Isomorphic factorization of trees of maximum degree three

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ABSTRACT. We prove that for any tree T of maximum degree three there is a subset S of $E(T)$ with $|S| = O(logn)$ and a two colouring of the edges of the forest $T \setminus S$ such that the two monochromatic forests are isomorphic, where n is the number of vertices of T of degree three.

1 Introduction

A graph G is called *even* or *bisectable* if its edges can be coloured by two colours such that the two monochromatic subgraphs are isomorphic.

It is a difficult problem to decide whether or not a given graph is even. The problem is probably NP-complete even for trees, [4]. In [6] Harary and Robinson conjectured that this decision problem is "easy" for trees with maximum degree three. Actually they conjectured that, except for two small trees, all trees of maximum degree three are even. Heinrich and Horak [5] disproved this conjecture by presenting an infinite class of trees of maximum degree three which are not even.

Let $B(G)$ be an even subgraph of G of maximum size. Erdös, Pach and Pyber [3] and Alon and Krasikov [2] studied the function $R(G) = e(G) - e(B(G))$. It was shown that any graph of size e contains an even subgraph with at least $\Omega(e^{\frac{2}{3}})$ edges, and that there are graphs of size e containing no even subgraphs of size bigger than $O(e^{\frac{2}{3}} loge/log(loge))$.

A better bound for trees was found by Alon, Caro and Krasikov [1]. They showed that $R(T) \leq O(\frac{n}{loglogn})$ for all trees T, where $n = e(T)$. They also presented a tree T of n edges such that $R(T) \geq \Omega(logn)$.

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The main result of our paper is

Theorem 1. *Let T be a tree with maximum degree three. Then $R(T) \leq O(logn)$, where n is the number of vertices of degree three.*

Although it is known that there are infinite many trees T with maximum degree three such that $R(T) > 0$, [5], we do not know any tree T with maximum degree three and $R(T) > 2$. It is likely that the method used in this paper could be modified to prove that $R(T) \leq c$ for all trees of maximum degree three, where c is a constant independent of T.

Our proof of Theorem 1 is constructive and gives a linear algorithm for finding an even subtree of T of the stated size.

2 Preliminaries

Throughout the paper by a tree we mean a tree of maximum degree three. A tree in which all vertices have degree one or three will be called a 3-tree. If T' is a tree and T is obtained from T' by subdividing its edges, then we say that T is reducible to T'. It is obvious that any tree T reduces to a unique 3-tree, and we denote this 3-tree by T^*.

Suppose that T is a tree and $e \in E(T^*)$ is an edge in the reduced 3-tree of T. We let $P_T(e)$ denote the path in T which replaces the edge e, and let $||e||_T$ be the length of $P_T(e)$. If there is no danger of misunderstanding the subscript T will be dropped. An edge $e \in E(T^*)$ is called even (odd) if $||e||$ is even (odd). A vertex v of T^* is called a $(3, i)$-vertex if $d(v) = 3$ and v is adjacent in T^* to i vertices of degree 3. An edge e of T is called a *pending* edge if it is incident to a degree one vertex of T. If two pending edges e, e' of T^* are adjacent, then we call the union $P(e) \cup P(e')$ an *end* of T.

Clearly it suffices to prove Theorem 1 for those trees T for which each pending edge e of T^* has length $||e|| \leq 2$, for if e has length greater than two, we can remove an even subpath of $P_T(e)$ and at the end we colour the removed edges alternately by two colours.

Definition 2: Let H_1, H_2 be subtrees of T^* as depicted in Fig. 1(a) and Fig. 1(b) respectively, where x, y, x_1, y_1, x_2, y_2 are pending edges. Let $P(z) = vv_1 \cdots v_{||z||-1}v_{||z||}$. If $||z|| > 1$ then we split T (not T^*) at the vertex $v_{||z||-1}$ into two trees, and call the one containing v a *type I removable end* of T. The path $vv_1 \cdots v_{||z||-1}$ is called the *neck* of this removable end. Let $P(z') = ww_1 \cdots w_{||z'||}$. If

- $||z_1|| = ||z_2|| = 1$, and
- at least one of x_1, y_1 is even and at least one of x_2, y_2 is even,

then we split the tree T at the vertex $w_{||z'||-1}$ ($w_{||z'||-1}$ could be w) into two trees and call the one containing w a *type II removable end* of T. Similarly the path $wu_1. \cdots u_{||z'||-1}$ is called the neck of the removable end.

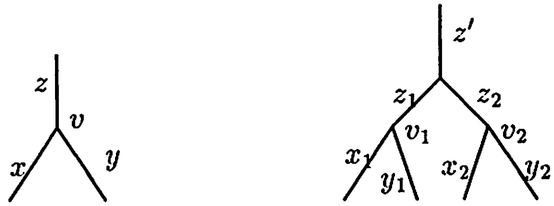


Figure 1

Note that we can assume that no vertex of T^* is incident to three ends of T. For otherwise the tree T contains at most four vertices of degree three, and it is easy to see that such a tree T has $R(T) \leq 5$. Therefore each edge of T is contained in at most one type II removable end.

Both type I and type II removable ends are called *removable ends* of T. If H is a removable end of T, then there is a unique edge of $T \setminus H$ which is incident to H. We denote this edge by e_H.

We remove from T all its removable ends, obtaining a tree T_1. Remove all removable ends of T_1 obtaining T_2 and so on. At the end we obtain a tree $\bar{T}$ which either has no removable ends or contains at most four vertices of degree three.

Theorem 3. *For any tree T we have $R(T) \leq R(\bar{T}) + c$, where c is a constant not depending on T.*

Proof: Suppose that we have coloured $e(\bar{T}) - R(\bar{T})$ edges of $\bar{T}$ by two colours such that monochromatic subgraphs are isomorphic. We extend this colouring to edges removed from T, i.e., the edges of the removable ends of $T, T_1, \cdots$, leaving at most c more edges uncoloured, so that the monochromatic subgraphs are still isomorphic.

