# Indecomposable triple systems with $\lambda = 5$

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**Abstract.** An NB$[k, \lambda; v]$ is a B$[b, \lambda; v]$ which has no repeated blocks. In this paper we prove that there exists an indecomposable NB$[3, 5; v]$ for $v \geq 7$ and $v \equiv 1$ or $3 \pmod 6$, with the exception of $v = 7$ and 9, and the possible exception of $v = 13, 15$.

## 1. Introduction

Let $v$, $k$ and $\lambda$ be positive integers. Let $X$ be a $v$-set (of points). **A** be a collection of some subsets of $X$ (called blocks) with size $k$. A B$[k, \lambda; v]$ is a pair $(X, \mathbf{A})$ such that every unordered pair of points of $X$ appears in exactly $\lambda$ blocks of **A**. If the design has no repeated blocks, then it will be referred to as an NB$[k, \lambda; v]$. An NB$[k, \lambda; v]$ will be called indecomposable (or irreducible) if there does not exist a subset $\mathbf{A}'$, $\mathbf{A}' \subset \mathbf{A}$, such that $\mathbf{A}'$is an NB$[k, \lambda'; v]$ for $1 \leq \lambda' < \lambda$. If $(Y, \mathbf{B})$ is a B$[k, \lambda; n]$ and $Y \subset X$, $\mathbf{B} \subset \mathbf{A}$ then we say the first design contains the second design as a subdesign.

It is known well that the necessary conditions for the existence of an NB$[3, \lambda; v]$ are

$$\lambda v(v-1) \equiv 0 \pmod 6, \quad \lambda(v-1) \equiv 0 \pmod 2 \quad \text{and} \quad \lambda \leq v-2.$$

Therefore the necessary conditions for the existence of an NB$[3, \lambda; v]$ containing an NB$[3, \lambda; n]$ as a subdesign are

$$\begin{array}{ll} \lambda v(v-1) \equiv 0 \pmod 6, & \lambda(v-1) \equiv 0 \pmod 2 \\ \lambda n(n-1) \equiv 0 \pmod 6, & \lambda(n-1) \equiv 0 \pmod 2 \\ v \geq 2n+1, & n \geq \lambda + 2 \end{array}.$$

Hanani [5] has obtained the following result:

**Theorem 1.** *There exists a $B[3, \lambda; v]$ if and only if $\lambda v(v-1) \equiv 0 \pmod 6$ and $\lambda(v-1) \equiv 0 \pmod 2$.*

Dehon [3] has proved that

**Theorem 2.** *There exists an $NB[3, \lambda; v]$ if and only if $\lambda v(v-1) \equiv 0 \pmod 6$, $\lambda(v-1) \equiv 0 \pmod 2$ and $\lambda \leq v-2$.*

H. Shen [3] has proved that

**Theorem 3.** *The necessary conditions for the existence of an* $NB[3,\lambda;v]$ *containing an* $NB[3,\lambda;v]$ *as a subdesign are also sufficient.*

Our knowledge of the existence of indecomposable designs about $k = 3$ can be summerised as follows.

**Theorem 4.** *There exists an indecomposable* $NB[3,2;v]$ *if and only if* $v \equiv 0,1 \ (mod 3)$, $v > 3$ *and* $v \neq 7$; *There exists an indecomposable* $NB[3,3;v]$ *if and only if* $v \equiv 1 \ (mod 2)$ *and* $v > 3$ *(see [1,6]); There exists an indecomposable* $NB[3,4;v]$ *if and only if* $v \equiv 0,1 \ (mod 3)$, $v \geq 10$ *(see [2]); There exists an indecomposable* $NB[3,6;v]$ *for* $v \geq 8$ *with the exception of* $v = 9$ *and the possible exception of* $v = 10,11,12,13,15,16$ *(see [4,7]).*

Dan Archdeacon and Jeff Dinitz have proved that

**Theorem 5.** *The necessary conditions for the existence of an indecomposable* $NB[3,\lambda;v]$ *are also sufficient for all sufficiently large* $v$. *(see [10])*

In this paper we shall prove the following result:

There exists an indecomposable NB$[3,5;v]$ for $v \equiv 1$ or $3 \ (\mathrm{mod}\ 6)$ and $v \geq 7$ with the exception of $v = 7$ and 9, and the possible exception of $v = 13, 15$.

