A NOTE ON CIRCULAR PERMUTATIONS

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Dedicated to the memory of Professor L. Kuipers

I. Introduction

Suppose we have r distinct objects, say $w_1, \cdots, w_r$, and for $i = 1, \cdots, r$, we have n_i copies of the i-th object w_i. Let $n = n_1 + \cdots + n_r$. We arrange all of the objects into n places numbered $1, 2, \cdots, n$. Each arrangement is in fact a function f from the set $D = \{1, 2, \cdots, n\}$ onto the set $\Omega = \{w_1, \cdots, w_r\}$ so that for each $1 \leq i \leq r$, there are exactly n_i elements of D mapped into w_i. Let F be the set of all such kind of functions f from D onto Ω.

Let S_n be the symmetric group defined on D and let S be a subgroup of S_n. S acting on F is defined to be that for any $f \in F$ and for any $\gamma \in S$, γf is an element of F so that for any $1 \leq j \leq n$, $j(\gamma f) = (j\gamma)f$. Thus, if $jf = w_{i_j}$ for all $1 \leq j \leq n$, then $j(\gamma f) = w_{i_{j_\gamma}}$ for all $1 \leq j \leq n$. For any $f \in F$, the set $Sf = \{\gamma f | \gamma \in S\}$ is called to be the orbit of f under S. The function f is a representative of the orbit Sf. The set of all orbits under S is denoted by F/S.

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Let $\alpha = (1,2,\cdots,n) \in S_n$ be the permutation on D so that $n\alpha = 1$ and $i\alpha = i+1$ for all $1 \le i < n$. Let C_n be the cyclic subgroup of S_n generated by α. Then $|C_n| = n$. Each orbit in F/C_n is called a circular permutation and the number n is called the length of any such circular permutation. For convenience, we may write and say a circular permutation f instead of a circular permutation C_nf. If $f : D \longrightarrow \Omega$ is a circular permutation so that there is a positive integer d satisfying $(j+d)f = jf$ for all $1 \le j \le n$ (we take $(j+d)$ modulo n instead of $j+d$ whenever $j+d > n$), the smallest such positive integer d is called the period of this circular permutation f. Trivially, if d is the period of some circular permutation then d divides the greatest common divisor of $n_1,\cdots,n_r$. See [4] for more details in this topic.

Let $\beta = (1,n)(2,n-1)\cdots \in S_n$ be the permutation on D so that $i\beta = n-i+1$ for all $1 \le i \le n$. Let D_n be the dihedral subgroup of order $2n$ of S_n generated by α and β. Each orbit D_nf in F/D_n is called to be a transposed circular permutation. We may also write and call a transposed circular permutation f instead of a transposed circular permutation D_nf.

The purpose of this note is to enumerate the number N of all transposed circular permutations (the general problem and techniques have been discussed intensively in any standard book of combinatorial theory; e.g., [1], [2], and [4]). We will give an explicit formula to compute N in the following theorem. In fact, the need for the results presented in this note have arised in dealing with orthogonal polygons for computational geometry (see [2]).

Theorem. Notations and terminologies are as stated above. Let $M(n_1,\cdots,n_r) = \frac{1}{n}\sum_{d|(n_1,\cdots,n_r)} \phi(d)\frac{(n/d)!}{(n_1/d)!\cdots(n_r/d)!}$, where $\phi(\cdot)$ is Euler phi function and $(n_1,\cdots,n_r)$ is the greatest common divisor of the numbers $n_1,\cdots,n_r$. Then

$$N = \begin{cases} \frac{1}{2}M(n_1,\cdots,n_r) + \frac{1}{2}\cdot\frac{(\frac{n}{2})!}{(\frac{n_1}{2})!\cdots(\frac{n_r}{2})!} & \text{if all } n_i\text{'s are even,} \\ \frac{1}{2}M(n_1,\cdots,n_r) + \frac{1}{2}\cdot\frac{(\frac{n-1}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2}{2})!\cdots(\frac{n_r}{2})!} & \text{if exactly one, say } n_1, \text{ of } n_i\text{'s is odd,} \\ \frac{1}{2}M(n_1,\cdots,n_r) + \frac{1}{2}\cdot\frac{(\frac{n-2}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2-1}{2})!(\frac{n_3}{2})!\cdots(\frac{n_r}{2})!} & \text{if exactly two, say } n_1 \text{ and } n_2, \text{ of } n_i\text{'s are odd,} \\ \frac{1}{2}M(n_1,\cdots,n_r) & \text{otherwise.} \end{cases}$$