As noted before, we can assume that each pending edge e of T^* has $||e|| \leq 2$. It is easy to see that under this assumption each pending edge e of T_i^* also has $||e|| \leq 2$. For a similar reason we can also assume the neck of each type I removable end has length one or two and the neck of each type II removable end has length zero or one. For if the neck of a removable end is too long we simply colour an even subpath of the neck alternately by two colours.

Depending on the length of the neck, and the parity of the two pending edges of a removable end, there are 6 different type I removable ends and 6 different type II removable ends. In Fig. 2 and Fig. 3 we list these 12 different removable ends $H_1, H_2, \cdots, H_{12}$, where the thick edge is e_H.

We will colour the edges (except at most a constant number of them) of all the ends so that the union of the ends induces two isomorphic monochromatic subgraphs. Furthermore the monochromatic graphs induced by these

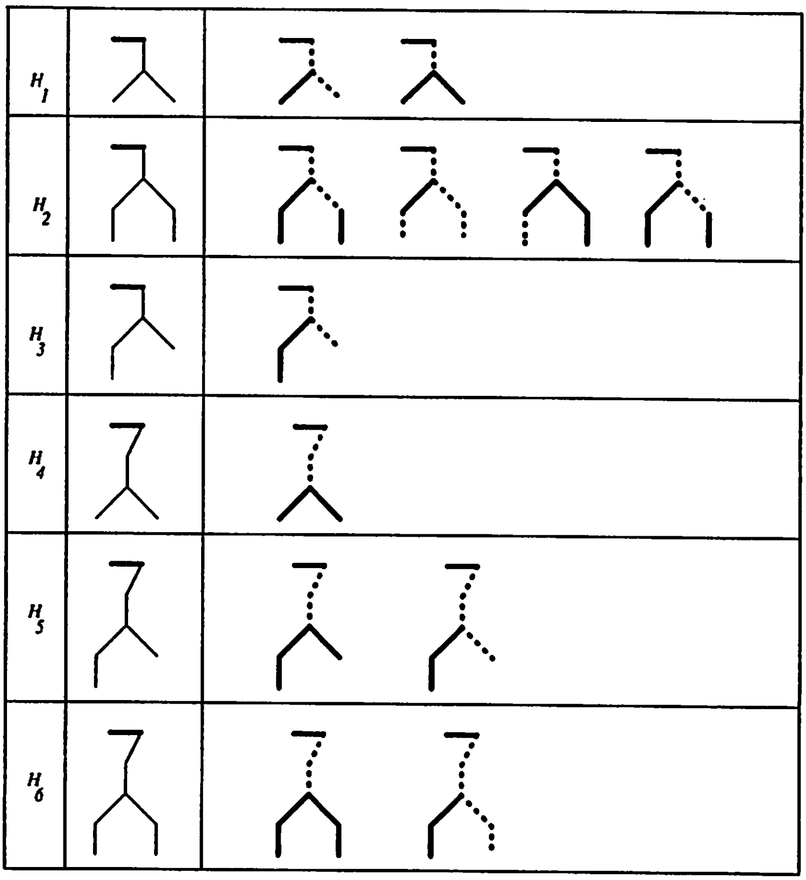


Figure 2

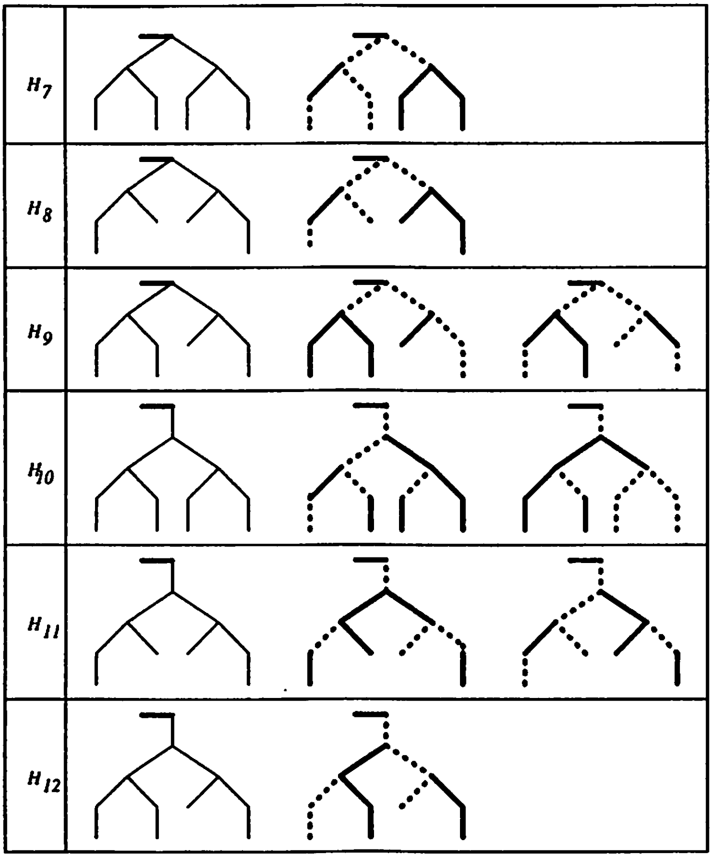


Figure 3

ends will not interfere with the monochromatic subgraphs induced by the edges of $\bar{T}$. This will be achieved by colouring the edges incident to e_H by the opposite colour of e_H. Therefore how we colour the edges of an end H depends on the colour of e_H. (Of course if e_H is an uncoloured edge of $\bar{T}$, we need not worry about the interference). By taking into consideration of the colour of e_H, we then have 24 different kinds of removable ends, $H_1, H_2, \cdots, H_{24}$.

We will use Figures 2 and 3 to colour all the removable ends. There are only 12 different removable ends $H_1, H_2, \cdots, H_{12}$ in these two figures. The other 12 removable ends are actually the same removable ends, only e_H are coloured in the opposite colour. Thus by interchanging the two colours, we obtain the solution table for the other 12 removable ends.

