 show that each Eulerian orientation of $K_{4,2n}$ leads to two identical partitions of n. We can use this information to count the total number of Eulerian orientations.

Lemma 6. *The number of Eulerian orientations of* $K_{4,2n}$ *is*

$$\sum_{\alpha_1+\alpha_2+\alpha_3=n} \frac{(2n)!}{(\alpha_1!\alpha_2!\alpha_3!)^2} P\{\alpha_1, \alpha_2, \alpha_3\}$$

where the sum is over all possible partitions $\{\alpha_1, \alpha_2, \alpha_3\}$ *of* n*, with zeros permitted, and* $P\{\alpha_1, \alpha_2, \alpha_3\}$ *is the number of distinct permutations of* $\{\alpha_1, \alpha_2, \alpha_3\}$.

Proof: Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be any 3-partition of n with zeros permitted. Choose α_1 columns of row 2 and insert ones in this row and the first row. Choose α_2 different columns of row 3 and insert ones in this row and the first row. Choose α_3 different columns of row 4 and insert ones in this row and the first row. Choose α_1 different columns of row 2 and insert zeros in this row and the first row. Choose α_2 different columns of row 3 and insert zeros in this row and the first row. For the remaining α_3 columns, insert zeros in row 4 and the first row. This can be done in

$$\frac{(2n)!}{\alpha_1!\alpha_2!\alpha_3!\alpha_1!\alpha_2!\alpha_3!}$$

ways. Any remaining empty cell has either two ones or two zeros in the same column and can be filled in with a zero or one respectively to complete an Eulerian orientation matrix. Now rows 2, 3 and 4 can be permuted in $P\{\alpha_1, \alpha_2, \alpha_3\}$ ways to obtain all possible Eulerian orientations associated with the given partition, and the result follows.

For example, to derive the number of Eulerian orientations of $K_{4,6}$ we note that $\{3,0,0\}, \{2,1,0\}, \{1,1,1\}$ are the only permissible partitions of 3, and that the corresponding values of $P\{\alpha_1, \alpha_2, \alpha_3\}$ are 3, 6, 1 respectively. Therefore the total number of Eulerian orientations is

$$\frac{6!}{(3!)^2}3 + \frac{6!}{(2!)^2}6 + 6! = 1860.$$

An alternative expression for the number of different Eulerian orientations of $K_{4,2n}$ follows.

$$3\binom{2n}{n}\sum_{i=0}^{n-1}\left[\binom{n-1}{i}\binom{n}{n-1-i}\binom{2n-2-2i}{n-1-i}\right].$$

This solution can be found by explicitly solving for the x's, y's and z's in Table 1, one row at a time. Unfortunately it provides no insights into the numbers of circuits, and so the proof is omitted.

In order to count the number of Eulerian circuits of $K_{4,2n}$ some additional results have to be established.

Lemma 7. *In an Eulerian orientation of $K_{4,2n}$, the number of paths of length 2 from H_i to H_j equals the number of paths of length 2 from H_j to H_i.*

Proof: Let p denote the number of paths from H_i to H_j. Then there are exactly p columns of the orientation matrix which have a one in row i and a zero in row j. Therefore there are $n-p$ different columns which have a one in rows i and j, and a further $n-p$ different columns which have a zero in rows i and j. Only p columns remain and all of these must have a zero in row i and a one in row j.

Lemma 8. *In an Eulerian orientation of $K_{4,2n}$, the number of paths of length 2 from H_i to H_j equals the number of paths of length 2 from H_r to H_s where H_r, H_s are the other two vertices.*

Proof: Let p denote the number of paths from H_i to H_j. Then there are exactly p columns of the orientation matrix which have a one in row i and a zero in row j. From lemma 7, there are exactly p different columns which have a zero in row i and a one in row j. Each of these $2p$ columns must each have exactly one zero in common with one of the remaining rows. Let H_r be one of these two remaining rows. One of the steps in the proof of lemma 7 established that there are $n-p$ different columns which have ones in rows H_i and H_j. Thus these columns must have zeros in row r. Similarly, there are another $n-p$ columns which must have ones in row r. Therefore exactly p of the $2p$ columns have a one in row r and a zero in row s.

