Existence of BIBDs with Block Size 8 and $\lambda = 7$

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Abstract. It is shown that the obvious nesessary condition for the existence of a $B(8,7;v)$ is sufficient, with the possible exception of $v \in \{48,56,96,448\}$.

1. Introduction

Let v, k and λ be positive integers. A *balanced incomplete block design* (BIBD) with parameters v, k and λ, denoted by $B(k,\lambda;v)$, is a pair $(X, \mathbf{A})$ where X is a v-set (of points) and $\mathbf{A}$ is a collection of k-subsets of X (called *blocks*) such that every pair of distinct points of X is contained in exactly λ blocks of $\mathbf{A}$. The notation $B(k,\lambda)$ denotes the set of all integers v for which there exists a $B(k,\lambda;v)$.

The obvious necessary condition for the existence of a $B(k,\lambda;v)$ is $\lambda(v-1) \equiv 0 \mod (k-1)$ and $\lambda v(v-1) \equiv 0 \mod k(k-1)$. When $\lambda = k-1$, this implies that $v(v-1) \equiv 0 \mod k$. For $3 \le k \le 7$, it was proved in [4] that this condition is also sufficient. The following result was also obtained in [4].

Lemma 1.1. $\{8,9,16,17,24,25,32,33,40,41\} \subset B(8,7)$.

In this paper, we are concerned mainly with the existence of a $B(8,7;v)$, where the necessary condition for existence is $v \equiv 0$ or $1 \mod 8$. We show that this condition is also sufficient, with the possible exception of $v \in \{48,56,96,448\}$.

2. Constructions

To obtain the required designs we employ some old and new constructions listed below. Fundamental to these constructions is a number of other designs which we define now.

Let X be a finite set (of points), λ a positive integer. A *group divisible design* (GDD) of index λ is a triple $(X, \mathbf{G}, \mathbf{A})$, where

1) $\mathbf{G}$ is a collection of subsets of X (called *groups*) which partition X,
2) $\mathbf{A}$ is a collection of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and
3) every pair of points from distinct groups occurs in exactly λ blocks of $\mathbf{A}$.

The *group-type* (or *type*) of a GDD is a listing of the group sizes using so-called "exponential" notation, i.e. $1^i 2^j 3^k \ldots$ denotes i groups of size 1, j groups of size 2, etc. Let K be a set of positive integers. We say that a GDD is a (K,λ)-GDD if $|A| \in K$ for every block A of $\mathbf{A}$. When $K = \{k\}$, the design is simply denoted by (k,λ)-GDD.

Three particular GDDs of which we will make use need to be mentioned. A (K,λ)- GDD of type 1^v is referred to as a *pairwise balanced design* (PBD), denoted by $B(K,\lambda;v)$. A $(k,1)$-GDD of type m^k is referred to as a *transversal design* (TD), denoted by $TD(k,m)$. A (k,λ)-GDD of type $1^u w^1$ is defined to be an *incomplete* BIBD, denoted by $IB(k,\lambda;u+w,w)$. The group of size w is thought of as a hole. Intuitively, an incomplete BIBD is a BIBD from which a sub-BIBD is missing (this is the hole). We wish to remark that a BIBD may be viewed as an incomplete BIBD with a hole of size 0 or 1. For more detailed information on PBDs and related designs, the interested reader may refer to [1,4,6].

We also require the notion of a resolvable BIBD. A resolvable $B(k,\lambda;v)$ briefly, $RB(k,\lambda;v)$) is defined to be a $B(k,\lambda;v)$ in which the block set can be partitioned into classes (called *parallel classes*) such that every point of the design occurs precisely once in each class.

Lemma 2.1. ([5]). *Let $m = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ be the factorization of m into powers of distinct primes p_i. Then a $TD(k,m)$ exists where $k \le 1 + \min\{p_i^{k_i}\}$.*

Lemma 2.2. ([3]). *If $n \in \{8,15,29,36,43,50,57,64,71,85\}$, then there exists an $RB(8,1;8n)$.*

We now give our constructions. We often construct BIBDs from GDDs by filling in the groups as follows (see, for example, [6]).

Lemma 2.3. *Let w be a non-negative integer. Suppose that the following designs exist:*

1) *a (k,λ)-GDD of type $\{t_1,t_2,\ldots,t_n\}$,*
2) *an $IB(k,\lambda;t_i+w,w)$, for $1 \le i \le n-1$, and*
3) *a $B(k,\lambda;t_n+w)$.*

Then there exists a $B(k,\lambda;t+w)$, where $t = \sum t_i$.

For our purpose, we mention a special form of Lemma 2.3 below.

Lemma 2.4. *Let q be a prime power not less than 8 and w a non-negative integer. Suppose that the following designs exist:*

1) *an $IB(8,7;q+w,w)$, and*
2) *a $B(8,7;u+w)$ where $0 \le u \le q$.*

Then there exists a $B(8,7;8q+u+w)$.

