Alternating cycles through fixed vertices in edge-colored graphs

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ABSTRACT. In an edge-colored graph, a cycle is said to be alternating, if the successive edges in it differ in color. In this work, we consider the problem of finding alternating cycles through p fixed vertices in k-edge-colored graphs, $k \geq 2$. We first prove that this problem is NP-Hard even for $p = 2$ and $k = 2$. Next, we prove efficient algorithms for $p = 1$ and k non fixed, and also for $p = 2$ and $k = 2$, when we restrict ourselves to the case of k-edge-colored complete graphs.

1 Introduction

The notion of alternating cycles, i.e. cycles such that no consecutive edges have the same color, was originally introduced by B. Bollobas and P. Erdös in [3]. In the same paper, the authors present conditions on colored degrees, sufficient for the existence of alternating hamiltonian cycles in edge-colored complete graphs. Results in almost the same vein are proved in [1,4]. For other works on alternating cycles and paths, the reader is encouraged to consult [2,5,10,11,14].

The aim of this work is to establish some further results on the existence of alternating cycles through a fixed set of p vertices. We recall that the analogous non colored version of this problem is polynomial for graphs for p fixed [13] and NP-complete otherwise [8]. It is also NP-complete for digraphs [7] even for $p = 2$.

Formally, in what follows, G^c denotes a k-edge-colored graph, of order n, with vertex-set $V(G^c)$ and edge-set $E(G^c)$, $k \geq 2$. If A and B denote subsets of $V(G^c)$, then $E(AB)$ denotes the set of edges between A and B, i.e. edges with one extremity in A and the other in B. Whenever the edges between A and B are monochromatic, then their color is denoted by $c(AB)$. If $A = \{x\}$ and $B = \{y\}$, then for simplicity we write xy (resp. $c(xy)$) instead of $E(AB)$ (resp. $c(AB)$). If v is a vertex of K_n^c and c_i is a color, then we define $\Gamma_{c_i}(v) = \{u \mid u \in V(K_n^c) - v \text{ and } c(vu) = c_i\}$. The c_i-degree of v is defined as $\mid \Gamma_{c_i}(v) \mid$ and is denoted $c_i(v)$. Whenever G^c is complete, then it is denoted by K_n^c. A bipartite tournament is an oriented complete bipartite graph.

In this work, we consider the following two problems:

Problem 1. *How easy is it to find an alternating cycle through p fixed vertices $x_1, x_2, ..., x_p$ in a k-edge-colored graph G^c or else to decide that such a cycle does not exist.*

Problem 2. *How easy is it to find an alternating cycle through p fixed ordered vertices $x_1, x_2, ..., x_p$ in a k-edge-colored graph G^c or else to decide that such a cycle does not exist?*

In the first result of this paper, we prove that both Problems 1 and 2 are NP-hard, even if $p = k = 2$. Next we study these problems by restricting ourselves to the case of edge-colored complete graphs. In particular, we give an $O(n^2)$ algorithm for finding an alternating cycle (if any) through a fixed vertex in a k-edge colored complete graph, for k non fixed, $k \geq 2$. We also give another $O(n^2)$ algorithm for finding an alternating cycle through two fixed vertices in a 2-edge-colored complete graph. This last algorithm improves an $O(n^3)$ algorithm of [12], since, as a corollary, we show that it can be used in order to find a directed cycle (if any) through two fixed vertices in a bipartite tournament.

2 Main results

We start this section by proving that the problem of finding an alternating cycle through two specified vertices in a 2-edge-colored graph G^c is NP-hard.

Theorem 2.1. *Deciding if there exists an alternating cycle through two fixed vertices in an edge-colored complete graph is NP-hard.*

Proof: The reduction is from the following "local cycle problem" (LC) : Given a directed graph D and two specified vertices x, y of D, decide if there is a cycle containing x and y in D. LC is known to be NP-hard [8]. Moreover, LC remains NP-hard even if D is restricted to be a bipartite graph. To see this, it suffices to add an intermediate vertex on each arc of D in order to obtain a directed bipartite graph.

Now consider any bipartite instance $D(X, Y)$ of LC, where X and Y are the bipartitions of D. Replace each XY-arc (YX-arc) by a red (blue) edge. Clearly, an alternating cycle containing x and y in the resulting edge-colored graph corresponds to a cycle through x, y in D. □

In the remaining part of this section, we study Problems 1 and 2 by restricting ourselves to the case of edge colored complete graphs. In paticular, we give a positive answer to these problems, for p=1, and for k=2, p=2 in Theorems 2.3 and 2.6, respectively.

The following Lemma will be used in the proof of Theorem 2.3.

Lemma 2.2.. *Let x be a fixed vertex in a k-edge-colored complete graph K_n^c. There exists an alternating cycle containing x, if and only if there exists an alternating cycle of length three or four containing x in K_n^c.*

Proof.: If there exists an alternating cycle of length three or four containing x, then, obviously, there exists an alternating cycle containing x in K_n^c.

Conversely, assume that there exists an alternating cycle $C : xx_1x_2...x_px$ through x in K_n^c. Choose C to be a shortest possible cycle through x. If $p \leq 3$, there is nothing to prove. Assume therefore $p \geq 4$. from now on, we suppose without loss of generality that the edge xx_1 is colored q and x_px is colored f. Since C is alternating, clearly $q \neq f$.

