On $(n-2)$-Extendable Graphs

N. Anunchuen and L. Caccetta

School of Mathematics and Statistics
Curtin University of Technology
GPO Box U1987
Perth, 6001
Western Australia

Abstract. Let G be a simple connected graph on $2n$ vertices with a perfect matching. G is *k-extendable* if for any set M of k independent edges, there exists a perfect matching in G containing all the edges of M. G is *minimally k-extendable* if G is k-extendable but $G - uv$ is not k-extendable for every pair of adjacent vertices u and v of G. The problem that arises is that of characterizing k-extendable and minimally k-extendable graphs. The first of these problems has been considered by several authors whilst the latter has only been recently studied. In a recent paper, we established several properties of minimally k-extendable graphs as well as a complete characterization of minimally $(n-1)$-extendable graphs on $2n$ vertices. In this paper, we focus on characterizing minimally $(n-2)$-extendable graphs. A complete characterization of $(n-2)$-extendable and minimally $(n-2)$-extendable graphs on $2n$ vertices is established.

1. Introduction

All graphs considered in this paper are finite, connected, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [3]. Thus G is a graph with vertex set $V(G)$, edge set $E(G)$, $\nu(G)$ vertices, $\varepsilon(G)$ edges, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph induced by V'. Similarly $G[E']$ denotes the subgraph induced by the edge set E' of G. $N_G(u)$ denotes the neighbour set of u in G and $\overline{N}_G(u)$ the non-neighbours of u. Note that $\overline{N}_G(u) = V(G) - N_G(u) - u$. The *join* $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

A *matching* M in G is a subset of $E(G)$ in which no two edges have a vertex in common. M is a *maximum matching* if $|M| \geq |M'|$ for any other matching M' of G. A vertex v is *saturated* by M if some edge of M is incident to v; otherwise, v is said to be *unsaturated*. A matching M is *perfect* if it saturates every vertex of the graph. For simplicity we let $V(M)$ denote the vertex set of the subgraph $G[M]$ induced by M.

Let G be a simple connected graph on $2n$ vertices with a perfect matching. For $1 \leq k \leq n-1$, G is *k-extendable* if for any matching M in G of size k, there exists a perfect matching in G containing all the edges of M. We say that G is *minimally k-extendable* or simply *k-minimal* if it is k-extendable but $G - uv$ is not k-extendable for any edge uv of G.

JCMCC 16 (1994), pp. 115-128

Observe that a cycle C_{2n} of order $2n \geq 4$ is 1-minimal. The complete graph K_{2n} of order $2n$ and the complete bipartite graph $K_{n,n}$ with bipartitioning sets of order n are each k-extendable. However, they are k-minimal if and only if $k = n - 1$ (see [2]).

A number of authors have studied k-extendable graphs; an excellent survey is the paper of Plummer [5]. Minimally k-extendable graphs have been recently studied by the authors. In [2] we established several properties of k-minimal graphs and proved that a graph G on $2n$ vertices is $(n-1)$-extendable if and only if $G \cong K_{n,n}$ or K_{2n}. It follows that G is $(n-1)$-minimal if and only if $G \cong K_{n,n}$ or K_{2n}. The problem of completely characterizing of k-extendable and k-minimal graphs on $2n$ vertices remains open for $1 \leq k \leq n-2$. In this paper we focus on a characterization of $(n-2)$-extendable and $(n-2)$-minimal graphs.

We establish that a graph G on $2n \geq 10$ vertices with a perfect matching is $(n-2)$-extendable if and only if G:

(1) is $K_{n,n}$ or K_{2n}, or
(2) is a bipartite graph with a minimum degree $n-1$, or
(3) has minimum degree $2n-3$ and contains a maximum independent set of order at most 2, or
(4) has minimum degree $2n-2$.

For $(n-2)$-minimal graphs on $2n$ vertices, we prove that an $(n-2)$-extendable graph G on $2n \geq 10$ vertices is minimal if and only if G:

(1) is an $(n-1)$-regular bipartite graph, or
(2) is a $(2n-3)$-regular graph, or
(3) contains one vertex of degree $2n-1$ and $2n-1$ vertices of degree $2n-3$, or
(4) contains $2n-2$ vertices of degree $2n-3$ and two vertices, u and v say, of degree $2n-2$ such that $N_G(u) - v = N_G(v) - u$.

Section 2 contains some preliminary results that we make use of in establishing our main results. The characterization of $(n-2)$-extendable and $(n-2)$-minimal graphs on $2n$ vertices is given in Section 3 and Section 4 respectively.

2. Preliminaries

In this section, we state a number of results on k-extendable and k-minimal graphs which we make use of in establishing our main results. We begin with fundamental results of k-extendable graphs proved by Plummer [4]:

Theorem 2.1. *Let G be a k-extendable graph on $2n$ vertices, $1 \leq k \leq n-1$. Then*

(a) *G is $(k-1)$-extendable.*
(b) *G is $(k+1)$-connected.*

Theorem 2.2. *Let G be a graph on $2n$ vertices and $1 \leq k \leq n-1$. If $\delta(G) \geq n+k$, then G is k-extendable.*

Anunchuen and Caccetta [1] established the following two results for k-extendable graphs.