Remark. One can also make use of the greatest integer function

$[\cdot]$ to obtain a simple explicit formula for N:

$$N = \begin{cases} \frac{1}{2}M(n_1, \cdots, n_r) & \text{if more than two of the } n_i \text{ are odd,} \\ \frac{1}{2}M(n_1, \cdots, n_r) + \frac{1}{2}\frac{[\frac{(n-a)}{2}]!}{[\frac{n_1}{2}]![\frac{n_2}{2}]!\cdots[\frac{n_r}{2}]!} & \text{if } a = 0, 1 \text{ or } 2 \text{ of the } n_i \\ & \text{are odd.} \end{cases}$$

II. Proof of the Theorem

Let $r, n_i, n, D, \Omega, \alpha, \beta, C_n$ and D_n have the same meanings as in Section I. In order to prove the theorem, we need the following two well-known results.

The first one is a formula for the number of circular permutations. It is known (see pp. 12-13, [4]) that the number of circular permutations of length and period n is $\frac{1}{n} \sum_{d|(n_1,\cdots,n_r)} \mu(d)\frac{(n/d)!}{(n_1/d)!\cdots(n_r/d)!}$, where $\mu(\cdot)$ is the Möbius function and $(n_1, \cdots, n_r)$ is the greatest common divisor of $n_1, \cdots, n_r$. Since one of our circular permutations of length n and period n/d arises from a concatenation of d circular permutations of length and period n/d so that for each $1 \le i \le r$, there are exactly n_i/d elements $j \in \{1, 2, \cdots, n/d\}$ mapped into w_i, the total number $M(n_1, \cdots, n_r)$ of circular permutations is

$$\begin{aligned} M(n_1, \cdots, n_r) &= \sum_{d|(n_1,\cdots,n_r)} \frac{d}{n} \sum_{t|\frac{(n_1,\cdots,n_r)}{d}} \mu(t)\frac{(n/dt)!}{(n_1/dt)!\cdots(n_r/dt)!} \\ &= \frac{1}{n}\sum_{d|(n_1,\cdots,n_r)} \frac{(n/d)!}{(n_1/d)!\cdots(n_r/d)!}\sum_{t|d} t\mu(\frac{d}{t}) \qquad (1) \\ &= \frac{1}{n}\sum_{d|(n_1,\cdots,n_r)} \phi(d)\frac{(n/d)!}{(n_1/d)!\cdots(n_r/d)!}, \end{aligned}$$

where $\phi(\cdot)$ is Euler phi function.

The second one is Burnside's Lemma (see Sec. 3, Chap. 8, [2]), namely, if the group G acts on a set X, then the number N of orbits of G acting on X is given by

$$N = \frac{1}{|G|}\sum_{\sigma\in G}(\#\text{ of elements fixed by } \sigma). \qquad (2)$$

For each $\sigma \in D_n$, let $F(\sigma) = \{f | f : D \longrightarrow \Omega, \sigma f = f\}$, i.e., $F(\sigma)$ be the set of elements of F fixed by σ. By Burnside's Lemma, we have

$$(3) \qquad N = \frac{1}{|D_n|} \sum_{\sigma \in D_n} |F(\sigma)|$$
$$= \frac{1}{2n}\Big(\sum_{\sigma \in C_n} |F(\sigma)| + \sum_{\sigma \in \beta C_n} |F(\sigma)|\Big)$$
$$= \frac{1}{2}\Big(\frac{1}{n} \sum_{\sigma \in C_n} |F(\sigma)|\Big) + \frac{1}{2n} \sum_{\sigma \in \beta C_n} |F(\sigma)|$$
$$= \frac{1}{2}M(n_1, \cdots, n_r) + \frac{1}{2n} \sum_{\sigma \in \beta C_n} |F(\sigma)|.$$

In the last equation, the number $M(n_1, \cdots, n_r)$ is known explicitly by the formula (1). We just need to evaluate the term $\frac{1}{2n} \sum_{\sigma \in \beta C_n} |F(\sigma)|$. There are now three cases to consider, but we first make a remark concerning the elements of βC_n.

