

Invariants for 2-Factorizations and Cycle Systems

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ABSTRACT. We propose several invariants for cycle systems and 2-factorizations of complete graphs, and enumerate the 4- and 6-cycle systems of K_9 .

1. Introduction

An r -factorization of the graph G is a partition of the edge-set of G into edge-disjoint regular factors of degree r ; an r -factorization is *uniform* if its factors are mutually isomorphic. Any 1-factorization of a graph G is necessarily uniform; the well-known Oberwolfach problem asks whether there exists, for a given 2-regular graph Q with n vertices, a uniform 2-factorization of the complete graph K_n with n odd into 2-factors, each isomorphic to Q . In spite of a considerable effort, the Oberwolfach problem has been solved only under the additional assumption that all cycles of Q are of the same length (see, e.g., [ABS], [LR]).

An m -cycle system is a partition of the edge-set of G into (edge-disjoint) cycles of length m . The existence problem for m -cycle systems has been settled for the case when m is a prime power, as well as for $m \leq 50$; for more details on the existence problem for cycle systems, see [LR].

Results on isomorphism and enumeration of uniform 2-factorizations and cycle systems are less numerous. The best known algorithm for isomorphism testing of 1-factorizations, and of 2-factorizations, respectively, is

subexponential [CC], [C2]. No polynomial time algorithm for isomorphism testing of 2-factorizations or cycle systems is known. But even when a polynomial time algorithm for isomorphism testing is known, such as in the case of Hamilton cycle decompositions [C], and even when the list of nonisomorphic designs is available (the 122 nonisomorphic Hamilton cycle decompositions of K_9 being a case in point), the problem of identifying designs on the list suggests the use of easily computable *isomorphism invariants* or simply *invariants*. An invariant on a set \mathbf{D} may be viewed as a function I on \mathbf{D} such that for $D_1, D_2 \in \mathbf{D}$, $I(D_1) = I(D_2)$ if D_1 and D_2 are isomorphic. When $I(D_1) = I(D_2)$ if and only if D_1 and D_2 are isomorphic, the invariant I is *complete*. The *sensitivity* of an invariant I for \mathbf{D} is the ratio of the number of isomorphism classes that I distinguishes to the number of nonisomorphic elements of \mathbf{D} . (For more details on isomorphism testing and invariant in general, see [C2].) The purpose of this paper is twofold: inspired somewhat by recent work on invariants of 1-factorizations of complete graphs ([DW], [GR]), we first propose several invariants for 2-factorizations of complete graphs and also for cycle systems. Second, we undertake an enumeration of all nonisomorphic cycle systems of the complete graph K_9 , making an essential use of the proposed invariants in the process.

2. Invariants

Let $\mathbf{F} = \{F_1, \dots, F_n\}$ be a (uniform) 2-factorization of K_{2n+1} on the vertex-set V . Call the 2-factors of \mathbf{F} *colours*. A set of invariants which proved fairly sensitive, provided the factors F_i involved do not contain too many triangles, is based on counting two-colored cycles of small length. More precisely, an i -cycle ($i \geq 3$) in K_n is said to be of *type* $[a, b]_i$ if it contains one edge of colour a and the remaining $i - 1$ edges of colour b . Let $\alpha[a, b]_i$ be the number of i -cycles of type $[a, b]_i$. For $j = 0, 1, \dots, n$, let $t_{i,j}$ be the number of ordered pairs (a, b) such that $\alpha[a, b]_i = j$, and let $T_i = (t_{i,j} : j = 0, 1, \dots, n)$. The vector T_i is the *bicolour vector* of rank i for \mathbf{F} .

For $a \in \{1, 2, \dots, n\}$, let $\sum_{x \neq a} \alpha[a, x]_i = s_{a,i}$. Let S_i be the sequence of n numbers $s_{a,i}$ arranged in nondecreasing order. The sequence S_i is the *sum-bicolour sequence* of rank i for \mathbf{F} .