Theorem 2.3. *Let G be a k-extendable graph on $2n$ vertices with $\delta(G) = k+t$, $1 \leq t \leq k \leq n-1$. If $d_G(u) = \delta(G)$, then the subgraph $G[N_G(u)]$ has a maximum matching of size at most $t-1$.*

Theorem 2.4. *Let G be a bipartite graph on $2n$ vertices with a perfect matching and $\delta(G) \geq n-1$. Then G is k-extendable for $1 \leq k \leq n-2$.*

For minimally k-extendable graphs, Anunchuen and Caccetta [2] proved the following three results which are very useful in establishing a characterization of minimally $(n-2)$-extendable graphs on $2n$ vertices.

Theorem 2.5. *Let G be a k-extendable graph on $2n$ vertices, $1 \leq k \leq n-1$. Then G is minimal if and only if for any edge $e = uv$ of G there exists a matching M of size k in $G-e$ such that $V(M) \cap \{u, v\} = \phi$ and for every perfect matching F, in G, containing M, $e \in F$.*

Theorem 2.6. *If $G \neq K_{2n}$ is a k-minimal graph on $2n$ vertices, $1 \leq k \leq n-1$, then $\delta(G) \leq n+k-1$.*

Theorem 2.7. (a) *K_{2n} is k-minimal, $1 \leq k \leq n-1$ if and only if $k = n-1$.*
(b) *$K_{n,n}$ is k-minimal, $1 \leq k \leq n-1$ if and only if $k = n-1$.*

We conclude this section by stating Dirac's Theorem (see [3], p. 54).

Theorem 2.8. *If G is a simple graph with $\nu(G) \geq 3$ and $\delta(G) \geq \frac{1}{2}\nu(G)$, then G is hamiltonian.*

3. A Characterization of $(n-2)$-Extendable Graphs

Our first result provides the possible values of the minimum degree of an $(n-2)$-extendable graph.

Theorem 3.1. *If G is an $(n-2)$-extendable graph on $2n \geq 6$ vertices, then $\delta(G) \leq n$ or $\delta(G) \geq 2n-3$.*

Proof: The assertion is obvious for $G = K_{2n}$. Assume $G \neq K_{2n}$ and suppose to the contrary that $n+1 \leq \delta(G) \leq 2n-4$. So we need only consider $n \geq 5$. Let u be a vertex of G with $d_G(u) = \delta(G) = r$ and M a maximum matching in $G[N_G(u)]$. Now by Theorem 2.3

$$|M| \leq \delta(G) - (n-2) - 1 = r - n + 1.$$

Hence, $r - 2|M| \geq 2n - r - 2 \geq 2$. Consequently, there exist vertices, a and b say, in $N_G(u)\backslash V(M)$. The maximality of M implies that there are at most $2|M|$ edges between the vertices of $V(M)$ and a, b. Consequently,

$$\begin{aligned} 2r \leq d_G(a) + d_G(b) &\leq 2 + 2|M| + 2(2n - r - 1) \\ &= 4n - 2r + 2|M| \\ &\leq 4n - 2r + 2(r - n + 1) = 2n + 2. \end{aligned}$$

Hence, $r \leq n+1$. So we have nothing to prove for $n+2 \leq r \leq 2n-4$. The only case to consider is $r = n+1$. Now the above inequality implies that $|M| = 2$.

Let $M = \{u_1u_2, v_1v_2\}$. The extendability of G implies the existence of a perfect matching F in G containing M. Let $uu' \in F$ and

$$F' = \{ww' \in F : w \in N_G(u)\backslash V(M), w' \in \overline{N}_G(u)\}.$$

Since M is a maximum matching in $G[N_G(u)]$ the vertices of $N_G(u)\backslash V(M)$ form an independent set. Consequently, $|F'| = n-4$. Now since $|\overline{N}_G(u)| = n-2$ there exists an edge $x_1x_2 \in F$ with x_1 and x_2 in $\overline{N}_G(u)$. The situation is depicted in Figure 3.1; the edges of F are shown in solid lines.

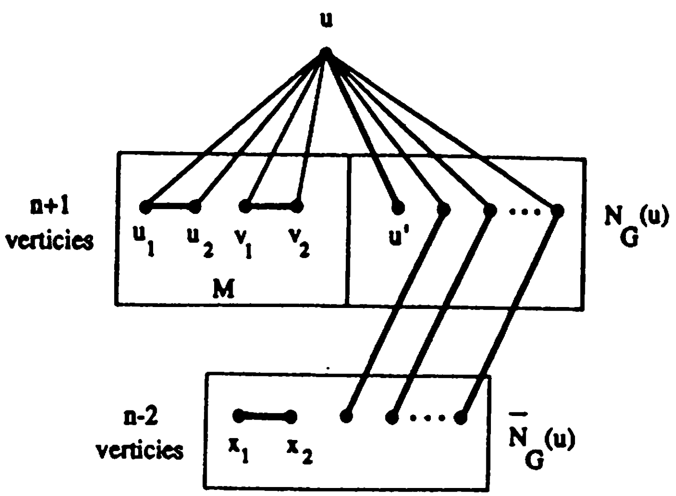


Figure 3.1

Suppose that there is a vertex $w_1 \in N_G(u)\backslash V(M)$ that is joined to both ends of an edge of M, say u_1u_2. Consider any vertex $w_2 \in N_G(u)\backslash V(M)$, $w_2 \neq w_1$. The maximality of M implies that w_2 is not joined to u_1 or u_2. The independence of the set $N_G(u)\backslash V(M)$ and the requirement that $d_G(w_2) \geq n+1$ implies that