Proof. From Lemma 2.1, a $TD(9,q)$ exists. Delete $q-u$ points from one group of a $TD(9,q)$, and then break the blocks of the resulting design by a $B(8,7;8)$ or a $B(8,7;9)$ (see Lemma 1.1). The result is an $(8,7)$-GDD of type q^8u^1. The conclusion then follows from Lemma 2.3.

Lemma 2.5. *Let n be a positive integer such that $6n+1$ is a prime power. Then there exists an $IB(8,7;7n+1,n)$.*

Proof. Let point set $X = [\mathrm{GF}(6n+1)] \cup \{\infty_0, \infty_1, \ldots \infty_{n-1}\}$, and let x be a primitive element of $\mathrm{GF}(6n+1)$. It was shown in [4, Lemma 4.2] that $\{B_i : 0 \leq i \leq n-1\}$ forms a $(6n+1, 7, 7)$ difference family based on the additive group of $\mathrm{GF}(6n+1)$ where

$$B_i = \{0, x^i, x^{i+n}, x^{i+2n}, x^{i+3n}, x^{i+4n}, x^{i+5n}\}, \qquad \text{for } 0 \leq i \leq n-1.$$

Let $\mathbf{D} = \{\{\infty_i\} \cup B_i : 0 \leq i \leq n-1\}$. The proof that $(X, \mathrm{dev}\mathbf{D})$ is indeed an $\mathrm{IB}(8, 7; 7n+1, n)$ is by standard difference set techniques, where $\mathrm{dev}\mathbf{D} = \{D + g : g \in \mathrm{GF}(6n+1) \text{ and } D \in \mathbf{D}\}$.

The following lemma is the variation and modification of Theorem 3.7 in [2].

Lemma 2.6. *Let q be an odd prime power not less than 9. Then there exists an $(8, 7)$-GDD of type 8^q.*

Proof. Suppose that x is a primitive element of $\mathrm{GF}(q)$. We construct an $(8, 7)$-GDD of type 8^q as follows. Let the point set be $\mathrm{GF}(q) \times (Z_7 \cup \{\infty\})$ and group set be $\{\{g\} \times (Z_7 \cup \{\infty\}) : g \in \mathrm{GF}(q)\}$. Then the $8q(q-1)$ blocks are obtained when the following base blocks are developed in the first component under the additive group of $\mathrm{GF}(q)$:

$$\{(x^i, \infty), (x^{i+1}, 0), (x^{i+2}, 1), (x^{i+3}, 3),$$
$$(-x^i, \infty), (-x^{i+1}, 0), (-x^{i+2}, 1), (-x^{i+3}, 3)\} \bmod (-, 7)$$

$$\{(x^i, 0), (x^{i+1}, 1), (x^{i+2}, 2), (x^{i+3}, 4),$$
$$(-x^i, 0), (-x^{i+1}, 1), (-x^{i+2}, 2), (-x^{i+3}, 4)\} \bmod (-, 7)$$

$$\{(x^{j+1}, \infty), (x^{j+2}, 0), (x^{j+3}, 1), (x^{j+4}, 2),$$
$$(x^{j+5}, 3), (x^{j+6}, 4), (x^{j+7}, 5), (x^{j+8}, 6)\}$$

where i runs from 0 to $(q-3)/2$ and j runs from 0 to $q-2$.

Lemma 2.7. *Let n be a positive integer relatively prime to $210 = 2.3.5.7$. Then there exists an $(8, 7)$-GDD of type 8^n.*

Proof. Let $n = p_1 p_2 \ldots p_r$ be the factorization of n into primes that $p_1 \leq p_2 \cdots \leq p_r$, and let $Q_i = p_i p_{i+1} \ldots p_r$. Start with a $\mathrm{TD}(p_1, Q_2)$ and successively partially break up the groups of size Q_i using a $\mathrm{TD}(p_i, Q_{i+1})$. Finally, fill with blocks of size $Q_r = P_r$ to produce a $\mathrm{B}(\{p_1, p_2, \ldots, p_r\}, 1; n)$. The condition that n is relatively prime to 210 implies that $p_j (1 \leq j \leq r)$ is an odd prime not less than 11. Give weight 8 to every point of the PBD and apply the Fundamental Construction (see [6]) with the necessary input designs from Lemma 2.6 to obtain an $(8, 7)$-GDD of type 8^n.

Lemma 2.8. *Suppose that an* RB$(8,1;v)$ *exists, that* $u \in B(8,7)$ *or* $u = 0$, *and that* $u \leq (v-1)/7$. *Then* $v+u \in B(8,7)$.

Proof. For $u = 0$, the proof is straightforward. Now assume $u > 0$. Take an RB$(8,1;v)$ and adjoin u infinite points to u of its parallel classes of blocks, where one infinite point is adioined to each of u parallel classes. This creates a PBD $B(\{8,9,u\},1;v+u)$. Since $\{8,9,u\} \subset B(8,7)$, we can then break up the blocks of the PBD to show that $v+u \in B(8,7)$.