Clearly, both the bicolour vector and the sum-bicolour sequence are (easily computable) invariants for 2-factorizations of K_{2n+1} .

For example, consider a Hamilton decomposition of K_9 with the four cycles=colours being (123456789), (135247968), (148295736), (158394627)—in that order. Then we have, e.g., $\alpha[2, 1]_3 = 6$, $\alpha[3, 1]_3 = 2$, $\alpha[4, 1]_3 = 1$. Also, $\alpha[2, 3]_3 = 6$, and since no other $\alpha[a, b]$ equals 6, we get $t_{3,6} = 2$. Calculations yield in this case $T_3 = (0420402000)$, $T_4 = (0216120000)$, $T_5 = (1123320000)$, $S_3 = (4, 6, 10, 16)$, $S_4 = (7, 8, 10, 11)$, $S_5 = (4, 9, 11, 12)$.

The following observations are immediate:

$$\sum_j t_{ij} = n(n-1) \quad (i \text{ fixed}) \quad (1)$$

$$\sum_{\substack{x \\ x \neq b}} \alpha[x, b]_i = 2n+1 \quad (2)$$

$$\sum_{\substack{i, j \\ i \neq j}} s_{ij} = n(2n+1) = \sum_{\substack{a, b \\ a \neq b}} \alpha[a, b] \quad (3)$$

I. For the standard Hamilton decomposition of K_{2n+1} given by $V = Z_{2n} \cup \{\infty\}$, $H = (\infty \ 0 \ 2n-1 \ 1 \ 2n-2 \ 2 \dots n+1 \ n-1 \ n) \pmod{2n}$, the bicolour vector of rank 3 is given by

- (1) if $n = 2$, $t_{3,5} = 2$, and $t_{3,j} = 0$ for all other j ,
- (2) if $n = 3$, $t_{3,2} = t_{3,5} = 3$, and $t_{3,j} = 0$ for all other j , and
- (3) if $n \geq 4$, $t_{3,2} = n(n-3)$, $t_{3,3} = t_{3,4} = n$, and $t_{3,j} = 0$ for other j .

Indeed, while (1) and (2) are easily checked, in order to show (3), it suffices to consider the edges of any colour i , $i \in Z_n$; w.l.o.g, let $i = 0$. In the two triangles with vertices $j, j+1, 2n-1-j$ and $j+n, j+n+1, n-1-j$, respectively, the edges $\{j, 2n-1-j\}$, $\{j+1, 2n-1-j\}$ and $\{j+n, n-1-j\}$, $\{j+n+1, n-1-j\}$ all have colour 0, while the edges $\{j, j+1\}$ and $\{j+n, j+n+1\}$ have colour $j+1$. Thus these two triangles are both of type $[j, 0]_3$. Further, the triangle with vertices $0, n, \infty$ is easily seen to be of type $[\lfloor n/2 \rfloor, 0]_3$, and the two triangles with vertices $\infty, j, j-1$ and $\infty, j+n, j+n-1$ are both of type $[1, 0]_3$. Thus for any fixed i , a triangle of type $[i+1, i]_3$ occurs four times, of type $[i + \lfloor n/2 \rfloor, i]_3$ three times, and of type $[q, i]_3$ for all other $q = i$ twice.

II. Similarly, let $2n+1$ be prime with $n > 3$, and let F be the Hamilton decomposition of K_{2n+1} given by $V = Z_{2n+1}$, $H_i = (0 \ i \ 2i \ 3i \dots 2n) \pmod{2n+1}$. It is easily checked that the bicolour vector T_3 for this decomposition is given by $t_{3,n} = n$, and $t_{3,j} = 0$ for all other j .

In general, however, there is a multitude of Hamilton decompositions of K_{2n+1} . We have tested the effectiveness of the above invariant on the set of 122 nonisomorphic Hamilton decompositions of K_9 , as determined in [C]. The invariant T_3 alone partitions the 122 solutions into 67 classes. As soon as one adjoins to it also T_4 and T_5 , the sensitivity increases considerably. The triple of invariants (T_3, T_4, T_5) fails to distinguish only two pairs of solutions ($x(8)$ and $x(10)$, and $x(71)$ and $x(107)$, respectively—see Table 1), thus its sensitivity is 0.984. Upon adjoining also S_3 , the quadruple (T_3, T_4, T_5, S_3) becomes a complete invariant in this case.