$$N_G(w_2) = \{u, v_1, v_2\} \cup \overline{N}_G(u).$$

Since $w_2 \neq w_1$ is any vertex of $N_G(u)\backslash V(M)$ and M is maximum, the only possibility is for $|\overline{N}_G(u)| = n+1 = 6$ and hence, $|N_G(u)| = 3$. Further, every vertex of $N_G(u)$ must be joined to every vertex of $\overline{N}_G(u)$. Let $x_3 \in \overline{N}_G(u)\backslash\{x_1, x_2\}$. Then

$$M' = \{uv_2, x_3v_1, x_1x_2\}$$

is a matching in $G[N_G(w_2)]$ of size greater than two, contradicting Theorem 2.3. Hence, each vertex of $N_G(u)\backslash V(M)$ is joined to at most one end of each edge of M.

The choice of M and the requirement that $\delta(G) = n+1$ implies that each vertex $w \in N_G(u)\backslash V(M)$ is joined to exactly one end of each edge of M. In fact,

$$N_G(w_1) \cap N_G(u) = N_G(w_2) \cap N_G(u)$$

for any $w_1 \neq w_2 \in N_G(u)\backslash V(M)$. Without any loss of generality we assume that $u_1w, v_1w \in E(G)$, for every $w \in N_G(u)\backslash V(M)$. Consequently,

$$N_G(w) = \{u, u_1, v_1\} \cup \overline{N}_G(u)$$

for every $w \in N_G(u)\backslash\{u_1v_1\}$.

Now if $u_1x \in E(G)$ for some $x \in \overline{N}_G(u)\backslash\{x_1, x_2\}$, then

$$M'' = \{uv_1, u_1x, x_1x_2\}$$

is a matching in G such that $N_G(u)\backslash V(M'')$ is an independent set of size $n-1$. But then since $|V(G)\backslash V(M'')| = 2n-6$, M'' does not extend to a perfect matching in G, contradicting the extendability of G. Hence, $u_1x \notin E(G)$ for any $x \in \overline{N}_G(u)\backslash\{x_1, x_2\}$. Similarly $v_1x \notin E(G)$ for any $x \in \overline{N}_G(u)\backslash\{x_1, x_2\}$.

Since $N_G(u)\backslash\{u_1, v_1\}$ is an independent set of $n-1$ vertices, $u_1v_1 \notin E(G)$, as otherwise, $\{u_1v_1, x_1x_2\}$ does not extend to a perfect matching. Let $x_3 \in \overline{N}_G(u)\backslash\{x_1, x_2\}$ and $wx_3 \in F'$. Note that $w \in N_G(u)\backslash V(M)$. Then

$$F'' = (F'\backslash\{wx_3\}) \cup \{u_2x_1, v_2x_2, uu'\}$$

is a matching of size $n-5+3 = n-2$ in G. But F'' does not extend to a perfect matching in G since $G - V(F'') = \{u_1, v_1, w, x_3\}$ is a $K_{1,3}$ with centre w. This contradiction completes the proof of the theorem. ∎

In the next three lemmas we establish a characterization of $(n-2)$-extendable graphs on $2n$ vertices with prescribed minimum degree.

Lemma 3.1. *Let G be a graph on $2n \geq 8$ vertices with a perfect matching and $\delta(G) = n-1$. Then G is $(n-2)$-extendable if and only if G is bipartite.*

Proof: The sufficiency follows from Theorem 2.4. We need only prove the necessity. So let G be an $(n-2)$-extendable graph with $\delta(G) = n-1$.

Let u be a vertex of degree $n-1$. By Theorem 2.3, $N_G(u)$ is an independent set of vertices. The subgraph $H = G[\overline{N}_G(u)]$ has at least one edge, since otherwise $\overline{N}_G(u) \cup \{u\}$ is an independent set of $n+1$ vertices implying that G has no perfect matching. If xy and $x'y'$ are independent edges of H, then the graph

$$G' = G - \{x, y, x', y'\}$$

has $2n-4$ vertices and contains $N_G(u)$ as an independent set of $n-1$ vertices. Thus G' cannot have a perfect matching, contradicting the fact that G is k-extendable, $k \geq 2$. Hence, H contains only one independent edge, xy say.

Now since G is $(n-1)$-connected (Theorem 2.1 (b)) and $|\overline{N}_G(u)| = n \geq 4$ at least one of x or y is adjacent to a vertex of $N_G(u)$. Suppose that $xz \in E(G)$ with $z \in N_G(u)$. If $yw \in E(G)$, $w \neq z \in N_G(u)$, then the graph $G - \{x, y, z, w\}$ contains two disjoint independent sets $\{u\} \cup (\overline{N}_G(u) \setminus \{x, y\})$ and $N_G(u) \setminus \{z, w\}$ of order $n-1$ and $n-3$, respectively, and so cannot have a perfect matching. This contradicts the fact that G is k-extendable, $k \geq 2$. Hence, $|N_G(y) \cap N_G(u)| \leq 1$.