It is easy to see that the necessary conditions for the existence of an NB$[3,5;v]$ containing an NB$[3,5;n]$ as a subdesign are

$$v \equiv 1,3 \ (\mathrm{mod}\ 6), \quad n \equiv 1,3 \ (\mathrm{mod}\ 6), \quad n \geq 7, \quad v \geq 2n+1$$

Let $m = v - n$, it is easy to see that $m$ is even. Let

$$E_i = \{\{g, g+i\} : g \in Z_m\} \quad for 1 \leq i \leq m/2 - 1$$
$$E_{m/2} = \{\{g, g+m/2\} : \quad g = 1,2,\ldots,m/2\}$$
$$E_i(j) = \{\{g, g+i\} : \quad g \equiv j \ (\mathrm{mod}\ 2), \quad g \in Z_m\} \quad \text{for } j = 0,1$$

Let $\mathbf{A}$ be a collection of blocks without repeated blocks, we define

$$d(x, \mathbf{A}) = |\{A : x \in A \in \mathbf{A}\}|, \quad \text{and}$$

$\lambda\mathbf{A}$ be a collection of blocks in which each $A \in \mathbf{A}$ appears $\lambda$ times.

## 2. Indecomposable NB$[3,5;v]$ (or INB$[3,5;v]$)

It is known that of the 13 non-isomorphic NB$[3,5;9]$ designs, all are decomposable by W. Harnau (see [9]), and it is clear that there does not exist an NB$[3,5;7]$.

In this section, we first construct an INB$[3,5;v]$ for $v = 19,21,25,27,31,33,37$.

Let $Y = \{\alpha_i : 1 \leq i \leq n\}$, $X = \{x_i : 1 \leq i \leq m\}$, $s = x_1 + x_2 + \cdots + x_m$, $(Y, \mathbf{B})$ be an NB$[3,\lambda;n]$ and $(X \cup Y, \mathbf{A} \cup \mathbf{B} \cup \mathbf{N})$ be an NB$[3,\lambda;v]$ where $v = m + n$ and $\mathbf{A}$ is a collection of blocks of $X$, and $N \cap Y \neq \emptyset$ for all $N \in \mathbf{N}$.

**Theorem 6.** *If* $a+b+c\equiv 0\ (mod\,\lambda)$ *for all* $A=\{a,b,c\}\in \mathbf{A}$ *and* $(\lambda,(m-n-1)s/2)=1$. *Then* $(X\cup Y,\mathbf{A}\cup\mathbf{B}\cup\mathbf{N})$ *is an INB*$[3,\lambda;v]$.

**Proof.** It is clear that for $x\in X, d(x,\mathbf{A}\cup\mathbf{B}\cup\mathbf{N})=\lambda(v-1)/2$ and $d(x,\mathbf{N})=\lambda n$, so $d(x,\mathbf{A})=\lambda(m-n-1)/2$. Suppose $(X\cup Y,\mathbf{A}'\cup\mathbf{B}'\cup\mathbf{N}')$ is an NB$[3,\lambda';v]$ where $\mathbf{A}'\subset A, \mathbf{B}'\subset B, \mathbf{N}'\subset N$, and $1\le\lambda'<\lambda$ then

$$d(x,\mathbf{A}')=\lambda'(m-n-1)/2.$$

Since $\mathbf{A}'\subset\mathbf{A}$ and $a+b+c\equiv 0\ (\mathrm{mod}\,\lambda)$ for all $A=\{a,b,c\}\in\mathbf{A}$ so

$$\sum_{\{a,b,c\}\in\mathbf{A}'}(a+b+c)=\lambda'(m-n-1)s/2\equiv 0\ (\mathrm{mod}\,\lambda).$$

But it is impossible that $\lambda'(m-n-1)s/2\equiv 0\ (\mathrm{mod}\,\lambda)$ since $1\le\lambda'<\lambda$ and $(\lambda,(m-n-1)s/2)=1$. Therefore we have completed the proof.