The same invariants—the bicolour vector and the sum-bicolour sequence—can be also used to distinguish m -cycle systems, again provided m is not “too small”. Actually, only the range of parameters and the equalities (1)–(3) above change, and have to be adjusted accordingly. However, already in the case of 6-cycle systems of order 9, the above quadruple of invariants, while still quite sensitive, is no longer a complete invariant. For details on the generation and isomorphism invariants for the 6-cycle systems of K_9 , see Section 3 below.

A yet different simple invariant proved effective for distinguishing the nonisomorphic 4-cycle systems of order 9. For a 4-cycle $f = (abcd)$ and its vertex, say a , let $N(a, f)$, the neighborhood of a in f , be the path P_3 induced by the vertices b, c and d . Given a 4-cycle system $\mathbf{F} = \{F_i : i = 1, \dots, 9\}$ of K_9 , let the neighborhood graph $N(a, \mathbf{F})$ of a in \mathbf{F} be the subgraph of K_9 induced by the union of the neighborhoods $N(a, F_i)$, $i = 1, \dots, 9$. It is a simple exercise to establish that there are exactly 3 nonisomorphic graphs (see Figure 1, graphs N_1, N_2, N_3) which may occur as neighborhood graphs of a vertex in a 4-cycle system of K_9 . If, for a 4-cycle system \mathbf{F} of K_9 , the symbol n_i , $i = 1, 2, 3$, gives the number of vertices having N_i as the neighborhood graph, then $N = (n_1, n_2, n_3)$ is clearly an invariant for 4-cycle systems of K_9 . Somewhat surprisingly, this simple invariant fails to distinguish only a pair of solutions (which, however, happen to be distinguished by the automorphism group order).

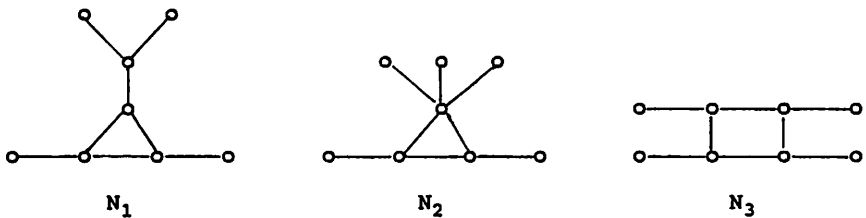


Figure 1.

3. Enumerating the cycle systems of K_9

A. The 6-cycle systems

The 640 nonisomorphic 6-cycle systems of K_9 were obtained in two ways, thus providing an independent check for correctness of the result. The first of these consisted in implementing an *orderly* generation algorithm [C], [Cal], [CRY], [R], in a manner which is a minor adaptation of the method used to generate Hamilton decompositions in [C].

A *form* of a 6-cycle system \mathbf{F} of K_9 is obtained as follows. Let $V(K_9) =$

$\{1, 2, \dots, 9\}$. For an ordered pair ab of vertices, we locate the unique 6-cycle F_i of F of the form $(abcdef)$. We then rename the i th element of F_i by j . The remaining three vertices not in F_i are assigned integers 7, 8, 9 in all 6 possible ways. For each of the resulting representations of F , we shift each 6-cycle so that it commences at the smallest integer, and reverse the order of the 6-cycle if this gives a lexicographically smaller vector representing the cycle. Next, the set of vectors is sorted in lexicographically increasing order, yielding a *form*. Since there are 72 ordered pairs of vertices of K_9 , we get in total $72 \times 6 = 432$ forms of F . The lexicographically smallest among these is the *canonical form* of F . Implementing now an exhaustive algorithm for generating forms of 6-cycle systems, we check each form for canonicity. Only canonical forms are kept; other forms are discarded since their canonical forms were encountered earlier. Altogether 640 canonical forms were obtained.