For each uncoloured removable end H_i listed on the left side of the figures, there are b_i coloured copies of the removable end on the right side, where b_i varies between 1 and 4. We refer to these copies as $A_{i1}, A_{i2}, \cdots, A_{ib_i}$. It is easy to verify that if we colour b_i copies of the removable end H_i as $A_{i1}, A_{i2}, \cdots, A_{ib_i}$ then the monochromatic subgraphs of $A_{i1} \cup A_{i2} \cup \cdots \cup A_{ib_i}$ are isomorphic.

We must know the colour of e_H before we colour the end H. If e_H is an edge of $\bar{T}$, then we can assume the colour of e_H is known. However e_H could be an edge of some other end, say H'. Thus we should colour H' before we colour H. Therefore we may not be able to choose b_i copies of a removable end H_i and colour them simultaneously (although there may be eventually a lot of them). So we colour the removable ends subsequently, as explained below.

Suppose that H is a copy of H_i which has not yet been coloured and for which e_H has already been coloured. Suppose that we have already coloured q copies of H_i before. Then we colour H as A_{ij}, where $j \equiv q + 1(mod(b))$. At the very end, we check the number m_i of coloured copies of H_i. If the number is not a multiple of b_i then we remove the colours of p copies of that kind of removable end, where $m = k \cdot b_i + p$ and $p < b_i$. Obviously the total number of such uncoloured edges is less than a constant. This proves Theorem 2.

Let T be a 3-tree of order n and e be a pending edge of T. Then there is a colouring of $T \setminus e$ by two colours such that each monochromatic component is a P_2 (i.e., a path of length 2). In fact such a colouring is unique. Let v be the vertex of degree three in T which is incident to e. We first colour the other two edges incident to v by one colour, say red. Then proceed to colour the rest of $T \setminus e$ as follows: Suppose that there are still uncoloured edges, then there must be a vertex x of degree three such that one of the three edges incident to x is coloured, say in the colour blue, and the other two edges are uncoloured. Then we colour these two edges red. It is easy to see that after we coloured all the edges, each monochromatic component is a P_2. Denote $\mathcal{P}_e(T)$ the set of all monochromatic P_2's of such a colouring. We note that if e, e' are two adjacent pending edges of T^*, then e and e' form a monochromatic P_2 of $\mathcal{P}_e(T)$. Thus if $P(e)$ and $P(e')$ form an end of T then e and e' form a monochromatic P_2.

Lemma 4. *Let T be a 3-tree of order n and e be a pending edge of T. Assign to each path P_2 from $\mathcal{P}_e(T)$ a weight $w \in \{\frac{1}{2}, \frac{3}{4}, 1\}$. Then there is a set S of paths of $\mathcal{P}_e(T)$ with $|S| = O(logn)$ such that the edges of $T \setminus S$ can be coloured by two colours red and blue where each monochromatic component of $T \setminus S$ is a $P_2 \in \mathcal{P}_e(T)$ and the total weight of red P_2's equals the total weight of blue P_2s.*

Proof: Let Q be a subset of $\mathcal{P}_e(T)$ and T_Q be a forest consisting of edges of Q. Denote by r_Q and b_Q the total weight of red and blue paths P_2 in Q respectively, set $d(T_Q) = d(Q) = r_Q - b_Q$. Put $T - e = T'$. Suppose $d(T') \geq 0$. If $d(T') \leq 4$, it can be easily seen that the removal of at most six P_2's in $\mathcal{P}_e(T)$ leads to the statement of this lemma. Therefore suppose that $d(T') > 4$. Let $P = u_1u_2u_3$ be a red path of $\mathcal{P}_e(T)$. Removing edges of P from T' we get three subtrees, T_1, T_2 and T_3 of T', where $u_i \in T_i$.

Claim: There is a red path P in $\mathcal{P}_e(T)$ and $1 \leq i \leq 3$ such that

$$\frac{1}{4}d(T') \leq d(T_i) \leq \frac{3}{4}d(T'). \qquad (*)$$

Take an arbitrary red path $P = u_1u_2u_3$. If one of the T_i's satisfies (*) we are done. Otherwise there is one subtree, say T_1, satisfying $d(T_1) > \frac{3}{4}d(T')$. Let B be the blue P_2 containing u_1. It is easy to verify that there is a red path which has a common vertex with B, and whose removal leads to three subtrees Q_1, Q_2, Q_3 such that either

- there is a $Q_i \supset T_2 \cup T_3$ and $\frac{1}{4}d(T') \leq d(Q_i) \leq \frac{3}{4}d(T')$, or
- there is a $Q_i \subset T_1$ and $d(Q_i) \geq \frac{1}{4}d(T')$.

In the first case the claim is proved. Thus we assume the second case occurs. If $d(Q_i) \leq \frac{3}{4}d(T')$ then the claim is also proved. If $d(Q_i) > \frac{3}{4}d(T')$ then we apply the same procedure to Q_i. Since $|Q_i| < |T_1|$ and T is finite, we will eventually find a red path satisfying the requirments of the claim. Otherwise we would arrive at Q_i with $d(Q_i) > 3/4d(T')$ and $|Q_i| < 3/4d(T')$.

Let $P \in \mathcal{P}_e(T)$ be a red path and T_1, T_2, T_3 are the three subtrees of $T \backslash P$ satisfying the requirments of the claim. Then either $\frac{1}{2}d(T') \geq d(T_1) \geq \frac{1}{4}d(T')$ or $\frac{1}{2}d(T') \geq d(T_2 \cup T_3) \geq \frac{1}{4}d(T')$. We interchange the colours of the edges of T_1 or the colours of the edges of $T_2 \cup T_3$. Under this new colouring, we have that each monochromatic component is a P_2 of $\mathcal{P}_e(T)$ and moreover

$$d(T_1 \cup T_2 \cup T_3) \le \frac{1}{2} d(T').$$

In general, for each subgraph F of T' whose edges are coloured by two colours such that each monochromatic component is a $P_2 \in \mathcal{P}_e(T)$ there is a path $P \in \mathcal{P}_e(T)$ and a new colouring of the edges of $F \setminus P$ such that $|d(F \setminus P)| \le \frac{1}{2} d(F)$.