**Theorem 7.** *There exists an INB*$[3,5;v]$ *for* $v=19,21,25,27,31,33,37$.

**Proof.** Let $Y=\{\alpha_i:1\le i\le n\}, X=Z_m\quad(m=v-n)$. From Theorem 2 we can let $(Y,\mathbf{B})$ be an NB$[3,5;v]$ for $n=7,9,13,15$ we then present a collection of blocks $\mathbf{A}$ of $X=Z_m$ for $m=12,16,18,22$ and $M_i$ for $1\le i\le n$. It is clear that $(X\cup Y,\mathbf{A}\cup\mathbf{B}\cup\mathbf{N})$ is an INB$[3,5;v]$ for $v=19,21,25,27,31,33,37$ by applying Theorem 6, where

$$\mathbf{N}=\bigcup_{1\le i\le n}\mathbf{N}_i,\quad \mathbf{N}_i=\{\{\alpha_i,a,b\}:\{a,b\}\in M_i\}\qquad \text{for } 1\le i\le n.$$

For $v=19, n=7, m=12$, let

| | | | | |
|---|---|---|---|---|
| **A** = { | 0 1 4 | 1 2 7 | 2 4 9 | 4 5 11 |
| | 0 1 9 | 1 3 6 | 2 5 8 | 4 6 10 |
| | 0 2 3 | 1 3 11 | 2 6 7 | 4 7 9 |
| | 0 2 8 | 1 4 5 | 2 7 11 | 4 10 11 |
| | 0 3 7 | 1 4 10 | 2 8 10 | 5 6 9 |
| | 0 4 6 | 1 6 8 | 3 5 7 | 5 7 8 |
| | 0 5 10 | 1 8 11 | 3 6 11 | 5 9 11 |
| | 0 6 9 | 1 9 10 | 3 7 10 | 6 8 11 |
| | 0 7 8 | 2 3 5 | 3 8 9 | 7 8 10 |
| | 0 9 11 | 2 3 10 | 4 5 6 | 9 10 11 } |

$M_1 = \{$ 0 3   1 2   2 4   3 8   5 6   6 10
0 4   1 5   2 6   3 9   5 8   7 9
0 5   1 7   2 9   3 10   5 10   7 11
0 10   1 10   2 11   4 7   6 7   8 9
0 11   1 11   3 4   4 8   6 9   8 11 $\}$

$M_2 = \{$ 0 1   1 2   2 9   3 11   5 7   6 10
0 2   1 3   2 11   4 7   5 10   7 9
0 5   1 6   3 4   4 8   5 11   7 10
0 10   1 8   3 6   4 9   6 7   8 9
0 11   2 5   3 9   4 11   6 8   8 10 $\}$

$M_3 = \{$ 0 6   1 5   2 4   3 4   4 8   7 11
0 7   1 7   2 6   3 5   5 9   9 10
0 8   1 9   2 10   3 8   6 11   10 11 $\} \cup E_3$

$M_4 = E_2 \cup E_1 \cup E_6$   $M_5 = E_2 \cup E_1 \cup E_6$
$M_6 = E_4 \cup E_5 \cup E_3(0)$   $M_7 = E_4 \cup E_5 \cup E_3(1)$

For $v = 21, n = 9, m = 12$, let

A $= \{$ 1 2 12   2 3 5   3 7 10   4 9 12
1 3 6   2 6 7   3 8 9   5 7 8
1 4 5   2 6 12   4 5 11   6 8 11
1 8 11   2 8 10   4 6 10   7 11 12
1 9 10   3 5 12   4 7 9   9 10 11 $\}$

$M_1 = \{$ 1 2   2 4   3 7   4 12   5 12   8 10
1 3   2 7   3 8   5 6   6 9   8 12
1 6   2 9   3 11   5 7   6 11   9 12
1 7   2 11   4 6   5 9   7 10   10 11
1 11   3 4   4 8   5 10   8 9   10 12 $\}$

$M_2 = \{$ 1 7   1 10   2 8   2 11   3 9   3 12   4 7   4 10
5 8   5 11   6 9   6 12 $\} \cup E_2 \cup E_5(0)$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $M_3 = \{$ | 1 4 | 1 5 | 1 8 | 1 9 | 1 12 | 2 3 | 2 4 | 2 5 | 2 9 |
| | 2 10 | 3 4 | 3 6 | 3 10 | 3 11 | 4 8 | 4 11 | 5 6 | 5 9 |
| | 5 10 | 6 7 | 6 8 | 6 10 | 7 8 | 7 9 | 7 11 | 7 12 | 8 12 |
| | 9 11 | 10 12 | 11 12 $\}$ | | | | | | |