The second method was based on identifying a reasonably small subset of the set of nonisomorphic 6-cycle systems consisting of solutions that satisfy an additional condition. Subsequently, several types of transformations, which use the invariant discussed in the preceding section, are applied to this subset repeatedly. The subset of solutions in question consisted of those 6-cycle systems (call these special) which contain two 6-cycles on the same set of 6 vertices. It is easily seen that these two cycles can be taken w.l.o.g. as (123456) and (136425). There are exactly 20 nonisomorphic special 6-cycle systems of K_9 , and they are listed in Table 2.

The following facts about the bicolour vector T and the sum-bicolour sequence S (cf. Section 2 above) for 6-cycle systems of K_9 are readily verified:

- (i) $t_{3,4} = 2$ if F is a special 6-cycle system of F_9 , and $t_{3,4} = 0$ otherwise; moreover, $t_{3,5} = t_{3,6}$.
- (ii) $t_{4,1} = t_{4,3} = t_{4,5} = 0$.
- (iii) $t_{4,2} + 2t_{4,4} = 18$, thus $t_{4,2}$ is even; moreover, $0 \leq t_{4,4} \leq 5$ and $8 \leq t_{4,2} \leq 18$.
- (iv) every element of S_4 is even.

The properties (i)–(iv) above suggest using the following reduced set of invariants in this case: $T'_3 = (t_{3,1}, t_{3,2}, t_{3,3})$, $T'_4 = (t_{4,4})$, S_3 and S_4 . This set of invariants partitions the 640 nonisomorphic 6-cycle systems of K_9 into 598 classes; thus its sensitivity is 0.934.

The transformations that were applied initially to the special 6-cycle systems, and then repeatedly to new obtained systems, were of four kinds:

1. Given a 6-cycle system \mathbf{F} and two of its 6-cycles F_i and F_j , assume there exist two vertices a and b at distance 2 both in F_i and in F_j . Say, we have $F_i = (acbdef)$ and $F_j = (agbhij)$. If $F'_i = (agbdef)$, $F'_j = (acbhi j)$ are also 6-cycles (this will happen precisely when g is not one of d, e, f , and c is not one of h, i, j) then the 6-cycle system \mathbf{F}' obtained from \mathbf{F} by replacing the 6-cycles F_i, F_j with F'_i, F'_j is said to be obtained from \mathbf{F} by *simple surgery of the first kind*.
2. Assume that the 6-cycles F_i and F_j of \mathbf{F} contain two vertices a and b at distance 3 both in F_i and F_j , say, $F_i = (acd b e f)$, $F_j = (a g h b i j)$. If $F'_1 = (a g h b e f)$ and $F'_2 = (a c d b i j)$ are also 6-cycles, then the 6-cycle system \mathbf{F}' obtained from \mathbf{F} by replacing F_1, F_2 with F'_1, F'_2 is said to be obtained from \mathbf{F} by *simple surgery of the second kind*.
3. Assume that the 6-cycle F_i of \mathbf{F} contains the paths ab and cde , and the 6-cycle F_j of \mathbf{F} contains the paths afb and ce . If replacing in F_i ab by the path afb and the path cde by ce yields a 6-cycle F'_i , and, simultaneously, replacing in F_j afb by ab and ce by cde yields a 6-cycle F'_j then the 6-cycle system \mathbf{F}' obtained from \mathbf{F} by replacing F_i, F_j with F'_i, F'_j is said to be obtained from \mathbf{F} by *double surgery*.
4. Finally, assume that \mathbf{F} contains three 6-cycles F_i, F_j, F_k such that F_i contains the paths (say) ab and cde , F_j contains the paths ce and fgh , and F_k contains the paths fh and aib . If replacing the above paths in F_i with aib and ce , in F_j with cde and fh , and in F_k with fgh and ab , respectively, results in new 6-cycles F'_i, F'_j, F'_k , then the 6-cycle system \mathbf{F}' obtained from \mathbf{F} by replacing F_i, F_j, F_k with F'_i, F'_j, F'_k is said to be obtained from \mathbf{F} by *triple surgery*.

