**Union-Closed Sets Conjecture: Improved Bounds**

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**Abstract.** A union closed family $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ is a non-empty finite collection of distinct non-empty finite sets, closed under union. It has been conjectured that for any such family, there is some element in at least half of its sets. But the problem remains unsolved. This paper establishes several results pertaining to this conjecture with the main emphasis on the study of a possible counterexample with minimal $n$.

## 1. Introduction

A union-closed set $\mathcal{A}$ is defined as a non-empty finite collection of distinct, non-empty finite sets, closed under union (i.e., if $S \in \mathcal{A}$ and $T \in \mathcal{A}$ then $S \cup T \in \mathcal{A}$).

The following conjecture is rephrased from [1].

**Conjecture.** *Let $A = \{A_1, A_2, \ldots, A_n\}$ be a union-closed set. Then there exists an element which belongs to at least half of its sets.*

This conjecture has been proved for $n \le 28$ ([4], Theorem 3).

In this paper we consider a possible counterexample with minimal $n$ and prove the conjecture for all families involving up to eight elements or having up to 36 sets extending the previously known results.

## 2. Preliminaries and notation

Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a union closed set. Assume, for convenience, that $|A_i| = \omega_i; \omega_1 \le \omega_2 \le \cdots \le \omega_n = q$ and $A_n = I_q = \{1, 2, \ldots, q\}$.

The support size of $\mathcal{A}$ is defined to be the number $q = \omega_n$. Let $\mathbf{A}(n, q)$ be the family of all union closed sets of $n$ finite sets with support size $q$. Let $\mathbf{F}(n, q) \subseteq \mathbf{A}(n, q)$ be the family of all counterexamples to the conjecture. Put

$$n_0 = \min\{n \in \mathbf{N} : \mathbf{F}(n, q) \neq \emptyset\}$$
$$q_0 = \min\{q \in \mathbf{N} : \mathbf{F}(n_0, q) \neq \emptyset\}$$

If $\mathbf{F}(n, q) \neq \emptyset$ for some $n$, then $n_0$ is odd ( cfr [6], Theorem 1) and $q_0 \ge 8$ (cfr. [4],Theorem 2).

Let $\mathcal{A} \in \mathbf{A}(n, q)$ and $x \in I_q$. Define $\mathcal{A}(x)$ to be the set of $A_i$ in $\mathcal{A}$ which contains $x$ and let $|\mathcal{A}(X)| = d_{\mathcal{A}}(x)$. Let $\mathcal{C}(x) = \{A_i \in \mathcal{A} : x \notin A_i\}$, $C(x) = \cup\{A_j : A_j \in \mathcal{C}(x)\}$ and $\mathcal{A}^*(x) = \{A_i - \{x\} : A_i \in \mathcal{A}\}$. It is clear that $\mathcal{A}(x)$, $\mathcal{C}(x)$ and $\mathcal{A}^*(x)$ are union closed sets with support size respectively $q$, $q-1$ and at most $q-1$.

We now list below some results from [2,4,6] which we use in the next sections.

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**Theorem 1 (Sarvate-Renaud [6]; Theorem 2).** *Let* $\mathcal{A} \in \mathbf{F}(n, q)$, *then* $\omega_1 \geq 3$.

**Theorem 2 (Lo Faro [2]; Theorem 2,4 and Corollary 3).** *Let* $\mathcal{A} \in \mathbf{F}(n_o, q_0)$, $x, y \in I_{q_0}$ *and* $x \neq y$ *then:*

(i) $|\mathcal{A}^*(x)| < |\mathcal{A}|$;
(ii) $\mathcal{A}(x) \neq \mathcal{A}(y)$;
(iii) $d_{\mathcal{A}}(x) \leq d_{\mathcal{A}}(y) \Rightarrow y \in \mathcal{C}(x)$;
(iv) $\mathcal{C}(x) \neq \mathcal{C}(y)$.