Suppose that $N_G(y) \cap N_G(u) \neq \phi$. Then $yz \in E(G)$. The above argument implies that $N_G(x) \cap N_G(u) = \{z\}$. Now, by Theorem 2.3, each of x, y and z has degree at least n in G. Consequently, x and y are joined to every vertex of $\overline{N}_G(u)$. But then since $n \geq 4$, H contains at least two independent edges. This contradiction establishes that $N_G(y) \cap N_G(u) = \phi$. Hence, $d_G(y) = n-1$ and the set of vertices $N_G(y) = \overline{N}_G(u) \setminus \{y\}$ must be (by Theorem 2.3) an independent set. Consequently, $N_G(u) \cup \{y\}$ and $\{u\} \cup (\overline{N}_G(u) \setminus \{y\})$ are independent sets of n vertices in G, proving that G is bipartite. ∎

Remark: Lemma 3.1 is best possible in the sense that there exists an $(n-2)$-extendable graph on $2n = 6$ vertices that is not bipartite. Figure 3.2 displays such a graph.

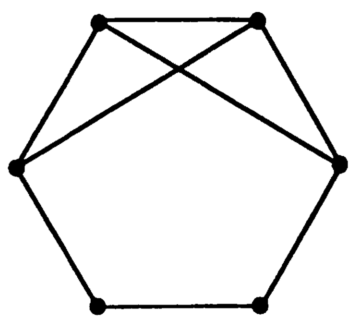


Figure 3.2

Lemma 3.2. *Let G be a graph on $2n \geq 10$ vertices with a perfect matching and $\delta(G) = n$. Then G is $(n-2)$-extendable if and only if $G \cong K_{n,n}$.*

Proof: The sufficiency is obvious as $K_{n,n}$ is k-extendable for $1 \leq k \leq n-1$. So we need to prove only the necessity. We do this by following a slmilar strategy to that used in the proof of the previous lemma. So let G be an $(n-2)$-extendable graph with $\delta(G) = n$.

If G contains an independent set X of n vertices, then $V(G)\backslash X$ is also an independent set, since otherwise G cannot be k-extendable, $k \geq 3$. But then, since $\delta(G) = n$, $G \cong K_{n,n}$. Hence, we may suppose that G contains at most $n-1$ independent vertices.

Let $d_G(u) = n$. Theorem 2.3 together with the above assumption implies that the subgraph $G[N_G(u)]$ contains only one independent edge, vw say. Now for the edge vw to be extendable to a perfect matching in G, the subgraph $H = G[\overline{N}_G(u)]$ must have edges. If H contains two independent edges xy and $x'y'$, then the graph $G' = G - \{x, y, x', y', v, w\}$ has $2n-6$ vertices and contains an independent set of order $n-2$ and hence, cannot contain a perfect matching. This contradicts the fact that G is k-extendable, $k \geq 3$. Hence, H contains only one independent edge, say xy. Consequently, either $d_H(x) = d_H(y) = 2$ or at least one of x or y, say x, has degree 1 in H. So x must be joined to at least $n-2 \geq 3$ vertices of $N_G(u)$. Hence, $xz \in E(G)$ for some $z \in N_G(u)\backslash\{v, w\}$. Further, since $n \geq 5$, y must be joined to a vertex $z' \in V(G)\backslash\{x, v, w, z\}$.

If $z' \notin \overline{N}_G(u)$, then $G - \{v, w, x, y, z, z'\}$ is a bipartite graph with bipartitioning sets $N_G(u)\backslash\{v, w, z, z'\}$ and $\{u\} \cup (\overline{N}_G(u)\backslash\{x, y\})$ of order $n-4$ and $n-2$, respectively. Hence, the edges $\{vw, xz, yz'\}$ do not extend to a perfect matching, a contradiction. Therefore, $z' \in \overline{N}_G(u)$ and hence, since $d_G(y) \geq n$, $t = |N_G(y) \cap N_G(u)|$ is 2 or 3. We claim that $\overline{N}_G(u) - y \subseteq N_G(y)$. This is clearly so when $t = 2$. If $t = 3$, then vw and xz are independent edges in $N_G(y)$ and hence, by Theorem 2.3, $d_G(y) = n+1$ and so $\overline{N}_G(u) - y \subseteq N_G(y)$. Now since $n \geq 5$, $\{u\} \cup (\overline{N}_G(u)\backslash\{y\})$ is an independent set of $n-1$ vertices in $G - \{v, w, y, z\}$. Hence, $yz \notin E(G)$. Consequently,

$$N_G(y) = \{v, w\} \cup (\overline{N}_G(u)\backslash\{y\}).$$

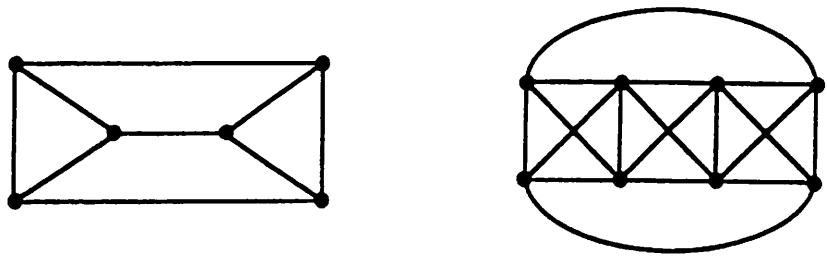
Further, $d_H(x) = 1$ and so x must be joined to at least $(n-1)$ vertices of $N_G(u)$. Hence, xv or $xw \in E(G)$.