Lemma 9.

$$\begin{vmatrix} t & b+c-t & -b & -c \\ b+c-t & t & -c & -b \\ 1-b & 1-c & t & b+c-t \\ 1-c & 1-b & b+c-t & t \end{vmatrix} = 8(t-b)(t-c)(b+c)$$

Proof: Denote the determinant by $|G|$. Since the upper and lower left 2×2

submatrices of G commute under multiplication,

$$
\begin{aligned}
|G| &= \left| \begin{pmatrix} t & b+c-t \\ b+c-t & t \end{pmatrix}^2 - \begin{pmatrix} -b & -c \\ -c & -b \end{pmatrix} \begin{pmatrix} 1-b & 1-c \\ 1-c & 1-b \end{pmatrix} \right| \\
&= B_2(t^2 + (b+c-t)^2 + b(1-b) + c(1-c), 2t(b+c-t) \\
&\qquad + b(1-c) + c(1-b)) \\
&= [t^2 + (b+c-t)^2 + b(1-b) + c(1-c) + 2t(b+c-t) \\
&\qquad + b(1-c) + c(1-b)] \\
&\times [t^2 + (b+c-t)^2 + b(1-b) + c(1-c) - 2t(b+c-t) \\
&\qquad - b(1-c) - c(1-b)] \\
&= [(t + (b+c-t))^2 + (b+c)(1-b) + (b+c)(1-c)] \\
&\times [(t - (b+c-t))^2 - b^2 + bc - c^2 + bc] \\
&= [(b+c)^2 - (b+c)^2 + 2(b+c)] \times [(2t - (b+c))^2 - (b-c)^2] \\
&= 2(b+c)[(2t - b - c + b - c)(2t - b - c - b + c)] \\
&= 8(b+c)(t-b)(t-c)
\end{aligned}
$$

Theorem 1. *The total number of Eulerian circuits of $K_{4,2n}$ is*

$$\sum_{\alpha_1+\alpha_2+\alpha_3=n} \frac{(2n)!}{(\alpha_1!\alpha_2!\alpha_3!)^2} P\{\alpha_1,\alpha_2,\alpha_3\}[(n-1)!]^4 2^{2n-2}(n+\alpha_1)(n+\alpha_2)(n+\alpha_3)$$

where the sum is over all possible partitions $\{\alpha_1,\alpha_2,\alpha_3\}$ of n with zeros permitted, and $P\{\alpha_1,\alpha_2,\alpha_3\}$ is the number of distinct permutations of $\{\alpha_1,\alpha_2,\alpha_3\}$.

Proof: Let $\{\alpha_1,\alpha_2,\alpha_3\}$ be an arbitrary partition of n with zeros permitted. Consider any Eulerian orientation matrix which generates the partition. (The proof of lemma 6 shows how to construct such a matrix). If necessary, permute the rows of the orientation matrix to ensure that the last column has zeros in the first two rows. Permuting the rows of the orientation matrix has the effect of carrying out identical permutations on the rows and columns of the degree matrix and so will not affect the number of circuits for the given orientation. By applying lemmas 7 and 8 we find that the mini-minor has the following form, where $(\alpha'_1,\alpha'_2,\alpha'_3)$ is a permutation of $\{\alpha_1,\alpha_2,\alpha_3\}$.