We continue this removing process until we obtain a subgraph F of T such that $d(F) \le 4$, and then we remove at most 6 more $P_2 \in \mathcal{P}_e(T)$ so that the total weight of red P_2's is equal to the total weight of blue P_2's. Clearly the total number of removed P_2's is $O(logn)$.

3 Proof of Theorem 1

According to Theorem 3 all which remains to be proved is the following statement:

Theorem 5. *Let T be a tree not containing a removable end. Then $R(T) \le O(logn)$, where n is the number of vertices of T of degree three.*

Proof: The idea of the proof is simple, however the details are quite involved. We first consider the 3-tree T^* reduced from T. We colour the edges of $T' = T^* - e$ by two colours, say red and blue, so that each monochromatic component is a P_2. Then we assign weights to these P_2's according to their "local enviroment", which we will explain below, and apply Lemma 4 to remove the colour of at most $O(logn)$ edges so that for the remaining coloured edges, the total weight of red P_2's is equal to the total weight of blue P_2's. We then transform this colouring into a partial colouring of the edges of T, so that each monochromatic component is still a path of length two. The final step is to extend this partial colouring to almost all edges of T, except for at most $O(logn)$ edges of T which will remain permanently uncoloured. It is easy to extend the partial colouring to colour all the edges of $P(e)$ if e is an odd edge of T^*. We simply colour the uncoloured edges of that $P(e)$ alternately by two colours. If e is an even edge of T^* then we need to treat several such edges together.

We start our detailed proof by partitioning even edges of T' into classes.

We consider T' as a rooted tree with the only degree two vertex v as its root. For a vertex u of T', define the level $\ell(u)$ of u to be the distance between u and v. Let $e = uw$ be an edge of T' then the difference between the levels of u and w is one. Suppose $\ell(u) = \ell(w) - 1$. Then we define the level $\ell(e)$ of the edge e to be $\ell(u)$. In this case we say that e is an edge of u. Therefore e is an edge of u if and only if e is incident to u and e and u have the same level. We will use the terms of *father, son, predecessor, successor* of a vertex of T' in the usual way. In addition, we say that an edge e is the father of another edge e' if they share a vertex and $\ell(e) = \ell(e') - 1$. We then use the term "predecessor, successor" of an edge in the natural way.

Now we partition even edges of T' into classes. We form the classes one by one, and follow the four procedures stated below in the given order. Once an even edge has been put into one class, it will stay in that class and will not be considered for the rest of the partition process. Each of the four procedures will be repeated as long as there exit unused even edges satisfying the conditions of that procedure. In case no such edges can be found, we proceed to the next procedure.

There is a general rule we will follow in the partition: If u is a $(3,1)$-vertex of T' and both edges of u are even, then these two edges will always belong to the same class. We will not refer to this rule explicitly in the procedure. It should be understood that once one of the even edge of u is put into one class, the other even edge of u goes to the same class automatically.

Procedure 1: Take an even edge x of the smallest level. If x has successors which are even edges, then let Y be the set of all such even edges of the smallest level. If Y contains a pending edge y then let x and y form a class. Otherwise choose an edge $y \in Y$ arbitrarily and let x and y form a class.

Procedure 2: Take a $(3,2)$-vertex u with even pending edge x of the smallest level. If the other son w of u is either a $(3,2)$-vertex with even pending edge y or a $(3,1)$-vertex with even pending edge y then let x and y form a class.

Procedure 3: Take a $(3,2)$-vertex u with even pending edge x. If there is a $(3,1)$-vertex w with even edge y such that w and u have the same father then x and y form a class.

Procedure 4: Each of the remaining even edges forms a class. (Note that if there is a $(3,1)$-vertex with two even edges then these two even edges form a class).

From the procedures defined above, we see that each class contains one, two or three even edges. All the even edges in the same class will be coloured simultaneously, and we may also need to consider several classes of even edges at the same time in the process of colouring. In the process of treating a class of even edges of T, we may need to change the colour of some already coloured edges. This may result in the loss of some monochromatic P_2's. Thus we assign weights to monochromatic P_2's according to their "local enviroment", and get prepared for these losses. It is in the treating of some pending even edges of $(3,2)$ vertices that will result in the loss of some coloured P_2's. For this purpose we define three types of special monochromatic paths of T'.

Suppose as depicted in Fig. 4, that u is a $(3,2)$-vertex with even pend-

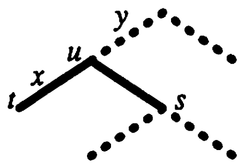


Figure 4

ing edge x and that x forms a class in the partition by itself. Then the monochromatic path $P = sut$ is a special path of

- type I if $||y|| = 1$,
- type II if y is odd and $||y|| > 1$,
- type III if y is even.

In case when there are two type I special paths sut and $s'u't'$, such that u and u' have the same father then these two special paths of type I are called *conjugate.*

Now we assign weight to each monochromatic path $P \in \mathcal{P}_e(T)$ as follows: $w(P) = \frac{3}{4}$ if P is a special path of type I, and $w(P) = \frac{1}{2}$ if P is a special path of type II or type III, and $w(P) = 1$ otherwise. By Lemma 4, there is a subgraph Q of T^* with $e(Q) \geq e(T^*) - O(logn)$, and a colouring of $e(Q)$ with two colours, say red and blue, such that each monochromatic component is a path P of length two, and $P \in \mathcal{P}_e(T)$, and the total weight of red P_2's is equal to the total weight of blue P_2's.

Also the change of the colour of already coloured edge may change a monochromatic P_2 into a monochromatic P_3, P_4 or a claw. It is important that these changes, if any, will be limited to "local" changes. To be precise, we associate to each class C of even edges a set $C(P)$ of monochromatic P_2's as follows:

If a class C is formed by Procedures 1, 2 or 3, then $C(P)$ consists those monochromatic P_2's which intersect (i.e., has at least one edge in common) a $x - y$-path for some edges $x, y \in C$ (the $x - y$-path is the unique path in T' connecting the two edges x and y, and x, y are included in this path). The treating of the classes C formed by Procedure 4 are divided further into cases, (cf. below). Depending on which case C belongs to, the set $C(P)$ consists all the monochromatic P_2's depicted in Fig. 4, or all the the monochromatic P_2's depicted in Fig. 12(a), or all the monochromatic P_2's depicted in Fig. 13(a), or all but the top monochromatic P_2's depicted in Fig. 14(a).