$M_4 = E_2 \cup E_1 \cup E_5(0)$ $\quad M_5 = E_2 \cup E_1 \cup E_5(1)$ $\quad M_6 = E_4 \cup E_3 \cup E_5(1)$

$M_7 = E_4 \cup E_3 \cup E_6$ $\quad M_8 = E_4 \cup E_3 \cup E_6$ $\quad M_9 = E_1 \cup E_5 \cup E_6$

For $v = 25, n = 9, m = 16$, let

| | | | | | | |
|---|---|---|---|---|---|---|
| $\mathbf{A}' = \{$ | 1 2 12 | 2 3 15 | 3 10 12 | 5 6 9 | 6 14 15 | 9 15 16 |
| | 1 3 16 | 2 6 12 | 3 13 14 | 5 7 8 | 7 11 12 | 5 10 15 |
| | 1 4 15 | 2 7 11 | 4 5 6 | 5 9 16 | 7 12 16 | |
| | 1 6 13 | 2 8 10 | 4 8 13 | 5 11 14 | 7 13 15 | |
| | 1 8 11 | 3 4 13 | 4 10 11 | 5 14 16 | 8 9 13 | |
| | 1 9 10 | 3 8 14 | 4 10 16 | 6 10 14 | 9 11 15 $\}$ | |

$\mathbf{A} = G(16) \setminus \mathbf{A}'$ where

$G(16) = \{\{a,b,c\} : a + b + c \equiv 0 \pmod 5,\ a,b,c \in \{1,2,\ldots,16\}\}$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $M_1 = \{$ | 1 2 | 1 9 | 1 11 | 1 12 | 1 16 | 2 6 | 2 10 | 2 11 |
| | 2 15 | 3 4 | 3 8 | 3 10 | 3 14 | 3 15 | 4 6 | 4 8 |
| | 4 12 | 4 13 | 5 6 | 5 8 | 5 10 | 5 15 | 5 16 | 6 7 |
| | 6 9 | 7 8 | 7 11 | 7 13 | 7 16 | 8 9 | 9 12 | 9 13 |
| | 10 15 | 10 16 | 11 12 | 11 14 | 12 14 | 13 14 | 13 15 | 14 16 $\}$ |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $M_2 = \{$ | 1 2 | 1 6 | 1 8 | 1 10 | 1 13 | 2 4 | 2 9 | 2 11 |
| | 2 14 | 3 4 | 3 12 | 3 13 | 3 14 | 3 16 | 4 10 | 4 13 |
| | 4 16 | 5 6 | 5 7 | 5 10 | 5 11 | 5 15 | 6 12 | 6 14 |
| | 6 15 | 7 9 | 7 11 | 7 15 | 7 16 | 8 9 | 8 11 | 8 13 |
| | 8 14 | 9 15 | 9 16 | 10 12 | 10 15 | 11 12 | 12 16 | 13 14 $\}$ |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $M_3 = \{$ | 1 3 | 1 4 | 1 7 | 1 12 | 1 15 | 2 3 | 2 6 | 2 8 |
| | 2 12 | 2 16 | 3 9 | 3 13 | 3 14 | 4 7 | 4 8 | 4 11 |
| | 4 13 | 5 9 | 5 10 | 5 14 | 5 15 | 5 16 | 6 10 | 6 12 |
| | 6 13 | 6 14 | 7 11 | 7 12 | 7 14 | 8 10 | 8 13 | 8 14 |
| | 9 11 | 9 13 | 9 16 | 10 11 | 10 15 | 11 15 | 12 16 | 15 16 $\}$ |

$M_4 = E_2 \cup E_1 \cup E_7(0)$ $\quad M_5 = E_2 \cup E_1 \cup E_7(0)$ $\quad M_6 = E_4 \cup E_3 \cup E_7(1)$
$M_7 = E_4 \cup E_3 \cup E_7(1)$ $\quad M_8 = E_6 \cup E_5 \cup E_8$ $\quad M_9 = E_6 \cup E_5 \cup E_8$