$$
|G \begin{vmatrix}
n & \frac{-(n-\alpha'_1)}{2} & \frac{-(n-\alpha'_2)}{2} & \frac{-(n-\alpha'_3)}{2} \\
\frac{-(n-\alpha'_1)}{2} & n & \frac{-(n-\alpha'_3)}{2} & \frac{-(n-\alpha'_2)}{2} \\
\frac{1-(n-\alpha'_2)}{2} & \frac{1-(n-\alpha'_3)}{2} & n & \frac{-(n-\alpha'_1)}{2} \\
\frac{1-(n-\alpha'_3)}{2} & \frac{1-(n-\alpha'_2)}{2} & \frac{-(n-\alpha'_1)}{2} & n
\end{vmatrix}
$$

From lemma 9,

$$\begin{aligned}|G| &= 2^{-4}8[2n-(n-\alpha_2')][2n-(n-\alpha_3')][n-\alpha_2'+n-\alpha_3')] \\ &= 2^{-1}[n+\alpha_2'][n+\alpha_3'][n+\alpha_1'] \\ &= 2^{-1}[n+\alpha_1][n+\alpha_2][n+\alpha_3]\end{aligned}$$

Therefore, the minor of the degree matrix corresponding to the particular Eulerian orientation is

$$2^{2n-2}(n+\alpha_1)(n+\alpha_3)(n+\alpha_3) \tag{1}$$

and the number of circuits for the particular orientation is

$$[(n-1)!]^4 2^{2n-2}(n+\alpha_1)(n+\alpha_2)(n+\alpha_3).$$

Now summing over all possible orientations associated with all possible partitions gives the required result.

5. Minors of the degree matrices

McKay (1983) discusses asymptotic numbers of Eulerian orientations of bipartite graphs. Schrijver (1983) and Las Vergnas (1990) impose bounds on numbers of Eulerian orientations of regular graphs. Our purpose here is to discuss bounds on the number of Eulerian circuits associated with an arbitrary Eulerian orientation. These bounds can be combined with the work of other authors in order to obtain estimates of the number of circuits of an arbitrary bipartite graph.

Equation (1) above generates all possible minors of $K_{4,2n}$. Bearing in mind that $\{\alpha_1, \alpha_2, \alpha_3\}$ is a partition of n, it is easy to show that $(n+\alpha_1)(n+\alpha_2)(n+\alpha_3)$ is a minimum when $\alpha_1 = n, \alpha_2 = 0, \alpha_3 = 0$, and is a maximum when $\alpha_1 = \alpha_2 = \alpha_3 = n/3$. The minimum is always achievable, and the maximum is achievable whenever $3 \mid n$. We therefore have $2^{2n-1}n^3$ as the minimum possible minor of any Eulerian orientation of $K_{4,2n}$, and $2^{2n+4}n^3/3^3$ as an upper bound on the maximum possible minor. Observe that the orientation matrix associated with the minimum has a very simple structure, viz. two submatrices of zeros and two submatrices of ones. When the upper bound is achieved, any pair of rows have ones in a common number of columns. For example, the following orientation matrix for $K_{4,6}$ has maximum minor and each pair of rows have ones in exactly one common column.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

We further note that the above orientation matrix is also the incidence matrix of a $(4,6,3,2,1)$ BIBD. We initially make use of these observations to obtain some minors for any bipartite graph.

Theorem 2. *$K_{2m,2n}$ has an Eulerian orientation with degree matrix minor equal to $m^{2n-1}n^{2m-1}$.*

Proof: Obviously Table 6 represents a valid Eulerian orientation.

	A_1	A_2	...	A_n	A_{n+1}	A_{n+2}	...	A_{2n}
H_1	1	1	...	1	0	0	...	0
H_2	1	1	...	1	0	0	...	0
...	...	...	...	...	...	...	...	...
H_m	1	1	...	1	0	0	...	0
H_{m+1}	0	0	...	0	1	1	...	1
H_{m+2}	0	0	...	0	1	1	...	1
...	...	...	...	...	...	...	...	...
H_{2m}	0	0	...	0	1	1	...	1

Table 6. An orientation matrix with simple structure.