Just as for Hamilton decompositions, computing isomorphisms or automorphisms of 6-cycle systems is easy (cf. [C]). The summary of automorphism group orders for the 640 nonisomorphic 6-cycle systems of K_9 is as follows.

| | | | | | | |
|-------------------|-----|----|---|----|---|----|
| group order | 1 | 2 | 3 | 4 | 6 | 36 |
| number of systems | 591 | 31 | 4 | 11 | 2 | 1 |

The 640 nonisomorphic 6-cycle systems of K_9 are listed in Table 3. The systems are listed according to the lexicographic order of the 598 distinct vectors of the reduced set of invariants as described above; this numbering is given in the last column of Table 3. The first column of Table 3 contains the number of the system in the lexicographic ordering of the canonical forms. The next five columns contain the last five cycles of the canonical form of the given system (whose first cycle is always 123456); the next column contains the order of its automorphism group. The next four columns list

the reduced set of invariants T'_3, S_3, T'_4, S_4 (with commas and parentheses omitted; here $10 = a, 11 = b, 12 = c$ etc.)

B. The 4-cycle systems

There are only 8 nonisomorphic 4-cycle systems of K_9 . The small number of solutions (both, expected and actual) makes it unnecessary to use orderly generation algorithm. The latter was nevertheless used as an independent check to a conventional generation method; a post-hoc observation suggests that this was a case where hand calculations alone would have succeeded. The vector $N = (n_1, n_2, n_3)$ described in Section 2 above together with the automorphism group order provide a complete invariant. Table 4 lists the 8 nonisomorphic 4-cycle systems of K_9 in lexicographic order of their canonical form, together with the automorphism group order, and the invariant N .

4. Conclusion

Table A below summarizes the information on the number of nonisomorphic m -cycle systems of K_n for $n \leq 9$.

It is hoped that the invariants described in Section 2, and their natural extensions, will prove useful in enumerating and distinguishing cycle systems and uniform 2-factorizations of K_n for $n > 9$, as well as cycle systems of other graphs, such as, for instance, of $K_n - F$, the cocktail-party graph on an even number of vertices.

| n | m | number of nonisomorphic systems | reference |
|-----|-----|---------------------------------|------------|
| 3 | 3 | 1 | trivial |
| 5 | 5 | 1 | trivial |
| 7 | 3 | 1 | PG(2,2) |
| | 7 | 2 | [C] |
| 9 | 3 | 1 | AG(2,3) |
| | 4 | 8 | this paper |
| | 6 | 640 | this paper |
| | 9 | 122 | [C] |

Table A.