**Theorem 3 (Lo Faro [2]; Theorem 6,8).** *Let* $\mathcal{A} \in \mathbf{F}(n_0, q_0)$, *there are at least four distinct elements* $x_1, x_2, x_3, x_4 \in I_{q_0}$ *such that* $\mathcal{C}(x_i) = I_{q_0} - \{x_i\}$, $i = 1, 2, 3$ *and* $\mathcal{C}(x_4) \supseteq I_{q_o} - \{x_3, x_4\}$.

**Theorem 4 (Poonen [4]; Corollary 4).** *If* $\mathcal{A} \in \mathbf{A}(n, q)$ *contains*

$$\mathcal{H} = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}\} \text{ then } \mathcal{A} \notin \mathbf{F}(n, q).$$

**Theorem 5 (Poonen [4]; Theorem 2).** $q_0 \geq 8$.

## 3. On a smallest counterexample

**THEOREM 6.** *Let* $\mathcal{A} = \{A_1, A_2, \ldots, A_n\} \in \mathbf{F}(n_0, q_0)$ *and let* $x, y$ *be distinct elements of* $I_{q_0}$. *If* $d(x) + 4 \geq d(y)$ *then* $x \in \mathcal{C}(y)$, *where* $d(i) = d_{\mathcal{A}}(i)$, *for each* $i \in I_{q_0}$.

Proof: By (iii) of Theorem 2, we can consider $d(x) + 1 \leq d(y) \leq d(x) + 4$. Suppose $x \notin \mathcal{C}(y)$, then $\mathcal{A}(x) \subseteq \mathcal{A}(y)$ and $\mathcal{C}(x) \in (\mathcal{A}(y) - \mathcal{A}(x)) = \mathcal{B}$. It is trivial to see that $\mathcal{A} - \mathcal{B}$ is a union closed set and there exists $z \in I_{q_o} - \{x, y\}$ such that $d^*(z) = |\{A_i \in \mathcal{A} - \mathcal{B} \text{ and } z \in A_i\}| \geq \frac{n_0 - (d(y) - d(x))}{2}$.

Case 1. $1 \leq d(y) - d(x) \leq 2$. Then $d^*(z) \geq \frac{n_0 - 2}{2}$ and being $n_0$ odd, $d^*(z) \geq \frac{n_0 - 1}{2}$.

Obviously if $d(x) < d(z)$, then $z \in \mathcal{C}(x)$ and so $d(z) \geq \frac{n_0 + 1}{2}$, contradicting the fact that $\mathcal{A} \in \mathbf{F}(n_0, q_0)$.

Case 2. $d(y) - d(x) = 3$. Let $\mathcal{B} = \{A, B, \mathcal{C}(x)\}$, then $d^*(z) \geq \frac{n_0 - 3}{2}$. Obviously $z \in \mathcal{C}(x)$ and so, if $A \cup B = \mathcal{C}(x)$ then $d(z) \geq \frac{n_0 + 1}{2}$, as above we obtain a contradiction.

Let $A \cup B = B$, then $A \subseteq B \subseteq \mathcal{C}(x)$.

If $B \supseteq \{w \in I_{q_o} - \{x, y\}: d(w) = \frac{n_0 - 1}{2}\}$, with a similar argument used in the previous case,we obtain a contradiction.

Suppose that there exists $w \in I_{q_0} - \{x, y\}$ such that $d(w) = \frac{n_0 - 1}{2}$ and $w \notin B$. Since $w \in \mathcal{C}(x) \cap \mathcal{C}(y)$, we can find a set $M_w$ such that $w \in M_w \in \mathcal{C}(y)$ and $|M_w| = \min |A_i \in \mathcal{C}(y): w \in A_i|$.

It is not hard to see that $\mathcal{A}-\{A,B,C(x),M_w\}$ is a union closed set in which we can find $t\in I_{q_0}-\{x,y\}$ such that $d^{\#}(t)=|A_i\in(\mathcal{A}-\{A,B,C(x),M_w\}):t\in A_i|\geq\frac{n_0-3}{2}$. Since $B\cup M_w=C(x)$ and $t\in C(x)$, it follows that $d(t)\geq\frac{n_0-3}{2}+2=\frac{n_0+1}{2}$, a contradiction.