Without any loss of generality suppose that $xv \in E(G)$. Then $\{uw, xv\}$ does not extend to a perfect matching in G, since the subgraph $G - \{u, v, w, x\}$ has an independent set $\{y\} \cup (N_G(u)\backslash\{v, w\})$ of order $n-1$ and so cannot have a perfect matching. This proves that $N_G(u)$ is an independent set and completes the proof of the lemma. ∎

Remark: Lemma 3.2 is best possible in the sense that there are $(n-2)$-extendable graphs on $6 \leq 2n \leq 8$ vertices with $\delta(G) = n$ which are not $K_{n,n}$. Two of these graphs are shown in Figure 3.3.

Our next lemma concerns the case when $\delta(G) = 2n-3$.

Lemma 3.3. *Let G be a graph on $2n \geq 8$ vertices with a perfect matching and $\delta(G) = 2n-3$. Then G is $(n-2)$-extendable if and only if G contains a maximum independent set of order at most 2.*



(a) (b)

Figure 3.3

Proof: First we prove the sufficiency. Let M be a matching of size $n-2$ in G. Consider the subgraph $G' = G - V(M)$. Clearly $\nu(G') = 4$ and $\delta(G') \geq 1$ since $\delta(G) = 2n-3$. If G' has no perfect matching, then $G' \cong K_{1,3}$. But then G contains an independent set of size 3, a contradiction. Hence, G' has a perfect matching. This implies that G is $(n-2)$-extendable.

Now we prove the necessity. Suppose G is an $(n-2)$-extendable graph with $\delta(G) = 2n-3$ having an independent set $S = \{u, v, w\}$. Then $d_G(u) = 2n-3$ and $\overline{N}_G(u) = \{v, w\}$. Let $u' \in N_G(u)$ and F a perfect matching containing the edge uu'. Then there exist vertices v' and w' of $N_G(u)$ such that $vv', ww' \in F$. Further, $G[N_G(u)\backslash\{u', v', w'\}]$ contains a subset F' of F of size $n-3$. If $G[\{u', v', w'\}]$ contains an edge e, then $\{e\} \cup F'$ is a matching of size $n-2$ that does not extend to a perfect matching in G since $G - V(\{e\} \cup F') \cong K_{1,3}$, contradicting the extendability of G. Hence, $\{u', v', w'\}$ is an independent set. Since $\delta(G) = 2n-3$, each vertex of $\{u', v', w'\}$ is adjacent to every vertex of F'. Let $ab \in F'$. Now $F'' = (F'\backslash\{ab\}) \cup \{v'a, w'b\}$ does not extend to a perfect matching in G, since $G - V(F'') \cong K_{1,3}$, again contradicting the extendability of G. This completes the proof of our lemma. ∎

Remark 1: Lemma 3.3 is best possible in the sense that $K_{3,3}$ is a 1-extendable graph on 6 vertices containing an independent set of order 3.

Remark 2: The necessity of Lemma 3.3 does not always hold for $(n-3)$-extendable graphs. For example, the graph $H = 3K_1 \vee K_{2n-3}$ is an $(n-3)$-extendable graph on $2n \geq 8$ vertices that contains an independent set of order 3.

In view of theorems 2.1(b), 2.2 and 3.1 and lemmas 3.1, 3.2 and 3.3 we can now state a characterization of $(n-2)$-extendable graphs.

Theorem 3.2. *Let G be a graph on $2n \geq 10$ vertices with a perfect matching. Then G is $(n-2)$-extendable if and only if G:*

(1) *is $K_{n,n}$ or K_{2n}, or*
(2) *is a bipartite graph with minimum degree $n-1$, or*
(3) *has minimum degree $2n-3$ and contains a maximum independent set of order at most 2, or*

(4) *has minimum degree* $2n-2$.

Remark 1: There exist $(n-2)$-extendable graphs for each type specified in Theorem 3.2. Clearly, $K_{n,n} \backslash \{$ a perfect matching $\}$ satisfies type (2). $2K_2 \vee K_{2n-4}$ satisfies type (3) and $K_{2n} - e$, for some edge e in K_{2n} is in type (4).

Remark 2: An $(n-2)$-extendable graph has order at least 6. Theorem 3.2 provides a characterization for $2n \geq 10$. The graphs displayed in Figures 3.2 and 3.3(b) indicate that this bound is best possible.

4. A Characterization of $(n-2)$-Minimal Graphs

From theorems 2.1(b), 2.6, 2.7 and 3.1, we conclude that an $(n-2)$-minimal graph G has $\delta(G) = n-1, n$ or $2n-3$ for $2n \geq 6$. Further, from Lemma 3.2, $\delta(G) \neq n$ for $2n \geq 10$. We thus have :

Lemma 4.1. *If G is an $(n-2)$-minimal graph on $2n \geq 6$ vertices, then* $\delta(G) = n-1, n$ *or* $2n-3$. *Furthermore, for* $2n \geq 10$, $\delta(G) \neq n$.

We establish our characterizations of $(n-2)$-minimal graphs by considering two cases according to the values of the minimum degree.