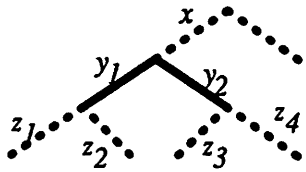


Figure 5

In the treating of a class C of even edges, only those monochromatic P_2's associated to C could be changed. Most monochromatic P_2's are associated to at most one class. For two classes C_1 and C_2, if there is no monochromatic P_2 associated to both classes, then the treating of C_1 and C_2 will not affect each other.

The only cases that a monochromatic P_2 could be associated to more than one classes are listed below:

(1): In Fig. 5, if x is even, z_1 or/and z_2, z_3 or/and z_4 are even pending edges, then it could happen that x, z_1 or/and z_2 form a class, z_3 or/and z_4 form a class. In this case, the monochromatic $P_2 = y_1y_2$ is associated to both classes.

(2): In Fig. 5, if x, z_1, z_2 are even edges, and z_2 is not a pending edge, (z_1 could be a pending edge), then it could happen that x and z_1 form a class, z_2 form a class by itself or with some successsor(s). In this case the monochromatic path z_1z_2 is associated to both classes.

(3): In Fig. 5, if z_2 is even but not a pending edge, x is even, z_1 is an even pending edge, z_3 or/and z_4 are even pending edge(s), then it could happen that x and z_1 form a class, z_3 or/and z_4 form a class, z_2 form a class by itself or with some successor(s). In this case the monochromatic path y_1y_2 is associated to two classes, and the monochromatic path z_1z_2 is associated to three classes.

(4): In Fig. 5, if z_1z_2 and z_3z_4 are two special paths, then the monochromatic path y_1y_2 is associated to the two classes, say $C_1 = \{z_1\}$ and $C_2 = \{z_3\}$.

We leave it to the reader to check that in almost all these cases, the treating below of the classes associated to a same monochromatic P_2 do not affect each other. The only case that we need to take special care of is for two conjugate special paths of type I. Special paths are coloured in groups, and each group consists four special paths of the same type, cf. Cases 4, 5 and 6 below. In case when two special paths of type I are conjugate, we always put them in the same group. This is because that in each group of four special paths of type I, for exactly one special path of type I, we need to change the top monochromatic P_2 into a monochromatic P_3, (see the fourth special path in Fig. 16(b)), and that monochromatic P_2 is associated to both conjugate special paths.

We have now finished our preparation work, and we start to colour the edges of T. First we transfer the colouring of Q into a partial colouring of T as follows:

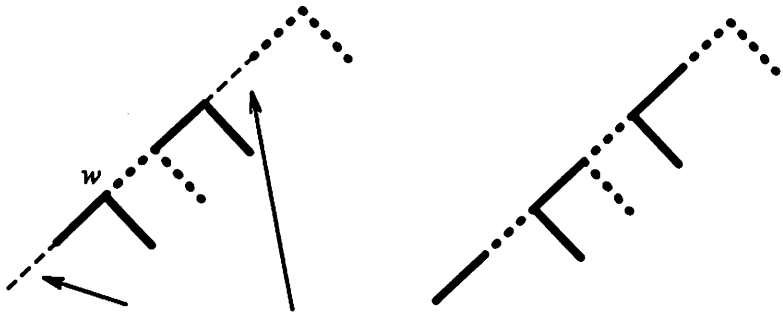
Let x, y be two edges that form a monochromatic P_2 in Q. Let v be their common vertex and let z be the other edge of T^* incident to v. Let $P(x) = vv_1 \cdots v_{||x||}, P(y) = vw_1 \cdots w_{||y||}$ and $P(z) = vu_1 \cdots u_{||z||}$. We colour the edges of $P(x)$ and $P(y)$ alternately by two colours with vv_1, vw_1 receiving the colour of x. If x or/and y is even then the edge $v_{||x||-1}v_{||x||}$ or/and the edge $w_{||y||-1}w_{||y||}$ will stay temporarily uncoloured. If $z \in Q$ then we will colour the edges of $P(z)$ when we consider the monochromatic path containing z, if $z \notin Q$ then we colour the edges of $P(z)$ only if $||z|| \geq 3$, and in this case we colour an even number of edges of $P(z)$ alternately by two colours, with the edge zu_1 receiving the opposite colour of x. The last edge or the last two edges (depending on whether z is odd or even) will stay permanently uncoloured. There are at most $O(logn)$ of such edges.

We treat each monochromatic P_2 of Q as above. In the end, we have coloured a subgraph of T such that each monochromatic component is either a P_1 or a P_2. The number of red P_1's is equal to the number of blue P_1's. Each monochromatic P_2's corresponds to a monochromatic P_2 of Q. Thus we can say that the total weight of red P_2's is equal to the total weight of blue P_2's.

We denote by U the set of the "temporarily" uncoloured edges. Note that each edge $x \in U$ is contained in an even edge of T'. The partition of the even edges of T' defined above induces a partition of U.

Note that we have removed the colour of some monochromatic P_2's of T' in order to balance the total weights of red P_2's and blue P_2's. However there are at most $O(logn)$ P_2's of T' for which the colour have been removed. Recall that each monochromatic P_2 is associated to at most three classes of the above partition. Thus there are at most $O(logn)$ classes C for which there is a monochromatic path $P \in C(P)$ whose colour has been removed. For uncoloured edges in such classes, we will let them remain uncoloured permanently. Obviously the total number of such uncoloured edges is at most $O(logn)$.