For $v = 27, n = 9, m = 18$, let

| | | | | | |
|---|---|---|---|---|---|
| A′ = { 1 2 17 | 2 3 5 | 3 7 10 | 5 7 18 | 6 11 13 | 9 14 17 |
| 1 3 16 | 2 3 10 | 3 12 15 | 5 8 17 | 6 13 16 | 10 12 18 |
| 1 4 5 | 2 7 11 | 3 13 14 | 5 9 16 | 7 8 15 | 10 13 17 |
| 1 6 18 | 2 8 10 | 4 5 6 | 5 11 14 | 7 16 17 | 10 14 16 |
| 1 8 11 | 2 12 16 | 4 6 10 | 5 12 13 | 8 12 15 | 11 12 17 |
| 1 9 10 | 2 13 15 | 4 7 9 | 6 7 12 | 8 14 18 | 11 16 18 |
| 1 14 15 | 3 4 8 | 4 11 15 | 6 9 15 | 9 13 18 | 15 17 18 |
| 5 10 15 } | | | | | |

$\mathbf{A} = G(18) \setminus \mathbf{A}'$ where
$G(18) = \{\{a, b, c\} : a + b + c \equiv 0 \pmod 5,\ a, b, c \in \{1, 2, \ldots, 18\}\}$

$M_1 = \{$ 1 5   1 9   1 11   2 3   2 10   2 12   3 11   3 13   4 6   4 7
4 12   5 10   5 15   6 14   6 16   7 15   7 17   8 16   8 17   8 18
9 17   9 18   10 15   11 14   12 13   13 16   14 18 $\} \cup E_7$

$M_2 = \{$ 1 2   1 3   1 8   1 9   1 18   2 4   2 10   2 13   2 16
3 7   3 10   3 12   3 14   4 7   4 8   4 12   4 17   5 6
5 10   5 13   5 15   5 16   6 7   6 9   6 12   6 18   7 9
7 18   8 11   8 12   8 15   9 13   9 17   10 15   10 16   11 13
11 15   11 17   11 18   12 14   13 14   14 16   14 17   15 16   17 18 $\}$

$M_3 = \{$ 1 4   1 10   1 14   1 15   1 17   2 3   2 8   2 9   2 11
2 14   3 4   3 8   3 13   3 16   4 5   4 6   4 11   5 6
5 10   5 11   5 15   6 10   6 13   6 15   7 9   7 8   7 11
7 14   7 16   8 14   8 18   9 12   9 16   9 17   10 15   10 18
11 12   12 15   12 16   12 18   13 17   13 18   14 17   16 17   16 18 $\}$

$M_4 = (E_1 \setminus \{5\ 6, 15\ 16\}) \cup \{5\ 15, 6\ 16\} \cup E_5 \cup E_7(0)$
$M_5 = E_2 \cup E_6 \cup E_5(0)$ $\quad M_6 = E_2 \cup E_8 \cup E_5(1)$ $\quad M_7 = E_4 \cup E_1 \cup E_7(1)$
$M_8 = E_4 \cup E_3 \cup E_9$ $\quad M_9 = E_6 \cup E_3 \cup E_9$

For $v = 31, n = 13, m = 18$, let

$A = \{$

| | | | | | |
|---|---|---|---|---|---|
| 1 2 7 | 2 3 5 | 3 5 17 | 4 11 15 | 6 8 16 | 8 10 17 |
| 1 3 6 | 2 4 9 | 3 7 10 | 4 12 14 | 6 9 15 | 8 13 14 |
| 1 4 5 | 2 5 18 | 3 9 13 | 4 13 18 | 6 11 13 | 9 10 16 |
| 1 6 18 | 2 6 12 | 3 10 12 | 4 14 17 | 6 11 18 | 10 11 14 |
| 1 8 11 | 2 8 10 | 3 11 16 | 5 6 14 | 7 8 15 | 10 12 13 |
| 1 9 10 | 2 9 14 | 3 12 15 | 5 7 13 | 7 9 14 | 11 13 16 |
| 1 12 17 | 2 11 12 | 3 14 18 | 5 8 12 | 7 10 18 | 12 15 18 |
| 1 13 16 | 2 13 15 | 4 5 16 | 5 8 17 | 7 11 17 | 13 15 17 |
| 1 14 15 | 2 16 17 | 4 6 10 | 5 9 11 | 7 12 16 | 14 15 16 |
| 1 16 18 | 3 4 8 | 4 7 9 | 6 7 17 | 8 9 18 | 15 17 18 $\}$ |