Case 3. $d(y)-d(x)=4$.
Let $\mathcal{B}=\{B_1,B_2,B_3,C(x)\}$ with $B_i\subseteq C(x)$, for each $i=1,2,3$.

If $B_1\cup B_2\cup B_3=C(x)$ or if $B_1\cup B_2\cup B_3=B_3\supseteq\{w\in I_{q_0}-\{x,y\}: d(w)=\frac{n_0-1}{2}\}$, we can derive a contradiction by a similar argument used in the previous cases.

Suppose that there exists $w\in I_{q_0}-\{x,y\}$ such that $d(w)=\frac{n_0-1}{2}$ and $w\notin B_3$. Since $w\in C(x)\cap C(y)$, we can find two sets $M_w^1$ and $M_w^2$ such that $w\in M_w^i\in C(y)$, for each $i=1,2$ and $\sup\{|M_w^1|,|M_w^2|\}\leq\min|A_i\in C(y)-\{M_w^1,M_w^2\}:w\in A_i|$

It is not hard to see that $\mathcal{A}-\{B_1,B_2,B_3,C(x),M_w^1,M_w^2\}$ is a union closed set in which we can find $t\in I_{q_0}-\{x,y\}$ such that:

$$d^{\#}(t)=|A_i\in\mathcal{A}-\{B_1,B_2,B_3,C(x),M_w^1,,M_w^2\}:t\in A_i|\geq\frac{n_0-5}{2}.$$

By Case 1, since $d(t)+2\geq d(x)$ it follows that $t\in C(x)$. Since $B_i\cup M_w^j=C(x)$, for each $i=1,2;\ j=1,2$, it follows that $d(t)\geq\frac{n_0-5}{2}+3=\frac{n_0+1}{2}$, a contradiction. ∎

For each $\mathcal{A}\in\mathbf{F}(n_0,q_0)$, put:

$$\begin{aligned}
d_{\mathcal{A}} &= \min\{d_{\mathcal{A}}(x):x\in I_{q_0}\};\\
d_0 &= \max\{d_{\mathcal{A}}:\mathcal{A}\in\mathbf{F}(n_0,q_0)\};\\
\mathbf{F}^{d_0}(n_0,q_0) &= \{\mathcal{A}\in\mathbf{F}(n_0,q_0):d_{\mathcal{A}}=d_0\};\\
\mathbf{F}_r^{d_0}(n_0,q_0) &= \{\mathcal{A}\in\mathbf{F}^{d_0}(n_0,q_0):|x\in I_{q_0}:d_{\mathcal{A}}(x)=d_0|=r\};\\
r_0 &= \min\{r\in\mathbf{N}:\exists\mathcal{A}\in\mathbf{F}_r^{d_0}(n_0,q_0)\};\\
\mathbf{F} &= \mathbf{F}_{r_0}^{d_0}(n_0,q_0).
\end{aligned}$$

Trivially $\mathbf{F}(n,q)\neq\emptyset$ if and only if $\mathbf{F}\neq\emptyset$.

Let $\approx$ be the equivalance relation on the set $\mathbf{F}$ such that $\mathcal{A}\approx\mathcal{A}'$ if and only if $d_{\mathcal{A}}(x)=d_{\mathcal{A}'}(x)$, for each $x\in I_{q_0}$. Let $[\mathcal{A}]$ be the equivalence class of $\mathcal{A}$ and denote by $\mathbf{F}/\approx$ the set of all equivalence classes.

Assume, for convenience, $d(1)\leq d(2)\leq\cdots\leq d(q_0)$ for each $\mathcal{A}\in\mathbf{F}$.