Theorem 4.1. *G is an $(n-2)$-minimal graph on $2n \geq 8$ vertices with* $\delta(G) = n-1$ *if and only if G is an $(n-1)$-regular bipartite graph.*

Proof: It follows from Lemma 3.1 and Theorem 2.1(b) that an $(n-1)$-regular bipartite graph G on $2n \geq 8$ vertices is $(n-2)$-minimal and so the sufficiency is immediate. We need to consider the necessity part.

Let G be an $(n-2)$-minimal graph with $\delta(G) = n-1$. Then, by Lemma 3.1, G is bipartite with bipartitioning sets, A and B say, of order n. We need to establish that G is $(n-1)$-regular. Suppose that this is not the case. Then, since $\delta(G) = n-1$ and $|A| = |B| = n$, G contains vertices $x \in A$ and $y \in B$ that have degree n. So $xy \in E(G)$. But then $G - xy$ is a bipartite graph with $\delta(G) = n-1$ and hence, by Theorem 2.4, is $(n-2)$-extendable. This contradicts the minimality of G and completes the proof of our theorem. ∎

Remark: The bound on n in Theorem 4.1 is best possible as an $(n-2)$-minimal graph G on 6 vertices exists which is neither bipartite nor regular; for example, the graph of Figure 3.2.

Characterizing the $(n-2)$-minimal graphs having minimum degree $2n-3$ is a more complicated exercise. We begin by establishing some sufficient conditions for $(n-2)$-extendable graphs to be minimal.

Lemma 4.2. *If G is a $(2n-3)$-regular $(n-2)$-extendable graph on $2n \geq 8$ vertices, then G is minimal.*

Proof: Let $e = uv \in E(G)$ and consider $G' = G[N_G(u) - v]$. Clearly $\nu(G') = 2n-4$ and $\delta(G') \geq 2n-7$. We claim that G' has a perfect matching M. For

$2n \geq 10$ this follows from Theorem 2.8 as $\delta(G') \geq 2n-7 \geq \frac{1}{2}\nu(G')$. For $2n = 8$, $\delta(G') \geq 1$ and so either G' has a perfect matching or an independent set of size 3. Now, by Lemma 3.3, G' must have a perfect matching as required. Now M is a matching of size $n-2$ which clearly does not extend to a perfect matching in $G - uv$. This completes the proof of the lemma. ∎

Remark: Lemma 4.2 is best possible in the sense that $K_{3,3}$ is 3-regular 1-extendable on 6 vertices but is not minimal, by Theorem 2.7.

Lemma 4.3. *Let G be an $(n-2)$-extendable graph on $2n \geq 8$ vertices. If G has only one vertex of degree $2n-1$ and $2n-1$ vertices of degree $2n-3$, then G is minimal.*

Proof: Follows from the proof of Lemma 4.2, since every edge of G is incident to at least one vertex of degree $2n-3$. ∎

Remark: Lemma 4.3 is best possible in the sense that the wheel W_6 (drawn in Figure 4.1) on 6 vertices satisfies our hypothesis but is not minimal since $W_6 - e$ is still 1-extendable.

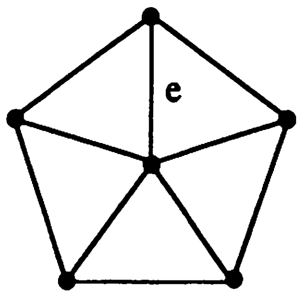


Figure 4.1

Lemma 4.4. *Let G be an $(n-2)$-extendable graph on $2n \geq 8$ vertices. If G has $2n-2$ vertices of degree $2n-3$ and 2 vertices, u and v say, of degree $2n-2$, such that $N_G(u) - v = N_G(v) - u$, then G is minimal.*

Proof: Let $e = xy$ be an edge of G. If $d_G(x) = 2n-3$, then the proof of Lemma 4.2 is valid and establishes that G is minimal. So the only case we need to consider is $d_G(x) = d_G(y) = 2n-2$. That is $x = u$ and $y = v$. Now since $N_G(u) - v = N_G(v) - u$, G contains a vertex w that is not joined to u or v. Clearly, $N_G(w) = N_G(u) - v$. Consider a vertex $z \in N_G(u) - v$ and the subgraph $G' = G[N_G(u) - v - z]$. Observe that $\nu(G') = 2n-4$ and $\delta(G') \geq 2n-7$ and so, as in the proof of Lemma 4.2, G' contains a perfect matching M. Now M does not extend to a perfect matching in $G - uv$ since the induced subgraph of $\{u, v, z, w\}$ in $G - uv$ is $K_{1,3}$. This completes the proof of the lemma. ∎

Remark: The condition $N_G(u) - v = N_G(v) - u$ in Lemma 4.4 is essential, since there exists an $(n-2)$-extendable graph which violates this condition and is not minimal. Let P_4 be a path on 4 vertices and $H = P_4 \vee K_{2n-4} \backslash \{$ a hamiltonian cycle $\}$, $n \geq 4$. It is easy to show that H is an $(n-2)$-extendable graph with 2 vertices, u and v say, of degree $2n-2$ and $2n-2$ vertices of degree $2n-3$. Clearly, u and v are internal vertices of P_4. Further, $N_G(u) - v \neq N_G(v) - u$. It is not difficult to show that $H - uv$ is $(n-2)$-extendable. Hence, H is not minimal.