References

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TABLE 1

| | | | | | |
|------------|-----------|------------|-----------|------------|-----------|
| x(2) | | x(42) | | x(87) | |
| 4111211-- | 135247968 | 1113111-- | 135274968 | 22231-1-- | 135284796 |
| 12332---- | 146285937 | 132311---- | 148592637 | 1234-1---- | 146295738 |
| 3123-11-- | 157294836 | -3221-1-- | 157938246 | 23411-1-- | 158639427 |
| x(3) | | x(45) | | x(90) | |
| 1223-11-- | 135247968 | 233211---- | 135279468 | 23122-1-- | 135286497 |
| 213311---- | 146293857 | 12621---- | 142857396 | 13431---- | 142736958 |
| 1324--1-- | 159482736 | 15132---- | 159263847 | 21242---- | 157483926 |
| x(6) | | x(46) | | x(93) | |
| -12211--1 | 135247968 | 3311-12-- | 135279468 | 23232---- | 135286497 |
| -22122---- | 146392857 | 151-22---- | 142859637 | 241311---- | 147583926 |
| 1123-1--1 | 159483726 | -6112-1-- | 157483926 | 133121---- | 159637248 |
| x(7) | | x(47) | | x(97) | |
| 411411---- | 135247968 | 213311---- | 135279468 | -43-1-2-- | 135286947 |
| -343-1---- | 146395827 | 214121---- | 142963857 | -6-14---- | 146297385 |
| 13321-1-- | 157384926 | 222221---- | 159374826 | 14131-1-- | 163957248 |
| x(11) | | x(49) | | x(100) | |
| 13-231--- | 135247968 | 2223-2--- | 135279468 | 24-42---- | 135286974 |
| -23311---- | 148592637 | 2145----- | 147382695 | -42-22---- | 157293648 |
| 1312111-- | 157283946 | 15213---- | 163924857 | 2-24-2--- | 162495837 |
| x(12) | | x(60) | | x(101) | |
| 112411---- | 135247968 | 31141-1-- | 135279648 | 22-222---- | 135286974 |
| 323121---- | 148593627 | 22214---- | 142936857 | 22-222---- | 158493627 |
| 141221---- | 157382946 | 12141-1-- | 159473826 | 22-222---- | 164295738 |
| x(14) | | x(63) | | x(102) | |
| -242-2---- | 135247968 | -4521---- | 135279648 | 14131-1-- | 135297468 |
| 152211---- | 149583627 | 1335----- | 147392685 | 231212---- | 142758396 |
| 12251---- | 157392846 | 1335----- | 163824957 | -23311---- | 159482637 |
| x(16) | | x(67) | | x(103) | |
| 412221---- | 135248697 | 2321111-- | 135279648 | 13321-1-- | 135297468 |
| 2145----- | 146275938 | 233211---- | 149268375 | 242121---- | 148593627 |
| 31251---- | 158294736 | -6-14---- | 163958247 | -6-14---- | 157382496 |
| x(17) | | x(69) | | x(104) | |
| -4-24---- | 135248697 | 22422---- | 135279648 | -4122-1-- | 135297468 |
| -35111---- | 146293758 | -52121---- | 149573826 | -7113---- | 149582637 |
| 242121---- | 159472836 | 2324-1---- | 158639247 | -1143---- | 157248396 |
| x(19) | | x(75) | | x(105) | |
| 133121---- | 135248697 | -3422---- | 135284697 | 3224--1-- | 135297468 |
| 234-21---- | 146372958 | 12332---- | 142957368 | 233211---- | 149628375 |
| 233211---- | 157493826 | 233211---- | 158394726 | 13223---- | 163958427 |
| x(20) | | x(78) | | x(106) | |
| 3-233---- | 135248697 | 21242---- | 135284697 | 113221---- | 135297486 |
| 11522---- | 146375928 | -262-1---- | 147385926 | 13431---- | 149583627 |
| 322311---- | 158394726 | 225111---- | 157249368 | -5412---- | 157396428 |
| x(21) | | x(80) | | x(108) | |
| 12222-1-- | 135248697 | 11233---- | 135284697 | -4221-1-- | 135729468 |
| 116111---- | 146392758 | 132311---- | 147583926 | -513-1---- | 142583597 |
| 33212-1-- | 159473826 | 233211---- | 159427368 | 23221-1-- | 159374826 |
| x(41) | | x(83) | | x(109) | |
| -4232---- | 135274968 | -3312-1-- | 135284697 | 132311---- | 135729648 |
| -43211---- | 146392857 | -2531---- | 149583726 | 12332---- | 147938526 |
| -433--1-- | 159738426 | 2252--1-- | 157429368 | 23411-1-- | 159428637 |