We can now define a total ordering $\ll$ on $\mathbf{F}/\approx$ by $[\mathcal{A}]\ll[\mathcal{A}']$ if and only if $d_{\mathcal{A}}(1)=d_{\mathcal{A}'}(1);\ d_{\mathcal{A}}(2)=d_{\mathcal{A}'}(2);\ \ldots;\ d_{\mathcal{A}}(i)=d_{\mathcal{A}'}(i)$ and $d_{\mathcal{A}}(i+1)<d_{\mathcal{A}'}(i+1)$.

Assume $[\mathcal{A}]=\mathcal{F}$ the maximum in $(\mathbf{F}/\approx,\ll)$.

Let $\mathcal{A}\in\mathcal{F}$ and let $x,y$ be distinct elements of $I_{q_0}$. We have the following:

**Theorem 7.** *If $x \notin C(y)$ then $A_i \cup \{x\} \in \mathcal{A}(x)$, for each $A_i \in \mathcal{A}(y)$.*

Proof: Let $\mathcal{B}_y = \{A_i \in \mathcal{A}(y) : A_i \cup \{x\} \notin \mathcal{A}(x)\}$ and $\mathcal{A}' = (\mathcal{A} - \mathcal{B}_y) \cup \{A_i \cup \{x\} : A_i \in \mathcal{B}_y\}$. It is not hard to see that $\mathcal{A}'$ is a union closed set, $d_{\mathcal{A}}(z) = d_{\mathcal{A}'}(z)$, for each $z \in I_{q_0} - \{x\}$ and $d_{\mathcal{A}}(x) \leq d_{\mathcal{A}'}(x) \leq d_{\mathcal{A}}(y)$.

We shall show that the assumption $d_{\mathcal{A}}(x) < d_{\mathcal{A}'}(x)$ (i.e. $\mathcal{B}_y \neq \emptyset$) leads to a contradiction and hence $\mathcal{B}_y = \emptyset$.

Case 1. $d_{\mathcal{A}}(x) = d_{\mathcal{A}}(1) = d_0$

If $r_0 = 1$ then $x = 1$ and so $d_0 < d_{\mathcal{A}'}$ contradicting the maximality of $d_0$. If $r_0 > 1$ then $\mathcal{A}' \in \mathbf{F}_r^{d_0}(n_0, q_0)$ with $r < r_0$, contradicting the minimality of $r_0$.

Case 2. $d_0 < d_{\mathcal{A}}(x)$

Obviously $\mathcal{A}' \in \mathbf{F}$. Reordering if necessary, assume $d_{\mathcal{A}'}(1) \leq d_{\mathcal{A}'}(2) \leq \cdots \leq d_{\mathcal{A}'}(q_0)$. Notice that $d_{\mathcal{A}}(x) < d_{\mathcal{A}'}(x)$ and $d_{\mathcal{A}}(i) = d_{\mathcal{A}'}(i)$, for each $i \in I_{q_0} - \{x\}$. Thus $[\mathcal{A}] < [\mathcal{A}']$, contradicting the maximality of $[\mathcal{A}]$. This completes the proof. ∎

**Corollary 1.** *If $x \notin C(y)$ then $d_{\mathcal{A}}(y) \leq 2 d_{\mathcal{A}}(x)$.*

Proof: By Theorem 7, $\mathcal{B}_y = \emptyset$. Let $A_i \in \mathcal{A}(y) - \mathcal{A}(x)$ then $A_i \cup \{x\} \in \mathcal{A}(x)$ and so $d_{\mathcal{A}}(y) - d_{\mathcal{A}}(x) \leq d_{\mathcal{A}}(x)$ gives the desidered result. ∎

**Corollary 2.** $1 \leq |C(i) : x \notin C(i); i \in I_{q_0}| \leq 2$, *for each* $x \in I_{q_0}$.

Proof: Since $x \notin C(x)$, $|C(i) : x \notin C(i); i \in I_{q_0}| \geq 1$ is trivial.

