# Some Simple Three-Designs of Small Order

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April 15, 1993

**Abstract**

We construct new simple 3-(17, 5, 3), 3-(19, 9, 56), 3-(19, 9, 140), and 3-(19, 9, 224) designs by combining disjoint designs.

## 1 Introduction

For positive integers $t \leq k \leq v$ and $\lambda$, a *t-design*, or more precisely a $t$-$(v,k,\lambda)$ *design*, is a pair $(X, \mathcal{B})$, where $X$ is a $v$-element set of *points* and $\mathcal{B}$ is a family of $k$-element subsets of $X$ called *blocks*. If $\mathcal{B}$ is a family of distinct blocks, then we say that the design is *simple*.

It is well known that the parameters of a $t$-$(v,k,\lambda)$ design satisfy:

$$\lambda\binom{v-j}{t-j} \equiv 0 \pmod{\binom{k-j}{t-j}}, \quad j = 0, 1, 2, \ldots, t-1.$$

These conditions are the *necessary conditions* for the existence of a $t$-$(v,k,\lambda)$ design.

For $v < 19$, the existence of simple 3-designs is settled completely except for the parameter situations 3-(16, 7, 5), 3-(17, 5, 3), and 3-(17, 7, 7) (see [1]). For $v = 19$, only the existence of simple 3-designs with $k = 9$ is not determined entirely. The necessary conditions imply that a 3-(19, 9, $\lambda$) design can exist only if $\lambda \equiv 0 \pmod{28}$. In [1], it was erroneously reported that the existence of a simple 3-(19, 9, 4004) design is known, and results on the existence of some simple 3-(19, 9, $\lambda$) designs obtained in [3] has not been included. In view of these observations, the existence of simple 3-(19, 9, $\lambda$) designs that should appear undetermined in [1] is for $\lambda \in \{56, 140, 224, 644, 1484, 2324, 3164, 4004\}$.

In this paper, we prove the existence of simple 3-(17, 5, 3), 3-(19, 9, 56), 3-(19, 9, 140), and 3-(19, 9, 224) designs. It follows that there remain only seven parameter situations with $v < 20$ for which the existence of simple 3-designs is not determined. We note that the existence of 3-(16, 7, 5) and 3-(17, 7, 7) designs is not known even if the 3-designs are not required to be simple [2].

| $\lambda$ | Starter blocks for $\mathcal{B}_\lambda$ |
|---|---|
| 28 | {0,1,2,3,5,6,8,10,13} {0,1,2,3,6,7,8,9,14} {0,1,2,3,5,7,12,13,16} |
| 112 | {0,1,2,3,5,6,8,10,13} {0,1,2,3,6,7,8,9,14} {0,1,2,3,5,7,12,13,16} {0,1,2,3,4,5,8,9,13} {0,1,2,3,4,5,8,10,13} {0,1,3,4,5,6,8,11,13} {0,1,2,3,4,5,6,9,16} |
| 196 | {0,1,2,3,5,6,8,10,13} {0,1,2,3,6,7,8,9,14} {0,1,2,3,5,7,12,13,16} {0,1,2,3,4,5,8,9,13} {0,1,2,3,4,5,8,10,13} {0,1,3,4,5,6,8,11,13} {0,1,2,3,4,5,6,9,16} {0,1,3,4,5,6,8,10,15} {0,1,3,4,6,7,8,10,18} {0,1,2,3,4,5,6,7,8} {0,1,3,4,5,7,9,14,17} |

Table 1: Some simple 3-(19, 9, $\lambda$) designs.

JCMCC 16 (1994), pp. 163-164

## 2 A Simple 3-(17,5,3) Design

In [4] a 3-$(17,5,1)$ design $(X,\mathcal{B})$ is constructed. The point set is $X = \mathbf{Z}_{17}$ and the blocks are obtained by developing $\{0,1,2,6,13\}$ into 136 blocks with the transformations of the form $x \mapsto mx + c$, where $m, c \in \mathbf{Z}_{17}$ and $m$ is a nonzero square. Applying individually the permutations $\pi_1$ and $\pi_2$ given below to $(X,\mathcal{B})$ constructs two further 3-$(17,5,1)$ designs that are pairwise disjoint and are both disjoint from $(X,\mathcal{B})$. The union of these three designs is a simple 3-$(17,5,3)$ design.

$$\pi_1 = (0\ 6\ 14\ 8\ 5\ 12\ 10\ 4)(1\ 13\ 3\ 16\ 9\ 11)(2)(7\ 15)$$

$$\pi_2 = (0\ 1\ 4)(2\ 5\ 10\ 13\ 8\ 3\ 16\ 11\ 15\ 7\ 14\ 9)(6\ 12)$$

## 3 Some Simple 3-(19,9,λ) Designs

In [3] simple 3-$(19,9,\lambda)$ designs $(X,\mathcal{B}_\lambda)$ with $\lambda \in \{28,112,196\}$ are constructed. The point set is $X = \mathbf{Z}_{19}$ and the blocks for each of these designs are obtained by developing the started blocks listed in Table 1 with the transformations of the form $x \mapsto mx + c$, where $m, c \in \mathbf{Z}_{19}$ and $m$ is nonzero. We list three permutations below. Applying the permutation $\tau_\lambda$ given below to $(X,\mathcal{B}_{28})$ constructs a further simple 3-$(19,9,28)$ design that is disjoint from $(X,\mathcal{B}_\lambda)$. The union of these two designs is a simple 3-$(19,9,\lambda+28)$ design.

$$\tau_{28} = (0\ 13\ 6\ 14\ 1\ 16\ 9\ 8\ 7\ 18\ 4\ 11\ 2\ 3\ 5\ 10\ 15\ 17\ 12)$$

$$\tau_{112} = (0\ 14\ 18\ 10\ 9\ 7\ 3\ 16\ 2\ 6\ 17\ 4\ 12\ 15\ 1\ 8\ 5\ 11\ 13)$$

$$\tau_{196} = (0\ 10\ 8\ 12\ 17\ 3\ 4\ 13\ 2\ 15\ 6\ 16\ 9\ 14\ 11\ 5\ 7\ 18)(1)$$

## References


[1] Y. M. Chee, C. J. Colbourn and D. L. Kreher, Simple $t$-designs with $v \le 30$, *Ars Combin.* 29 (1990) 193–258.

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[3] D. L. Kreher, Y. M. Chee, D. de Caen, C. J. Colbourn and E. S. Kramer, Some new simple $t$-designs, *J. Combin. Math. Combin. Comput.* 7 (1990) 53–90.

[4] T. Skolem, Note 16, in: E. Netto, *Kombinatorik*, 2.Auflage, Teubner, Leipzig, 1927.