If we obtain the minor of the degree matrix by deleting the row and column associated with A_{2n}, we find that the minor has value $m^{(2n-1)}|G|$, where G is the following $2m \times 2m$ determinant.

$$\begin{vmatrix} nI_m & -\frac{n}{m}J_{m,m} \\ \frac{-(n-1)}{m}J_{m,m} & nI_m \end{vmatrix}$$

By adding appropriate columns on the left-hand side of G to columns on the right-hand side, we eliminate the upper right matrix and obtain

$$\begin{vmatrix} nI_m & 0_{m,m} \\ \frac{-(n-1)}{m}J_{m,m} & nI_m - \frac{(n-1)}{m}J_{m,m} \end{vmatrix}.$$

Therefore

$$\begin{aligned}
\text{minor} &= m^{(2n-1)}|G| \\
&= m^{(2n-1)}\left|nI_m\left(nI_m - \frac{(n-1)}{m}J_{m,m}\right)\right| \\
&= m^{(2n-1)}\left|B_m\left(n\left(n - \frac{(n-1)}{m}\right), \frac{-n(n-1)}{m}\right)\right| \\
&= m^{(2n-1)}n^m\left(n - \frac{(n-1)}{m} + \frac{(n-1)}{m}\right)^{(m-1)} \\
&\quad \left(n - \frac{(n-1)}{m} - (m-1)\frac{(n-1)}{m}\right) \\
&= m^{(2n-1)}n^m n^{(m-1)}\left(n - m\frac{(n-1)}{m}\right) \\
&= m^{(2n-1)}n^{(2m-1)}
\end{aligned}$$

Lemma 10.

$$\left|\begin{matrix} \left(n+\frac{n}{(2m-1)}\right)I_m-\frac{n}{(2m-1)}J_{m,m} & -\frac{n}{(2m-1)}J_{m,m} \\ \left(\frac{1}{m}-\frac{n}{(2m-1)}\right)J_{m,m} & \left(n+\frac{n}{(2m-1)}\right)I_m-\frac{n}{(2m-1)}J_{m,m} \end{matrix}\right|$$
$$=\frac{4^{(m-1)}m^{(2m-1)}n^{(2m-1)}}{(2m-1)^{(2m-1)}}$$

Proof: Denote the determinant by G. (The connection with the mini-minors will be discussed below). Since the upper and lower left $m\times m$ submatrices of G commute under multiplication,

$$|G|=\left|\left(\left(n+\frac{n}{(2m-1)}\right)I_m-\frac{n}{(2m-1)}J_{m,m}\right)^2\right.$$
$$\left.-\left(\frac{-n}{(2m-1)}J_{m,m}\right)\left(\left(\frac{1}{m}-\frac{n}{(2m-1)}\right)J_{m,m}\right)\right|$$
$$=\left|B_m\begin{pmatrix} n^2+(m-1)\frac{n^2}{(2m-1)^2}-m\left(\frac{n^2}{(2m-1)^2}-\frac{n}{m(2m-1)}\right), \\ \frac{-2n^2}{(2m-1)}+(m-2)\frac{n^2}{(2m-1)^2}-m\left(\frac{n^2}{(2m-1)^2}-\frac{n}{m(2m-1)}\right)\end{pmatrix}\right|$$
$$=\left|B_m\left(n^2-\frac{n^2}{(2m-1)^2}+\frac{n}{(2m-1)},\frac{n-2n^2}{(2m-1)}-\frac{2n^2}{(2m-1)^2}\right)\right|$$
$$=\left(n^2-\frac{n^2}{(2m-1)^2}+\frac{n}{(2m-1)}-\frac{(n-2n^2)}{(2m-1)}+\frac{2n^2}{(2m-1)^2}\right)^{(m-1)}$$
$$\times\left(n^2-\frac{n^2}{(2m-1)^2}+\frac{n}{(2m-1)}+(m-1)\left(\frac{n-2n^2}{(2m-1)}-\frac{2n^2}{(2m-1)^2}\right)\right)$$

and the result follows.