| | | |
|--|--|--|
| x(110) 13-121---1 135729648 13-231--- 149738625 -4-13---1 163958247 x(111) -543----- 135824796 2235----- 149258375 5-421----- 172593648 | x(118) 3-36----- 135842796 22422----- 146375928 22422----- 152683947 x(119) -363----- 135842796 -63-3----- 147593628 -363----- 152946837 | x(121) 2226----- 135842796 13512----- 149528637 1254----- 157462938 |
| x(1) 23-321--- 135247968 -23311--- 146283957 1332-2--- 158492736 x(25) 23-321--- 135248697 21612--- 149275836 22341--- 159374628 | x(15) -5123----- 135248697 217-11--- 146273958 2226----- 157492836 x(26) -5123----- 135268497 -5331----- 142759638 -363----- 158293746 | x(22) 32223----- 135248697 2226----- 146395728 311321--- 158374926 x(24) 32223----- 135248697 133121--- 147582936 3224--1-- 159462738 |
| x(18) 32113-1-- 135248697 13431---- 146295738 3-152---- 158274936 x(23) 32113-1-- 135248697 15213---- 147395826 213311--- 157294638 | x(28) 3215-1--- 135268497 2324-1--- 142963758 13431---- 159382746 x(51) 3215-1--- 135279468 234-21--- 148573926 14322---- 159638247 | x(43) 332112--- 135279468 21531---- 142693857 14322---- 159284736 x(44) 332112--- 135279468 31332---- 142695837 15132---- 157482936 |
| x(59) 31251---- 135279648 323-4---- 142863957 121321--- 158374926 x(113) 31251---- 135827946 151-22--- 148396257 1341-11-- 159247368 | x(61) 15213---- 135279648 -282----- 147368295 13431---- 162493857 x(91) 15213---- 135286497 12332---- 142758396 213311--- 159263748 | x(65) 14322---- 135279648 1433-1--- 147583926 13431---- 159428637 x(85) 14322---- 135284796 12332---- 142957368 24221-1-- 158394627 |
| x(73) 12621---- 135279684 -6222---- 158293746 1335----- 175942638 x(82) 12621---- 135284697 1335----- 149386275 124211--- 163742958 | x(71) (compare x(107)) -444----- 135279684 2422-2--- 157493628 2226----- 164295837 x(107) (compare x(71)) -444----- 135726948 2422-2--- 142597386 2226----- 158293647 | |

| | | |
|--|--|---|
| x(5) 322311--- 135247968 2226----- 146385927 213311--- 157394826 x(9) 322311--- 135247968 23232---- 148395726 124211--- 158294637 x(13) 322311--- 135247968 21242---- 149285736 124211--- 159384627 | x(4) 42-4-2--- 135247968 1254----- 146295837 12332---- 157284936 x(8) (compare x(10)) 42-4-2--- 135247968 21612---- 148295736 12332---- 158394627 x(10) (compare x(8)) 42-4-2--- 135247968 21612---- 148573926 12332---- 159463827 | x(32) 2324-1--- 135269748 21612---- 142863957 152211--- 158372946 x(48) 2324-1--- 135279468 1352-1--- 147369285 13512---- 162483957 x(50) 2324-1--- 135279468 324-12--- 147582936 25-221--- 159624837 |
| x(35) 14241---- 135269748 1173----- 149273685 15331---- 164283957 x(53) 14241---- 135279486 1332-2--- 142695738 3215-1--- 158293647 x(64) 14241---- 135279648 2341-2--- 147582936 2324-1--- 159426837 | x(52) -6222---- 135279486 1335----- 142693758 1335----- 159283647 x(62) -6222---- 135279648 15213---- 147386295 13-231--- 163942857 x(95) -6222---- 135286497 -5331---- 148572936 13431---- 159624738 | |
| x(54)₁ 1254----- 135279486 31251---- 142963758 23151---- 159382647 x(56) 1254----- 135279486 22422---- 146293758 14322---- 159638247 x(79) 1254----- 135284697 13431---- 147386295 116111--- 163942758 x(98) 1254----- 135286947 31332---- 146372958 22422---- 157938426 | x(68) 13431---- 135279648 241311--- 149362857 241311--- 159247386 x(89) 13431---- 135284796 2226----- 149362758 2226----- 159246837 x(94) 13431---- 135286497 2145----- 148396275 323121--- 163742958 x(99) 13431---- 135286947 231131--- 146392758 25112-1-- 159738426 | x(76) -282----- 135284697 -3133---- 147293685 3313-2--- 162495738 x(86) -282----- 135284796 3117----- 142958637 242121--- 157264938 x(96) -282----- 135286947 161121--- 146275938 12251---- 158429736 x(120) -282----- 135842796 1352-1--- 149268375 13431---- 174639528 |