Now we can establish a characterization of $(n-2)$-minimal graphs on $2n$ vertices with minimum degree $2n-3$. We begin with the following lemma.

Lemma 4.5. *Let G be a k-minimal graph on $2n$ vertices, $1 \leq k \leq n-2$ with $\Delta(G) = 2n-1$. If $d_G(u) = 2n-1$, then $d_G(v) \leq 2k+1$ for every $v \in V(G)\backslash u$.*

Proof: Suppose to the contrary that there exists a vertex $v \neq u$ of G with $d_G(v) \geq 2k+2$. Since G is minimal and $uv \in E(G)$ there exists a matching M of size k in $G - uv$ with $V(M) \cap \{u, v\} = \phi$. Let F be a perfect matching, in G containing M. Thus $uv \in F$ (Theorem 2.5). Since $k \leq n-2$, $|V(G)\backslash(V(M) \cup \{u, v\})| \geq 2n-2k-2 \geq 2$. Because $d_G(v) \geq 2k+2$, there exists a vertex $x \in V(G)\backslash(V(M) \cup \{u, v\})$ with $vx \in E(G)$. Let $xy \in F$. Clearly, $y \notin V(M) \cup \{u, v\}$. Further, $uy \in E(G)$ since $d_G(u) = 2n-1$. Thus

$$F_0 = (F\backslash\{uv, xy\}) \cup \{vx, uy\}$$

is a perfect matching containing M and $uv \notin F_0$. But this contradicts Theorem 2.5 and hence proves our lemma. ∎

Lemmas 4.3 and 4.5 together yield the following theorem.

Theorem 4.2. *Let G be an $(n-2)$-extendable graph on $2n \geq 8$ vertices with $\delta(G) = 2n-3$ and $\Delta(G) = 2n-1$. Then G is minimal if and only if G has only one vertex of degree $2n-1$ and $2n-1$ vertices of degree $2n-3$.*

Theorem 4.3. *Let G be an $(n-2)$-extendable graph on $2n \geq 8$ vertices with $\delta(G) = 2n-3$ and $\Delta(G) = 2n-2$. Then G is minimal if and only if G has $2n-2$ vertices of degree $2n-3$ and 2 vertices, u and v say, of degree $2n-2$ such that $N_G(u) - v = N_G(v) - u$.*

Proof: The sufficiency follows from Lemma 4.4. So we need only prove the necessity. Let G be an $(n-2)$-minimal graph with $\delta(G) = 2n-3$ and $\Delta(G) = 2n-2$. Then the number of vertices of degree $2n-3$ must be even and hence, the number of vertices of degree $2n-2$ is also even. Thus G contains at least two vertices of degree $2n-2$. We need to prove that there are exactly 2 such vertices in G. Suppose to the contrary that u, v and w are three vertices of degree $2n-2$. We distinguish into two cases according to whether or not $uv \in E(G)$.

Case 1: $uv \notin E(G)$. Then $N_G(u) = N_G(v) = V(G)\backslash\{u, v\}$ and $w \in N_G(u)$. Let M be any matching in G of size $n-2$ with $V(M) \cap \{u, w\} = \phi$; such an

M exists since G is extendable. Consider the subgraph $G' = G - V(M)$. Let $V(G') = \{u, w, x, y\}$. Note that v could be x or y.

If $v = x$, then $F = \{xw, uy\} \cup M$ is a perfect matching in $G - uw$, contradicting the minimality of G (Theorem 2.5). Hence, $v \notin \{x, y\}$. This implies that $x, y \in N_G(u)$. Since $d_G(w) = 2n-2$, $wy \in E(G)$ or $wx \in E(G)$. Without any loss of generality, assume $wy \in E(G)$. Then $F' = \{wy, ux\} \cup M$ is a perfect matching in $G - uw$, again contradicting the minimality of G (Theorem 2.5). This proves Case 1.

Case 2: $uv \in E(G)$. Let $a \in \overline{N}_G(u)$ and M' a matching in G with $V(M') \cap \{u, v\} = \phi$. If $av \in E(G)$, then an argument similar to that used in Case 1 establishes the existence of a perfect matching F'', in $G - uv$, containing M' such that $uv \notin F''$. Hence, by Theorem 2.5, $av \notin E(G)$. This implies that $N_G(u) - v = N_G(v) - u$. Since $\delta(G) = 2n-3$, $N_G(a) = N_G(u) - v$ and $w \neq a$. Thus $w \in N_G(u) - v$. Again a similar argument to that used in Case 1 establishes the existence of a perfect matching in $G - uw$ containing a matching M'' of size $n-2$ with $V(M'') \cap \{u, w\} = \phi$. This contradicts the minimality of G. Hence, u and v are only two vertices of degree $2n-2$ in G. Moreover, $N_G(u) - v = N_G(v) - u$ follows directly from the proof and completes the proof of the theorem. ∎

The following result which follows from Lemma 3.3 and theorems 4.2 and 4.3 gives us information on the induced subgraph of a neighbour set of a vertex having maximum degree in $(n-2)$-minimal graphs with $\delta(G) = 2n-3$.