Theorem 3. *If there exists a* $(2m,2k(2m-1),k(2m-1),m,k(m-1))$ *BIBD for* $m\geq 2$ *and* $k\geq 1$, *then there is an Eulerian orientation of* $K_{2m,2k(2m-1)}$ *with minor equal to* $4^{m-1}m^{2m+2k(2m-1)-2}k^{2m-1}$.

Proof: If such a BIBD exists, then its incidence matrix is the orientation matrix for an Eulerian orientation of $K_{2m,2k(2m-1)}$. Furthermore, any two rows will have exactly $k(m-1)$ columns with ones in common. Therefore the number of paths of length 2 from any H_i to any H_j will be

$$k(2m-1)-k(m-1)=mk.$$

Therefore the mini-minor takes the form of the determinant in Lemma 10, so that the full minor is

$$\frac{m^{(2n-1)}4^{(m-1)}m^{(2m-1)}n^{(2m-1)}}{(2m-1)^{(2m-1)}}=\frac{4^{(m-1)}m^{(2m+2n-2)}n^{(2m-1)}}{(2m-1)^{(2m-1)}} \quad (2)$$

But $n = k(2m-1)$ and so $n^{(2m-1)} = k^{(2m-1)}(2m-1)^{(2m-1)}$. Substituting in the right-hand side of the above equality gives the result.

Many BIBD's with the properties expressed in theorem 3 are known. Any such BIBD for $2m$ treatments will exist whenever there is a Hadamard matrix of order $4m$.

5.1 Conjectures

For all bipartite graphs examined in Table 5, the formula in theorem 2 generates the smallest minor. Also, BIBD's as in theorem 3 exist for the 4×6 and 6×10 graphs. In these two cases, the formula in theorem 3 generates the largest minor. In all other cases, the associated formula in equation 2 slightly overestimates the largest minor. We are obviously led to the following conjectures.

Conjecture 1. *For arbitrary $K_{2m,2n}$, the minimum minor over all possible Eulerian orientations is $m^{2n-1}n^{2m-1}$.*

Conjecture 2. *For arbitrary $K_{2m,2n}$, the maximum minor over all possible Eulerian orientations of $K_{2m,2n}$ is bounded by*

$$\frac{4^{(m-1)}m^{(2m+2n-2)}n^{(2m-1)}}{(2m-1)^{(2m-1)}}$$

and equality occurs whenever $2m-1$ divides n.

The bases of these conjectures have some intuitive appeal, and it is not difficult to generalize the concepts to non-bipartite graphs, e.g. a maximum minor may exist for an orientation which tends to make all off-diagonal elements of the degree matrix "as equal as possible".

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Appendix 1

Equivalence classes of some complete bipartite graphs.

Size	Value of minor (Δ)	Relative frequency
4×4	64	1
	72	4
4×6	864	1
	960	18
	1 024	12
4×8	8 192	1
	8 960	32
	9 216	36
	9 600	144
4×10	64 000	1
	69 120	50
	71 680	200
	73 728	400
	75 264	900

Size	Value of minor (Δ)	Relative frequency
6 × 6	59 049	1
	64 881	81
	68 445	216
	68 607	432
	70 785	648
	72 000	108
6 × 8	2 239 488	1
	2 426 112	144
	2 488 320	162
	2 540 160	576
	2 571 264	2 592
	2 576 448	2 592
	2 612 736	2 592
	2 624 832	2 592
	2 654 208	972
	2 661 120	10 368
	2 664 144	20 736
	2 667 168	864
	2 692 170	5 184
	2 709 504	6 480
	2 713 284	20 736
	2 737 800	10 368

Appendix 2.

Minimum and maximum minors for some complete bipartite graphs.

Size	Minimum minor (Δ)	Maximum minor (Δ)
6 × 10	61 509 375	76 527 504
8 × 8	268 435 456	335 936 160
8 × 10	20 480 000 000	25 906 839 552