| | | | | | |
|-----------|-----------|-----------|-----------|------------|-----------|
| x(58) | | x(29) | | x(34) | |
| 22341---- | 135279648 | -5331---- | 135268497 | 1335----- | 135269748 |
| 1254----- | 142683957 | -363----- | 146372958 | 3-522---- | 146829375 |
| 1335----- | 158294736 | 1173----- | 157428396 | 24123---- | 163859427 |
| x(74) | | x(31) | | x(39) | |
| 22341---- | 135284697 | -5331---- | 135269748 | 1335----- | 135274968 |
| 1335----- | 142759368 | 1335----- | 142759386 | 1254----- | 146283795 |
| 222221--- | 158374926 | 132311--- | 158294637 | 15213---- | 163924857 |
| x(81) | | x(33) | | x(70) | |
| 22341---- | 135284697 | -5331---- | 135269748 | 1335----- | 135279684 |
| 2145----- | 149362758 | -343-1--- | 146385927 | 21612---- | 157382946 |
| 322311--- | 159247386 | -352--1-- | 157394286 | 241311--- | 174263958 |
| x(88) | | x(57) | | x(77) | |
| 22341---- | 135284796 | -5331---- | 135279486 | 1335----- | 135284697 |
| 3-36----- | 146385927 | 22341---- | 146293857 | 2226----- | 147362958 |
| 31332---- | 157394268 | 213311--- | 159637428 | 323121--- | 157249386 |
| x(92) | | x(66) | | x(112) | |
| 22341---- | 135286497 | -5331---- | 135279648 | 1335----- | 135824796 |
| 22422---- | 147263958 | -4521---- | 147592836 | 22341---- | 149526837 |
| 323121--- | 157384296 | 1335----- | 158624937 | 22214---- | 157293648 |
| | | x(84) | | x(115) | |
| | | -5331---- | 135284796 | 1335----- | 135842697 |
| | | 12332---- | 142936857 | 2341-2---- | 146375928 |
| | | 1223-11-- | 159462738 | 22422---- | 152749386 |
| | | | | x(122) | |
| | | | | 1335----- | 135847926 |
| | | | | 3411-3---- | 146938257 |
| | | | | 312212--- | 159427368 |

| | | | | | |
|-----------|-----------|-----------|-----------|------------|-----------|
| x(27) | | x(38) | | x(114) | |
| -444----- | 135268497 | -444----- | 135274968 | -444----- | 135829647 |
| 152211--- | 142857396 | 22422---- | 142857936 | -8--4----- | 142597386 |
| 22341---- | 159274638 | 2243-1--- | 159264837 | ---4----- | 157263948 |
| x(30) | | x(40) | | x(116) | |
| -444----- | 135268497 | -444----- | 135274968 | -444----- | 135842697 |
| 1335----- | 146395728 | -3133---- | 146283957 | 1433-1--- | 149375286 |
| 12621---- | 158374296 | -13212--- | 158429736 | 13431---- | 159274638 |
| x(36) | | x(55) | | x(117) | |
| -444----- | 135269748 | -444----- | 135279486 | -444----- | 135842796 |
| -444----- | 149285736 | 323121--- | 142963857 | -363----- | 146283957 |
| 1335----- | 159386427 | 13142---- | 159374628 | -282----- | 152947368 |
| x(37) | | x(72) | | | |
| -444----- | 135274968 | -444----- | 135279684 | | |
| -444----- | 142638597 | --4--4--- | 158293647 | | |
| -444----- | 157392846 | 44--4--- | 162495738 | | |