We now proceed to colour the other uncoloured edges in U. The discussions are divided into cases, and the solutions for the cases are given by figures. For the sake of simplicity, in all figures from Fig. 6 to Fig. 18, the odd edges of T' are depicted by a single edge, and even edges of T' are depicted by a path of length two. It is a matter of routine to check that the



a b

Figure 6

figures can be used as patterns also when the depicted edges of T' represent longer paths. In case the odd edge of T' must be actually a single edge of T, we state that fact explictly and explain why this must be the case. In case the odd edge of T' must be a path of length at least three, we also state that explictly, and give explanation if needed.

1) Classes of uncoloured edges formed by Procedure 1.

Each class contains either two edges or three edges. If a class contains two edges, then the uncoloured edges are as depicted in Fig. 6(a), and we re-colour the edges as in Fig. 6(b).

If a class contains three edges then the uncoloured edges are as depicted in Fig. 7(a), or 8(a), where w is a $(3,1)$-vertex with two even pending edges. Then $||y|| = 1$, for otherwise we would have a removable end. Depending on whether x and z have different colours or not, we divide this into two subcases.

Case 1. x and z have different colours, as depicted in Fig. 7(a).

We take two copies of such classes and re-colour the edges as in Fig. 7(b).

Case 2. x and y have the same colour, as depicted in Fig. 8(a).

We take two copies of such classes and re-colour the edges as in Fig. 8(b).

After this re-colouring, we see that if the original monochromatic subgraphs are isomorphic, the new monochromatic subgraphs are still isomorphic. In other owrds, we may have added some monochromatic paths of various lengths, we also may have destroyed some monochromatic paths of various lengths. However the same change happened to both monochromatic subgraphs. This is true for all the re-colouring process, except for the last three cases of the classes formed by Procedure 4. We will explain what happens there when we deal with those cases.

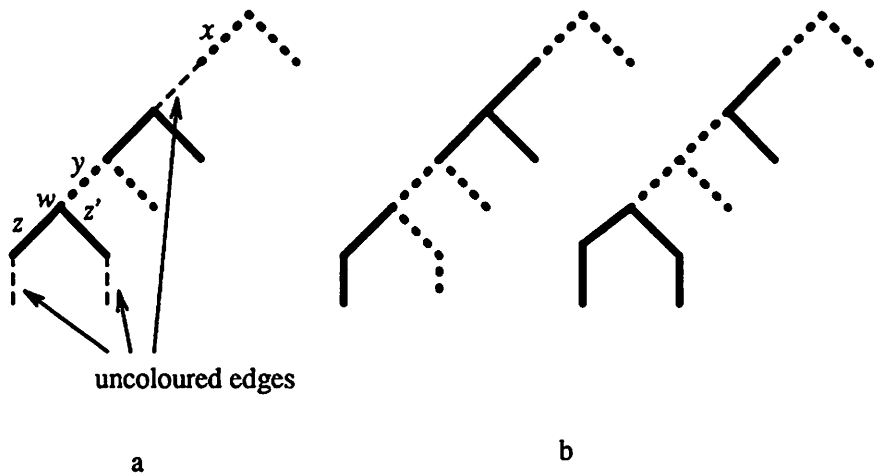


Figure 7

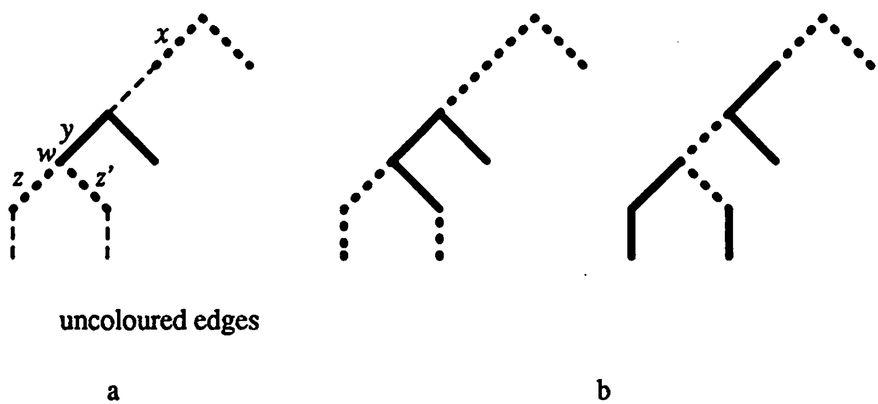


Figure 8

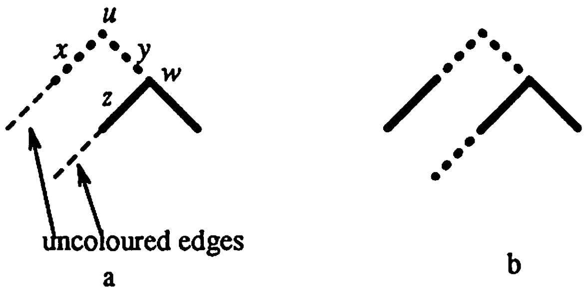


Figure 9

Note that since we need to take two copies of the same classes and colour together, it may happen that the total number of such classes is odd, and we cannot colour every such class this way. In this case, we just leave the uncoloured edges in one class uncoloured permanently.

2) Classes of uncoloured edges formed by Procedure 2.

Again each such class contains either two edges or three edges. If a class contains two edges, then the uncoloured edges are as depicted in Fig. 9(a), where u and w are $(3,2)$-vertices, and x and z are even pending edges. We colour the uncoloured edges as in Fig. 9(b).

Note that the edge y must be odd. For otherwise y and z should form a class by Procedure 1. We should point out here that y cannot form a class with a predecessor by Procedure 1, because any predecessor of y is also a prodecessor of x, and x has the priority to form a class with their predecessor by Procedure 1.

If a class contains three edges then the uncoloured edges are as depicted in Fig. 10(a), and we take four copies of such classes and re-colour the edges as in Fig. 10(b).

Note that in this case we must have $||y|| = 1$, for otherwise the monochromatic path containing y would be a removable end.

3) Classes of uncoloured edges formed by Procedure 3.