$M_1 = \{$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 2 | 2 15 | 3 16 | 4 8 | 5 10 | 7 11 | 9 13 | 11 14 | 13 17 |
| 1 3 | 2 16 | 3 17 | 4 10 | 6 12 | 8 9 | 9 17 | 11 15 | 13 18 |
| 1 4 | 2 17 | 3 18 | 5 6 | 6 13 | 8 11 | 10 11 | 12 13 | 14 16 |
| 1 5 | 2 18 | 4 6 | 5 7 | 6 14 | 8 15 | 10 15 | 12 14 | 14 18 |
| 1 7 | 3 15 | 4 7 | 5 9 | 7 8 | 9 12 | 10 17 | 12 16 | 16 18 $\}$ |

$M_2 = \{$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 8 | 2 3 | 3 4 | 4 12 | 5 16 | 6 10 | 8 14 | 10 18 | 13 14 |
| 1 9 | 2 4 | 3 11 | 4 13 | 5 18 | 7 15 | 9 12 | 11 12 | 14 17 |
| 1 10 | 2 6 | 3 13 | 5 10 | 6 7 | 7 16 | 9 15 | 11 17 | 15 16 |
| 1 11 | 2 7 | 3 14 | 5 14 | 6 8 | 7 18 | 9 16 | 12 17 | 16 17 |
| 1 13 | 2 8 | 4 11 | 5 15 | 6 9 | 8 13 | 10 15 | 12 18 | 17 18 $\}$ |

$M_3 = \{$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 12 | 2 10 | 3 6 | 4 15 | 5 11 | 6 15 | 7 13 | 8 18 | 10 13 |
| 1 14 | 2 11 | 3 7 | 4 16 | 5 12 | 6 16 | 7 14 | 9 11 | 10 14 |
| 1 15 | 2 13 | 3 8 | 4 17 | 5 13 | 6 17 | 8 12 | 9 17 | 10 16 |
| 1 17 | 2 14 | 3 9 | 4 18 | 5 15 | 7 12 | 8 16 | 9 18 | 11 18 $\} \cup E_1(0)$ |

$M_4 = E_2 \cup E_8 \cup E_7(0)$ $\quad M_5 = E_2 \cup E_8 \cup E_7(0)$ $\quad M_6 = E_2 \cup E_1 \cup E_7(1)$

$M_7 = E_4 \cup E_3 \cup E_7(1)$ $\quad M_8 = E_6 \cup E_5 \cup E_1(0)$ $\quad M_9 = E_6 \cup E_5 \cup E_1(1)$

$M_{10} = E_8 \cup E_7 \cup E_1(1)$ $\quad M_{11} = E_4 \cup E_3 \cup E_9$ $\quad M_{12} = E_4 \cup E_3 \cup E_9$

$M_{13} = E_6 \cup E_5 \cup E_9$

For $v = 33, n = 15, m = 18$, let

| | | | | | | |
|---|---|---|---|---|---|---|
| A = { | 1 2 7 | 2 3 10 | 3 7 15 | 4 13 18 | 6 11 18 | 9 15 16 |
| | 1 3 6 | 2 5 18 | 3 9 13 | 4 12 14 | 6 12 17 | 10 11 14 |
| | 1 8 11 | 2 9 14 | 4 5 16 | 5 7 8 | 7 16 17 | 10 12 13 |
| | 1 13 16 | 2 11 12 | 4 6 10 | 5 9 11 | 8 12 15 | 10 17 18 |
| | 1 14 15 | 3 5 17 | 4 7 9 | 6 8 16 | 8 14 18 | 13 15 17 } |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $M_1$ = { | 1 9 | 2 4 | 3 4 | 4 8 | 5 13 | 6 14 | 7 18 | 9 12 | 12 16 |
| | 1 10 | 2 8 | 3 11 | 4 15 | 5 14 | 6 15 | 8 9 | 10 15 | 12 18 |
| | 1 12 | 2 13 | 3 14 | 4 17 | 5 15 | 7 10 | 8 10 | 11 13 | 14 16 |
| | 1 17 | 2 16 | 3 16 | 5 6 | 6 7 | 7 11 | 8 13 | 11 15 | 14 17 |
| | 1 18 | 2 17 | 3 18 | 5 12 | 6 9 | 7 13 | 9 10 | 11 17 | 16 18 } |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $M_2$ = { | 1 4 | 2 6 | 3 8 | 4 11 | 6 13 | 7 14 | 9 17 | 10 16 | 13 14 |
| | 1 5 | 2 15 | 3 12 | 5 10 | 7 12 | 8 17 | 9 18 | 11 16 | 15 18 } $\cup E_2 \cup E_1(0)$ |