Suppose that there are distinct elements $y$ and $z$ belonging to $I_{q_0} - \{x\}$ such that $x \notin C(y)$ and $x \notin C(z)$. Let $A_i \in \mathcal{A}(y)$. By Theorem 7, $A_i \cup \{x\} \in \mathcal{A}(x)$. Since $x \notin C(z)$, it follows that $\mathcal{A}(x) \subseteq \mathcal{A}(z)$ and so $A_i \cup \{x\} \in \mathcal{A}(z)$. Hence $A_i \in \mathcal{A}(z)$. This shows that $\mathcal{A}(y) \subseteq \mathcal{A}(z)$.

Interchanging $y$ and $z$ we obtain $\mathcal{A}(y) = \mathcal{A}(z)$, contradicting (ii) of Theorem 2. ∎

**Corollary 3.** $d_{\mathcal{A}}(1) = d_0 \geq 2q_0 - 5$.

Proof: We can find $z \in I_{q_0} - \{1\}$ such that $|A_i \in C(1) : z \in A_i| \geq \frac{n_0 - d_{\mathcal{A}}(1)}{2}$.

By Corollary 2, it follows that $|C(i) : \{1, z\} \subseteq C(i), i \in I_{q_0} - \{1, z\}| \geq q_0 - 4$. Since $I_{q_0} = A_n \supseteq \{1, z\}$, we have $\frac{n_0 - d_{\mathcal{A}}(1)}{2} + q_0 - 3 \leq \frac{n_0 - 1}{2}$ and so the desired result. ∎

**Theorem 8.**

(a) *If $C(x) = I_{q_0} - \{x\}$, for each $x \in I_{q_0}$, then $n_0 \geq 4q_0 - 1$;*
(b) *If there exist $y \in I_{q_0}$ such that $C(y) \neq I_{q_0} - \{y\}$ then $n_0 \geq 4q_0 + 1$.*

Proof:

(a) The same argument used in Corollary 3 works here to show that $d_A(1) \geq 2q_0 - 1$ and so $\frac{n_0-1}{2} \geq 2q_0 - 1$. Thus $n_0 \geq 4q_0 - 1$.

(b) By Corollary 3, $d_A(1) \geq 2q_0 - 5$. Let $x \in I_{q_0} - \{y\}$ such that $x \notin C(y)$. By Theorem 6, $d_A(x) + 5 \leq d_A(y)$. Thus $\frac{n_0-1}{2} \geq d_A(y) \geq d_A(x) + 5 \geq d_A(1) + 5 \geq 2q_0$ and so $n_0 \geq 4q_0 + 1$.

∎

**4. Restrictions on $q_0$ and $n_0$**

Let $\mathcal{A} \in \mathrm{A}(n, q)$, then $d_A(1) + d_A(2) + \cdots + d_A(q) = \omega_1 + \omega_2 + \cdots + \omega_n$. Let $n_j$ be the number of sets of $\mathcal{A}$ of cardinality $j$, it results:

$$\begin{aligned}\omega_1 + \omega_2 + \cdots + \omega_n &= \sum_{j=1}^{q}(j \cdot n_j) = \sum_{j=1}^{q}\left(j - \frac{q}{2}\right) \cdot n_j + \\ &+ \sum_{j=1}^{q}\frac{q}{2} \cdot n_j = \sum_{j=1}^{q}\left(j - \frac{q}{2}\right) \cdot n_j + \frac{n \cdot q}{2} = \\ &= \sum_{j=1}^{q-1}\left(j - \frac{q}{2}\right) \cdot n_j + \frac{q}{2} + \frac{n \cdot q}{2} = \\ &= \sum_{j=1}^{q-1}\left(j - \frac{q}{2}\right) \cdot n_j + q + \frac{(n-1) \cdot q}{2}.\end{aligned}$$

If $\mathcal{A} \in \mathbf{F}(n_0, q_0)$ then $n_1 = n_2 = 0$ and $n_{q_0} = 1$, so $\Sigma\left(j - \frac{q_0}{2}\right) \cdot n_j + q_0 \leq 0$ (Compare [4]).

**Theorem 9.** $q_0 \geq 9$.

Proof: By Theorem 5, $q_0 \geq 8$. Suppose $q_0 = 8$.