Again each such class contains either two edges or three edges. If a class contains two edges, then the uncoloured edges are as depicted in Fig. 11(a), where u is a $(3,2)$-vertex with an even pending edge z_1, w is a $(3,1)$-vertex with exactly one even pending edge. The vertex a cannot be a $(3,1)$ or $(3,2)$-vertex with even pending edge, for otherwise suppose z_3 is an even pending edge, then z_1 and z_3 should have formed a class by Procedure 2. The edge y must have $||y|| = 1$, for otherwise we would have a removable end.

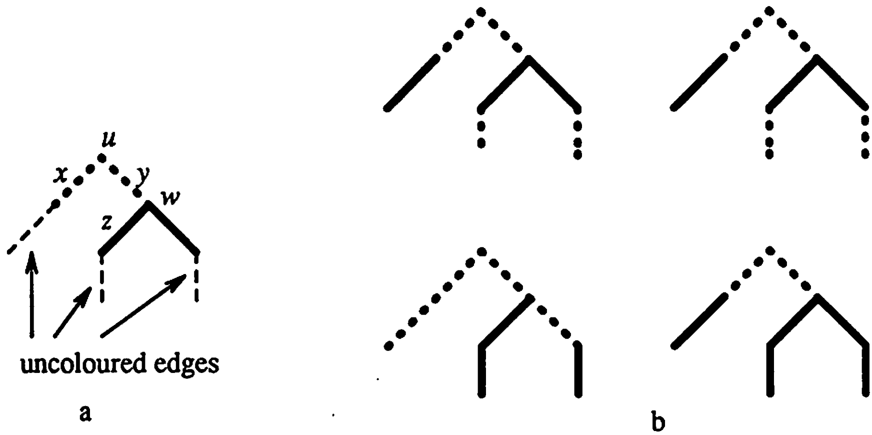


Figure 10

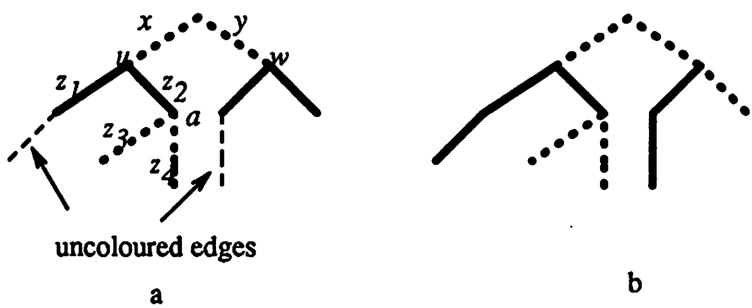


Figure 11

We then re-colour the edges as in Fig. 11(b).

If a class contains three edges then the uncoloured edges are as depicted in Fig. 12(a), and we take four copies of such classes and colour three copies as A and colour one copy as B in Fig. 12(b).

4) Classes of uncoloured edges formed by Procedure 4.

We divide this case into six subcases.

Case 1. The class contains a single edge as depicted in Fig. 13(a), where u is a $(3,1)$-vertex. We take two copies of such classes, and colour them as in Fig. 13(b).

Case 2. The class contains two uncoloured edges as depicted in Fig. 14(a), where u is a $(3,1)$-vertex. We take two copies of such classes, and colour them as in Fig. 14(b).

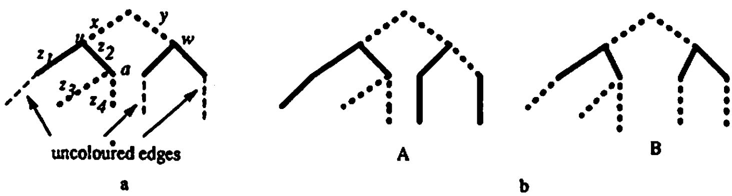


Figure 12

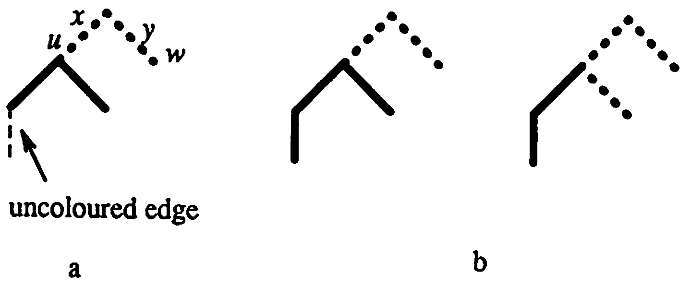


Figure 13

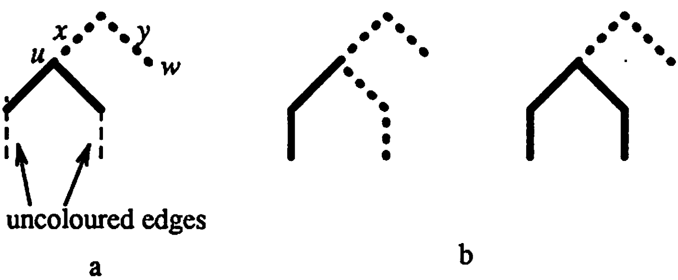


Figure 14

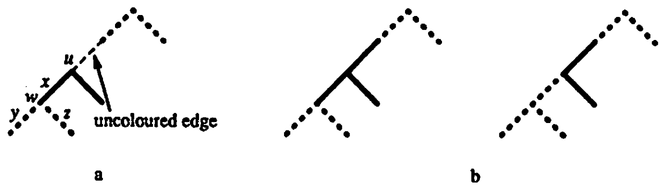


Figure 15

Observe that w cannot be a $(3,1)$-vertex, for otherwise the tree T would contain a removable end. The vertex w also cannot be a $(3,2)$-vertex with even pending edges, for otherwise this even edge and the even pending edge of u would have been put into one class by Procedure 3.

Case 3. The class contains a single uncoloured edge as depicted in Fig. 15(a), where x, y, z are all odd edges, for otherwise we would have formed a class by Procedure 1. We take two copies of such classes, and colour them as in Fig. 15(b).

Note that w cannot be a $(3,1)$-vertex, for otherwise T would have a removable end.