$M_3 = E_2 \cup E_8 \cup E_7(0)$ $\quad M_4 = E_2 \cup E_8 \cup E_7(0)$ $\quad M_5 = E_2 \cup E_1 \cup E_7(0)$
$M_6 = E_4 \cup E_1 \cup E_7(0)$ $\quad M_7 = E_4 \cup E_1 \cup E_7(1)$ $\quad M_8 = E_4 \cup E_3 \cup E_1(1)$
$M_9 = E_4 \cup E_3 \cup E_7(1)$ $\quad M_{10} = E_6 \cup E_3 \cup E_7(1)$ $\quad M_{11} = E_6 \cup E_3 \cup E_7(1)$
$M_{12} = E_6 \cup E_5 \cup E_9$ $\quad M_{13} = E_6 \cup E_5 \cup E_9$ $\quad M_{14} = E_8 \cup E_5 \cup E_9$
$M_{15} = E_8 \cup E_5 \cup E_9$

For $v = 37, n = 15, m = 22$, let

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A = { | 1 2 7 | 2 3 5 | 3 7 15 | 4 11 15 | 5 19 21 | 7 13 20 | 9 19 22 |
| | 1 2 17 | 2 3 10 | 3 8 19 | 4 11 20 | 6 7 12 | 7 17 21 | 10 11 14 |
| | 1 3 6 | 2 4 9 | 3 9 18 | 4 12 19 | 6 7 17 | 8 10 17 | 10 12 13 |
| | 1 4 5 | 2 6 22 | 3 10 22 | 4 13 18 | 6 8 16 | 8 11 21 | 10 14 16 |
| | 1 4 10 | 2 8 10 | 3 10 17 | 4 14 22 | 6 9 10 | 8 14 18 | 10 16 19 |
| | 1 5 14 | 2 8 15 | 3 11 16 | 4 15 21 | 6 9 20 | 8 15 17 | 10 18 22 |
| | 1 6 8 | 2 11 12 | 3 12 20 | 5 6 19 | 6 11 18 | 8 20 22 | 11 12 22 |
| | 1 7 22 | 2 11 17 | 3 13 19 | 5 7 13 | 6 12 17 | 8 13 14 | 11 13 21 |
| | 1 8 11 | 2 12 16 | 3 15 22 | 5 7 18 | 6 13 21 | 8 18 19 | 11 15 19 |
| | 1 9 10 | 2 13 15 | 3 16 21 | 5 8 12 | 6 14 20 | 9 15 16 | 12 14 19 |
| | 1 12 22 | 2 14 19 | 4 5 16 | 5 9 11 | 6 15 19 | 9 13 18 | 12 16 17 |
| | 1 13 16 | 2 18 20 | 4 6 15 | 5 9 21 | 7 8 20 | 9 14 17 | 12 18 20 |
| | 1 14 15 | 2 16 22 | 4 7 9 | 5 11 14 | 7 10 18 | 9 15 21 | 13 15 17 |
| | 1 18 21 | 3 4 13 | 4 7 14 | 5 13 22 | 7 11 22 | 9 16 20 | 14 15 16 |
| | 1 19 20 | 3 5 12 | 4 10 21 | 5 18 22 | 7 12 21 | 9 17 19 | 14 20 21 |
| | 13 17 20 | 16 18 21 | 16 19 20 | 17 18 20 | 17 21 22 } | | |