We shall show that this assumption leads to a contradiction. If $\mathcal{A} \in \mathbf{F}(n_0, 8)$, then

$$n_5 + 2n_6 + 3n_7 + 8 \leq n_3 \qquad (*)$$

By Theorem 3, $n_7 \geq 3$ and if $n_7 = 3$ then $n_6 \geq 1$. Thus $n_3 \geq 19$.

Let $s_j$ be the number of 4-set of $I_8$ containing exactly $j$ 3-sets of $\mathcal{A}$. If $s_j \neq 0$ for $j \geq 3$ then $\mathcal{H} \subseteq \mathcal{A}$ (where $\mathcal{H}$ is the set defined in Theorem 4) and by Theorem 4, $\mathcal{A} \notin \mathbf{F}(n_0, 8)$, so we can suppose $j = 0, 1, 2$.

Notice that $s_2 \leq n_4 \leq \binom{8}{4} = 70$. Appling the counting principle we have:

$$\begin{cases} s_1 + 2 \cdot s_2 = 5 \cdot n_3 \\ s_0 + s_1 + s_2 = 70 \end{cases} \text{and so} \begin{cases} s_2 = 5 \cdot n_3 - 70 + s_0 \\ s_1 = 140 - 5 \cdot n_3 - 2s_0 \end{cases}$$

Thus $n_4 \geq s_2 \geq 5n_3 - 70$.

If $n_3 \geq 29$, then $n_4 \geq 75$, a contradiction. So we may assume $19 \leq n_3 \leq 28$. If $n_3 \in \{23, 24, 25, 26, 27, 28\}$ then $n_4 \geq 45$ and so there exist a 3-set $\{x_1, x_2, x_3\}$ of $I_8$ contained at least in four 4-sets of $\mathcal{A}$. Thus $n_5 \geq 6$ and $n_6 \geq 4$, contrary to the $(*)$. If $n_3 \in \{19, 20, 21, 22\}$ then there exists $x_1 \in I_8$ which appears at least in eight 3-sets of $\mathcal{A}$.

Let $x_1 \in G_i$, $|G_i| = 3$, $i = 1, 2, \ldots, 8$ and let $\mathcal{G} = \{G_1, G_2, \ldots, G_8\} \subseteq \mathcal{A}$. Note that every $y \in \bigcup_{i=1}^{8} G_i - \{x_1\}$ appears at most in three 3-sets of $\mathcal{G}$, otherwise $n_5 \geq 4$, contradicting $(*)$ and then there exists $x_2 \in I_8 - \{x_1\}$ exactly in three 3-sets of $\mathcal{G}$. Let $G_1 = \{x_1, x_2, x_3\}$; $G_2 = \{x_1, x_2, x_4\}$; $G_3 = \{x_1, x_2, x_5\}$. It is easy to see that we can suppose $G_4 = \{x_1, x_6, y\}$ with $x_6 \notin \{x_1, x_2, x_3, x_4, x_5\}$.

If $y \neq x_i$, for each $i = 1, 2, \ldots, 6$ then $n_5 \geq 4$ and $n_6 \geq 3$, contradicting $(*)$. If $y \in \{x_3, x_4, x_5\}$ then we can assume $G_5 = \{x_1, x_7, z\}$ with $x_7 \neq x_i$, $i = 1, 2, \ldots 6$ and so $n_5 \geq 4$, contradicting $(*)$. ∎

**Theorem 10.** $n_0 \geq 37$.

Proof: Combining together Theorems 8, 9 we have $n_0 \geq 35$.

Suppose $n_0 = 35$. By Theorem 8, $q_0 = 9$ and $C(x) = I_9 - \{x\}$ for each $x \in I_9$. This implies that:

$$(8 \cdot 9) + 9 + 3 \cdot (35 - 9 - 1) \leq \omega_1 + \omega_2 + \cdots + \omega_{35} \leq \frac{35 - 1}{2} \cdot 9$$

and so $156 \leq 153$, which is a contradiction.

Since $n_0$ is odd, the theorem holds. ∎

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