Case 4. The class contains a single uncoloured edge as depicted in Fig. 16(a), where xy form a special path of type I. We take four copies of such classes, and re-colour the edges as in Fig. 16(b).

As noted before, if two special paths of type I are conjugate, we always put them in the same group of four so that the colouring will not affect each other.

Unlike the previous cases, after the re-colouring, the change happened to the red subgraph is not the same as the change happened to the blue subgraph. (For a moment suppose that the dotted lines represent red edges and solid lines repredent blue edges). We count the losses and gains of monochromatic paths of various lengths of the two colours, we see that "net" loss is a red P_2. However xy is a special path of type I, the weight of this path is $\frac{3}{4}$. Therefore the four copies of such a path in the four classes are actually counted as three red P_2's in the weight function. Thus the loss of one red P_2 is exactly what we needed to make the red subgraph isomorphic to the blue subgraph.

Case 5. The class contains a single uncoloured edge as depicted in Fig. 17(a), where xy form a special path of type II. We take four copies of such

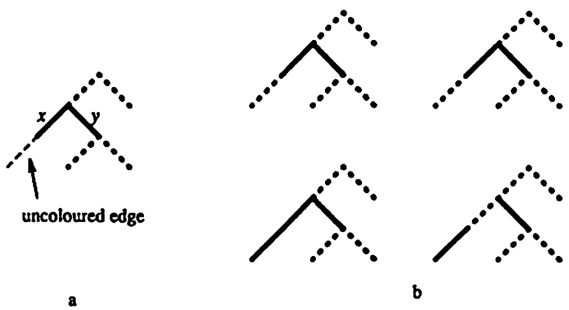


Figure 16

classes, and re-colour the edges as in Fig. 17(b).

Again, the "net" loss of the re-colouring is two red P_2's. This is why we assign weight $\frac{1}{2}$ to a special path of type II.

Case 6. The class contains a single uncoloured edge as depicted in Fig. 18(a), where xy form a special path of type III. We take four copies of such classes, and re-colour the edges as in Fig. 18(b).

The "net" loss of the re-colouring is two red P_2's. This is why we assign weight $\frac{1}{2}$ to a special path of type III.

Up to now we have coloured a subgraph of T, with at most $O(logn)$ edges of T uncoloured. Each monochromatic component of the coloured subgraph is a P_1, or a P_2, or a P_3, or a P_4, or a claw. It is easy to see from the process of colouring that the numbers of red P_1's, P_3's, P_4's and claws are equal to the number of blue P_1's, P_3's, P_4's and claws respectively.

For the number of monochromatic P_2's, we have that the total weight of red P_2's equal to the total weight of blue P_2's. Then through the process of treating special monochromatic paths, cf. Cases 4, 5 and 6, we see that the weights are transformed into actual numbers of copies of red P_2's and blue P_2's. However there could be some (at most $O(logn)$) special monochromatic P_2's which we did not treat.

There are two circumstances under which we did not treat a special monochromatic path P_2:

(1): Recall that we removed the colour of some monochromatic P_2's of T' in order to balance the weights of red P_2's and blue P_2's. If such a monochromatic P_2 is associated to a class C of even edges, then that class is not treated. If such a class C consists of a single even edge belonging to

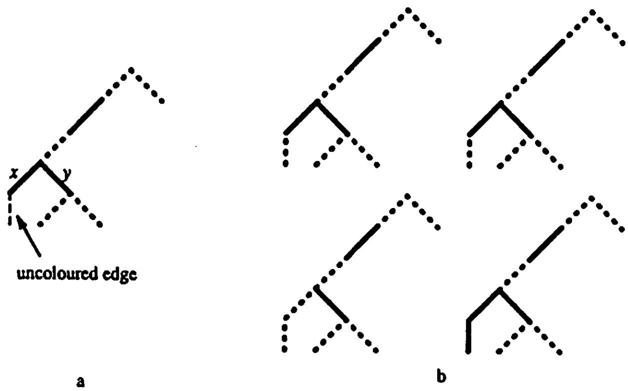


Figure 17

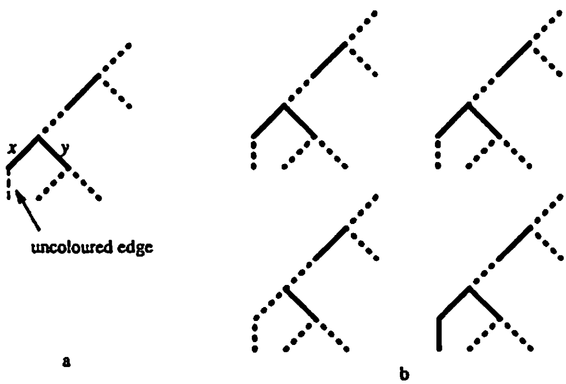


Figure 18

a special path, then the weight of that special path did not transform into actual numbers of copies of monochromatic P_2's. Therefore that monochromatic P_2 is still counted as $\frac{3}{4}$ or $\frac{1}{2}$ monochromatic P_2's. This would cause the number of red P_2's be different from the number of blue P_2's. However there are at most $O(logn)$ such classes C. Therefore the difference between the number of red P_2's and blue P_2's caused by these classes is at most $O(logn)$.

(2): In the treating of classes consisting a single even edge belonging to a special monochromatic path, cf. cases 4, 5 and 6, we always treat 4 classes simultaneously. If the number of that type of classes is not a multiple of 4, then there could be 1, 2, or 3 classes remain untreated. It is easy to see that there are at most 9 such untreated classes for each colour, and therefore the difference between the number of red P_2's and blue P_2's caused by these classes is at most 9.

Summing up the discussion above, we have that the difference between the number of red P_2's and blue P_2's is at most $O(logn)$. Thus we can remove the colour of some (at most $O(logn)$) monochromatic P_2's, and the resulting coloured subgraph of T, has two isomorphic monochromatic subgraphs. Moreover the number of uncoloured edges of T is at most $O(logn)$. This completes the proof of Theorem 5, as well as Theorem 1.

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