Remark. Let $1 \leq k < n$. If k is odd, then $\beta\alpha^k = (\frac{k+1}{2} - 1, \frac{k+1}{2} + 1)(\frac{k+1}{2} - 2, \frac{k+1}{2} + 2) \cdots$, where $\frac{k+1}{2} + i$ represents $\frac{k+1}{2} + i - n$ if $\frac{k+1}{2} + i > n$, and $\frac{k+1}{2} - i$ represents $n + (\frac{k+1}{2} - i)$ if $\frac{k+1}{2} - i \leq 0$. In this case, $\frac{k+1}{2}$ is fixed by $\beta\alpha^k$, and $\frac{k+1}{2} + \frac{n}{2}$ is also fixed by $\beta\alpha^k$ whenever n is even. If k is even, then $\beta\alpha^k = (\frac{k}{2}, \frac{k}{2} + 1)(\frac{k}{2} - 1, \frac{k}{2} + 2) \cdots$. In this case, either $\beta\alpha^k$ fixes no elements if n is even, and $\beta\alpha^k$ fixes only the element $\frac{k}{2} + \frac{n+1}{2}$ if n is odd.

Using this remark, we can evaluate the sum $\frac{1}{2n} \sum_{\sigma \in \beta C_n} |F(\sigma)|$ in the following three cases:

Case 1. n is odd. First consider k odd. Then $\frac{k+1}{2}$ is fixed by $\beta\alpha^k$, and every element $f \in F(\beta\alpha^k)$ is completely determined by the images $\frac{k+1}{2}f, (\frac{k+1}{2} + 1)f, \cdots, (\frac{k+1}{2} + \frac{n-1}{2})f$. Therefore, $|F(\beta\alpha^k)| \neq 0$ if and only if there is only one odd number, say n_1, among $n_1, \cdots, n_r$. Moreover, if $|F(\beta\alpha^k)| \neq 0$, then $|F(\beta\alpha^k)| = \frac{(\frac{n-1}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2}{2})!\cdots(\frac{n_r}{2})!}$. This is also true for k even and hence, $\frac{1}{2n} \sum_{\sigma \in \beta C_n} |F(\sigma)| = 0$ if at least two of the n_i's are odd, and

$$\frac{1}{2n} \sum_{\sigma \in \beta C_n} |F(\sigma)| = \frac{1}{2n} \cdot \frac{(\frac{n-1}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2}{2})! \cdots (\frac{n_r}{2})!} \cdot n = \frac{1}{2} \frac{(\frac{n-1}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2}{2})! \cdots (\frac{n_r}{2})!}$$

if only one, say n_1, of the n_i's is odd. Substituting in (3), we have

$$N = \begin{cases} \frac{1}{2}M(n_1,\cdots,n_r) + \frac{1}{2}\frac{(\frac{n-1}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2}{2})!\cdots(\frac{n_r}{2})!} & \text{if exactly one, say } n_1, \\ & \text{of the } n_i\text{'s is odd.} \\ \frac{1}{2}M(n_1,\cdots,n_r) \text{ otherwise.} \end{cases}$$

Case 2. n is even and all n_i's are even. If k is odd, then both $\frac{k+1}{2}$ and $\frac{k+1}{2}+\frac{n}{2}$ are fixed by $\beta\alpha^k$. Every element $f \in F(\beta\alpha^k)$ is completely determined by the images $\frac{k+1}{2}f, (\frac{k+1}{2}+1)f, \cdots, (\frac{k+1}{2}+\frac{n}{2})f$. Since all n_i's are even, $\frac{k+1}{2}f = (\frac{k+1}{2}+\frac{n}{2})f$ and this could be equal to any one of the w_i's. Hence,

$$\begin{aligned}|F(\beta\alpha^k)| &= \frac{(\frac{n-2}{2})!}{(\frac{n_1-2}{2})!(\frac{n_2}{2})!\cdots(\frac{n_r}{2})!} + \frac{(\frac{n-2}{2})!}{(\frac{n_1}{2})!(\frac{n_2-2}{2})!\cdots(\frac{n_r}{2})!} \\ &\quad + \cdots + \frac{(\frac{n-2}{2})!}{(\frac{n_1}{2})!\cdots(\frac{n_{r-1}}{2})!(\frac{n_r-2}{2})!} \\ &= \frac{(\frac{n}{2})!}{(\frac{n_1}{2})!\cdots(\frac{n_r}{2})!}\end{aligned}$$