Lemma 2.9. *If there is an* IB$(8,7;v+u,u)$ *and* $8u+e \in B(8,7)$, *where* $e = 0$ *or* 1, *then* $8u+8v+e \in B(8,7)$.

Proof. By definition, an IB$(8,7;v+u,u)$ is an $(8,7)$-GDD of type $1^v u^1$. Give weight 8 to every point of an IB$(8,7;v+u,u)$. Since a TD$(8,8)$ exists by Lemma 2.1, the Fundamental Construction (see [6]) gives us an $(8,7)$-GDD of type $8^v(8u)^1$. The required result then follows from Lemma 2.3 and the fact that $\{8+e, 8u+e\} \subset B(8,7)$.

3. Main result

In this section,we work towards determining the set $B(8,7)$.

Lemma 3.1. *If* n *is a positive integer relatively prime to 210, then* $\{8n, 8n+1\} \subset B(8,7)$

Proof. This is immediate consequence of Lemmas 2.3 and 2.7.

Lemma 3.2. $\{49,57,97,449\} \subset B(8,7)$.

Proof. For $v = 57$, the result follows from adding one infinite point to every group of a TD$(8,7)$. For the other values of v, a $B(8,7;v)$ was constructed by H. Hanani [4, Lemma 4.1].

Lemma 3.3. *If* $s \in \{8,9,15-17,29-33,36-41,43-48,50-55,57-62,64-69,71-76,79,80,85-90\}$, *then* $\{8s, 8s+1\} \subset B(8,7)$.

Proof. In view of Lemma 1.1, we can apply Lemma 2.8 with resolvable BIBDs from Lemma 2.2 to establish the lemma.

Lemma 3.4. *If* $s \in \{13,14,19-21,26-28,34,35,42,63,70,77-78,81-84\}$, *then* $\{8s, 8s+1\} \subset B(8,7)$.

Proof. For $s = 13$ or 19, the result follows from Lemma 3.1. For the remaining values of s, we apply Lemma 2.4 with $q \in \{13,19,25,31,37,61,67,73,79\}$ and $w = (q-1)/6$ such that $u+w \in \{8,9,16,17,24,25,32,33,40,41\}$. This guarantees that the conclusion holds because of Lemmas 1.1 and 2.5.

Lemma 3.5. *If* $s \in \{10, 11, 18, 22, 23, 24, 25, 49\}$, *then* $\{8s, 8s+1\} \subset B(8,7)$.

Proof. For $s = 10$, the result follow from deleting e points from one group of a TD(9,9), where $e = 0$ or 1. Take $v = 19$ and $u = 3$ in Lemma 2.9. This takes care of the case $s = 22$, where an IB(8,7;22,3) exists by Lemma 2.5. Lemma 1.1, 3.2 and 2.4 work for the cases where $s \in \{18, 24, 25, 49\}$. The remaining cases where $s = 11$ and 23 are contained in Lemma 3.1.

Lemma 3.6. *Suppose that a* TD(11, t) *exists and* $0 \leq a, b, c \leq t$. *Then* $8(8t + a + b + c) + e \in B(8,7)$ *if* $\{8t+e, 8a+e, 8b+e, 8c+e\} \subset B(8,7)$, *where* $e = 0, 1$.

Proof. In a TD(9,9), we delete one point to obtain a (9,1)-GDD of type 8^{10}. We then break the block of the resulting GDD by a B(8,7;9) to obtain a (8,7)-GDD of type 8^{10}. From Lemmas 2.1 and 2.6 we have also (8,7)-GDD of type 8^8, 8^9 and 8^{11}. In a TD(11, t), we delete $t - a$, $t - b$ and $t - c$ points from three group respectively to obtain an $(\{8, 9, 10, 11\}, 1)$-GDD of type $t^8 a^1 b^1 c^1$. Now we apply the Fundamental Construction with weight 8 to the last GDD. This creates an (8,7)-GDD of type $(8t)^8 (8a)^1 (8b)^1 (8c)^1$. The result then follows from Lemma 2.3.

We are now in the position to give our main result.

Theorem 3.1. *For every positive integer* $v \equiv 0$ *or* 1 mod 8, *with the possible exception of* $v \in \{48, 56, 96, 448\}$, *there exists a* B(8,7; v).

Proof. From lemmas 3.2–3.5 and 1.1, we know that the conclusion holds for $v \leq 721$. Note that for the sequence of integers $n, n+1, n+2, \ldots, n+9$ at least one integer is relatively prime to 210. Simple calculation shows that each value of $v \geq 721$ can be written in the form $8(8t + a + b + c) + e$ where $e = 0$ or 1 and a, b, c and t are chosen so that

1) t is a positive integer relatively prime to 210,
2) $\{8a+e, 8b+e, 8c+e\} \subset B(8,7)$, where $0 \leq a, b, c \leq t$.

Now apply Lemma 3.6 and the result follows,where we used Lemmas 2.1, 3.1 and already determined values.

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