$$M_1=\{\begin{array}{llllllllll}
1\,3 & 2\,4 & 3\,7 & 4\,17 & 5\,20 & 7\,9 & 9\,11 & 11\,16 & 13\,14 & 15\,20\\
1\,12 & 2\,5 & 3\,11 & 4\,18 & 6\,10 & 7\,10 & 9\,12 & 11\,20 & 13\,19 & 16\,22\\
1\,15 & 2\,6 & 3\,14 & 5\,6 & 6\,14 & 7\,16 & 10\,13 & 12\,13 & 14\,17 & 17\,19\\
1\,16 & 2\,9 & 3\,17 & 5\,8 & 6\,21 & 8\,9 & 10\,15 & 12\,18 & 14\,21 & 17\,18\\
1\,19 & 2\,21 & 4\,8 & 5\,10 & 7\,8 & 8\,16 & 11\,13 & 12\,15 & 15\,18 & 19\,22\\
4\,22 & 18\,19 & 20\,21 & 20\,22 & 21\,22\,\} & & & & &
\end{array}$$

$$M_2=\{\begin{array}{llllllllll}
1\,13 & 2\,14 & 3\,4 & 4\,8 & 5\,16 & 6\,13 & 7\,14 & 8\,21 & 10\,12 & 12\,14\\
1\,18 & 2\,18 & 3\,8 & 4\,17 & 5\,17 & 6\,16 & 7\,16 & 9\,12 & 10\,19 & 12\,15\\
1\,17 & 2\,19 & 3\,9 & 4\,19 & 5\,20 & 6\,22 & 7\,19 & 9\,13 & 10\,20 & 13\,16\\
1\,20 & 2\,20 & 3\,20 & 5\,10 & 4\,6 & 7\,11 & 8\,9 & 9\,14 & 11\,17 & 13\,22\\
1\,21 & 2\,21 & 3\,21 & 5\,15 & 6\,11 & 7\,15 & 8\,12 & 10\,11 & 11\,18 & 14\,22\\
15\,18 & 15\,22 & 16\,18 & 17\,22 & 19\,21\,\} & & & & &
\end{array}$$

$$M_3=\{\begin{array}{lllllllllll}
1\,9 & 2\,7 & 3\,6 & 4\,12 & 5\,15 & 6\,18 & 8\,13 & 9\,22 & 10\,21 & 11\,19 & 15\,20\\
1\,11 & 2\,13 & 3\,14 & 4\,16 & 5\,17 & 7\,19 & 8\,22 & 10\,20 & 12\,21 & 16\,17 & 14\,18\,\}
\end{array}$$
$$\cup E_2 \cup E_1(1)$$

$M_4=\{E_2 \cup E_{11} \cup E_3(0) \cup E_7(1) \cup B\} \setminus A$ where

$$A=\{3\ 10, 4\ 15, 18\ 20\}, \qquad B=\{3\ 18, 4\ 20, 10\ 15\}$$

$M_5=E_2 \cup E_{10} \cup E_9(0)$ $\quad M_6=E_4 \cup E_{10} \cup E_9(0)$ $\quad M_7=E_6 \cup E_5 \cup E_1(0)$
$M_8=E_8 \cup E_7 \cup E_7(0)$ $\quad M_9=E_8 \cup E_7 \cup E_5(0)$ $\quad M_{10}=E_4 \cup E_1 \cup E_9(1)$
$M_{11}=E_4 \cup E_1 \cup E_9(1)$ $\quad M_{12}=E_8 \cup E_5 \cup E_3(1)$ $\quad M_{13}=E_{10} \cup E_9 \cup E_5(1)$
$M_{14}=E_6 \cup E_3 \cup E_{11}$ $\quad M_{15}=E_6 \cup E_3 \cup E_{11}$

It is easy to see that if NB$[k,\lambda;v]$ contains an INB$[k,\lambda;n]$ as a subdesign then the NB$[k,\lambda;v]$ is also indecomposable. From Theorem 3 there exists an NB$[3,5;v]$ containing an NB$[3,5;19]$ as a subdesign for $v \geq 39$ and $v \equiv 1,3 \pmod 6$. From Theorem 7 there exists an INB$[3,5;v]$ for $v = 19,21,25,27,31,33,37$. Therefore we have

**Theorem 8.** *There exists an indecomposable NB$[3,5;v]$ for $v \equiv 1$ or $3 \pmod 6$ and $v \geq 7$ with the exception of $v = 7$ and $9$, and the possible exception of $v = 13, 15$.*

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