If k is even, then $\beta\alpha^k$ fixes no elements of D. Every element f of F is completely determined by $(\frac{k}{2}+1)f, \cdots, (\frac{k}{2}+\frac{n}{2})f$ and so $|F(\beta\alpha^k)| = \frac{(\frac{n}{2})!}{(\frac{n_1}{2})!\cdots(\frac{n_r}{2})!}$.

Combining all of these results together, we have $\frac{1}{2n}\sum_{\sigma\in\beta C_n}|F(\sigma)| = \frac{1}{2}\cdot\frac{(\frac{n}{2})!}{(\frac{n_1}{2})!\cdots(\frac{n_r}{2})!}$, and then, $N = \frac{1}{2}M(n_1,\cdots,n_r) + \frac{1}{2}\cdot\frac{(\frac{n}{2})!}{(\frac{n_1}{2})!\cdots(\frac{n_r}{2})!}$

Case 3. n is even and at least one of the n_i's is odd. In fact, the number of the n_i's which are odd is even. Without loss of generality, let $n_1, n_2, \cdots, n_{2t}$ be odd.

Let k be odd. As in Case 2, $f \in F(\beta\alpha^k)$ is completely determined by $\frac{k+1}{2}f, (\frac{k+1}{2}+1)f, \cdots, (\frac{k+1}{2}+\frac{n}{2})f$, so that $|F(\beta\alpha^k)| \neq 0$ if and only if $t = 1$. In the case $t = 1$, $\frac{k+1}{2}f \neq (\frac{k+1}{2}+\frac{n}{2})f$, and moreover, $\frac{k+1}{2}f = w_1$ if and only if $(\frac{k+1}{2}+\frac{n}{2})f = w_2$. Hence, if $t = 1$, then $|F(\beta\alpha^k)| = 2\cdot\frac{(\frac{n-2}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2-1}{2})!(\frac{n_3}{2})!\cdots(\frac{n_r}{2})!}$.

Let k be even. Then $\beta\alpha^k$ fixes no elements of D and $f \in F(\beta\alpha^k)$ is completely determined by $(\frac{k}{2}+1)f, \cdots, (\frac{k}{2}+\frac{n}{2})f$. Since some n_i are odd, $|F(\beta\alpha^k)| = 0$.

Combining all of these results together, we have

$$\frac{1}{2n}\sum_{\sigma\in\beta C_n}|F(\sigma)| = \frac{1}{2n}\left(\sum_{i=0}^{\frac{n}{2}-1}|F(\beta\alpha^{2i+1})| + \sum_{i=0}^{\frac{n}{2}-1}|F(\beta\alpha^{2i})|\right)$$
$$= \frac{1}{2n}\left(\frac{n}{2}\cdot 2\cdot\frac{(\frac{n-2}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2-1}{2})!(\frac{n_3}{2})!\cdots(\frac{n_r}{2})!} + 0\right)$$
$$= \frac{1}{2}\cdot\frac{(\frac{n-2}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2-1}{2})!(\frac{n_3}{2})!\cdots(\frac{n_r}{2})!}.$$

Therefore,

$$N = \begin{cases} \frac{1}{2}M(n_1,\cdots,n_r) + \frac{1}{2}\cdot\frac{(\frac{n-2}{2})!}{(\frac{n_1-1}{2})!(\frac{n_2-1}{2})!(\frac{n_3}{2})!\cdots(\frac{n_r}{2})!} & \text{if exactly two, say } n_1 \text{ and } n_2, \text{ of the } n_i\text{'s are odd,} \\ \frac{1}{2}M(n_1,\cdots,n_r) & \text{otherwise.} \end{cases}$$

This concludes the proof.

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