Lemma 4.6. *Let G be a non-regular $(n-2)$-minimal graph on $2n \geq 8$ vertices with $\delta(G) = 2n-3$. If $d_G(u) = \Delta(G)$ and $H = G[N_G(u)]$, then $\overline{H}$, the complement of H, is a 2-regular triangle-free graph or a 2-regular triangle free graph plus an isolated vertex.*

Lemmas 4.1 and 4.2 and theorems 4.1, 4.2 and 4.3 together allow us to state the following characterization of $(n-2)$-minimal graphs on $2n$ vertices.

Theorem 4.4. *Let G be an $(n-2)$-extendable graph on $2n \geq 10$ vertices. Then G is minimal if and only if G:*

(1) *is an $(n-1)$-regular bipartite graph, or*
(2) *is a $(2n-3)$-regular graph, or*
(3) *contains one vertex of degree $2n-1$ and $2n-1$ vertices of degree $2n-3$, or*
(4) *contains $2n-2$ vertices of degree $2n-3$ and two vertices of degree $2n-2$, u and v say, such that $N_G(u) - v = N_G(v) - u$.*

Remark: There exists $(n-2)$-minimal graphs for each type specified in Theorem 4.4. Examples are: $K_{n,n} \backslash \{$a perfect matching$\}$; $K_{2n} \backslash \{$ a hamiltonian cycle$\}$; $K_1 \vee K_{2n-1} \backslash \{$ a hamiltonian cycle$\}$; and $\overline{K}_2 \vee K_{2n-2} \backslash \{$ a hamiltonian cycle $\}$, respectively.

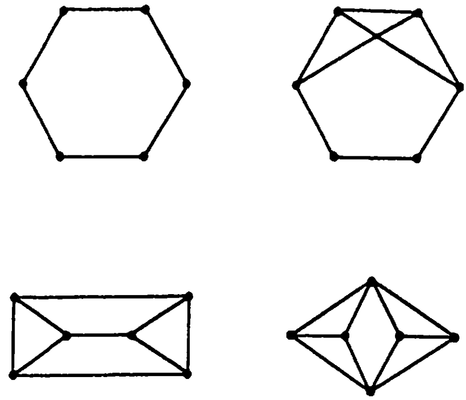


Figure 4.2

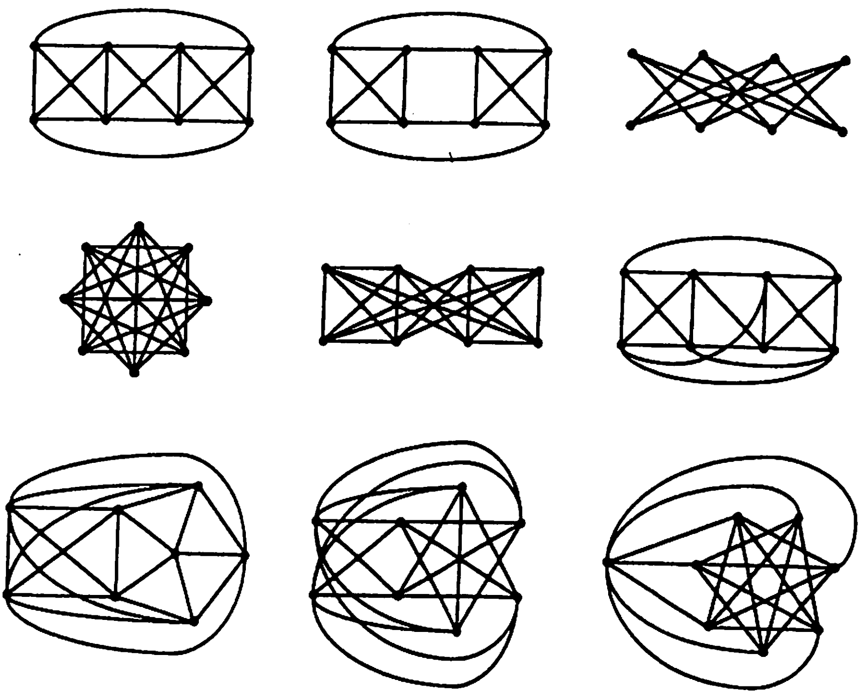


Figure 4.3

We have observed that an $(n-2)$-extendable graph on $2n$ vertices must have $2n \geq 6$. Theorem 4.4 characterizes $(n-2)$-minimal graphs of order $2n \geq 10$. We have completely characterized all $(n-2)$-minimal graphs on 6 and 8 vertices. As the proofs are somewhat tedious we simply state the results.

Theorem 4.5. *Let G be an $(n-2)$-minimal graph on $2n$ vertices, $n = 3$ or 4. Then*

(a) *if $n = 3$, G is one of the graphs displayed in Figure 4.2.*
(b) *if $n = 4$, G is one of the graphs displayed in Figure 4.3.*

Acknowledgement

This work has been supported, in part, by an Australian Research Grant (A48932119).

References

1. N. Anunchuen and L. Caccetta, *On Critically k-Extendable Graphs*, Australasian Journal of Combinatorics **6** (1992), 39–65.
2. N. Anunchuen and L. Caccetta, *On Minimally k-Extendable Graphs*. (submitted for publication).
3. J.A. Bondy and U.S.R. Murty, "Graph Theory with Applications", The MacMillan Press, London, 1976.
4. M.D. Plummer, *On n-Extendable Graphs*, Discrete Mathematics **31** (1980), 201–210.
5. M.D. Plummer, *Extending Matchings in Graphs:A Survey*. (submitted for publication).