Assume first that for some i, $1 \leq i \leq p-1$, $c(x_ix_{i+1})$=t, where $t \neq q$ and $t \neq f$. Since C is alternating, both the edges $x_{i-1}x_i$ and $x_{i+1}x_{i+2}$ are colored by colors other than t. Throughout, for technical reasons we can suppose that $i \neq p-1$, for otherwise we may consider the cycle obtained from C by interchanging q and f and having opposite orientation. Let r be the color of xx_{i+1}. If $r \neq q$ and $r \neq f$, then either $xx_{i+1}x_{i+2}...x_{p-1}x_px$ or $xx_{i+1}x_ix_{i-1}...x_2x_1x$ is a cycle shorter than C, depending upon if r is the color of $x_{i+1}x_{i+2}$ or not. If $r = f$, then $xx_{i+1}x_ix_{i-1}...x_2x_1x$ is again a cycle shorter than C. Finally, if $r = q$, then, the cycle $xx_{i+1}x_{i+2}...x_{p-1}x_px$ is shorter than C unless the color of $x_{i+1}x_{i+2}$ is q. Assume therefore that $c(x_{i+1}x_{i+2}) = q$. If $c(xx_{i+2}) \neq q$ (resp. $c(xx_{i+2}) = q$), then the cycle $xx_{i+2}x_{i+1}x_i...x_2x_1x$ (resp. $xx_{i+2}x_{i+3}...x_{p-1}x_px$) is shorter than C, a contradiction to the choice of C.

Assume next that all edges of C are in colors f and q. Clearly, since C is colored by two colors, its length is even and therefore p is odd. We look now if there exists a smallest even integer i, $1 \leq i < \frac{p-1}{2}$, such that the edge xx_{2i+1} is in a color other than q. If i exists, then it follows from its minimality that the color of $xx_{2(i-1)+1}$ is q and consequently the cycle $xx_{2(i-1)+1}x_{2(i-1)+2}x_{2i+1}x$ is of length four through x, a contradiction to the choice of C. If, on the other hand, the integer i does not exist, it follows that the edge xx_{p-2} is colored q and therefeore, once more, the alternating cycle $xx_{p-2}x_{p-1}x_px$ contains x; also a contradiction. This completes the proof. □

Theorem 2.3.. *Let x be a vertex in a k-edge-colored complete graph K_n^c. There exists an alternating cycle containing x in K_n^c, if and only if (i) or (ii) below holds.*

(i) There are two colors i and j and two vertices u and v so that $u \in \Gamma_i(x), v \in \Gamma_j(x)$, $i \neq j$ and $c(uv) \neq i$, $c(uv) \neq j$.

(ii) For some colors i, j, q and h, $i \neq j$, $q \neq i$, $h \neq j$, there exist two vertices w and z in $\Gamma_i(x)$ and a vertex s in $\Gamma_j(x)$ such that $c(wz) = q$ and $c(ws) = h$.

Proof: If either (i) or (ii) is satisfied, respectively, then clearly there exists a cycle of length three or four through x.

Conversely, assume that there exists an alternating cycle C through x in K_n^c. By Lemma 2.2, we can consider that this cycle has length three or four. Now, if its length is three, then (i) is satisfied. Assume therefore that C has length four. Set $C : xx_1x_2x_3x$. Suppose that $c(xx_1) = i$, $c(x_1x_2) = q$, $c(x_2x_3) = h$ and $c(x_3x) = j$. Now, if $c(xx_2) \neq i$ and $c(xx_2) \neq j$, then either xx_2x_3x or xx_2x_1x is shorter than C, a contradiction to the fact that C is as short as possible. It follows that the edge xx_2 is either in color i or in color j. Consequently, the conclusion of (ii) is satisfied by identifying z, w and s by x_1, x_2 and x_3 or x_3, x_2, x_1 respectively. This completes the proof of the theorem. □

We notice that from Lemma 2.2 and Theorem 2.3, an $O(n^2)$ algorithm can easily be derived in order to find an alternating cycle through a given vertex or else to decide that such a cycle does not exist in K_n^c.

In Theorem 2.6 given later, we prove an $O(n^2)$ algorithm for finding an alternating cycle through two fixed vertices in a 2-edge-colored complete graph. In order to facilitate the discussion, the following notation will be useful.

Notation: Let x and y be two specified vertices in K_n^c. We let P_1 : $(x =)q_0q_1q_2...q_mq_{m+1}(= y)$ and P_2 : $(x =)f_0f_1f_2\ldots f_lf_{l+1}(= y)$ denote two alternating paths (if any) between x and y satisfying the following properties :

i) P_1 and P_2 are internally vertex-disjoint,

ii) $c(xq_1) \neq c(xf_1)$, $c(xq_1) = c(f_ly) = c(xy)$ and $c(q_my) \neq c(f_ly)$ and

iii) subject to (i) and (ii), the sum of the lengths of P_1 and P_2 is as small as possible.

In what follows we suppose without loss of generality that P_1 is no longer than P_2.

With the notation above, the following lemma is trivial.

Lemma 2.4. *There is an alternating cycle containing x and y if and only if either there is an alternating path $P: xp_1p_2...p_iy$ from x to y such that $c(xp_1) \neq c(xy)$ and $c(p_iy) \neq c(xy)$ or else both P_1 and P_2 exist in K_n^c.*

Lemma 2.5. *Let x and y be two fixed vertices in a k-edge-colored complete graph K_n^c, $k \geq 2$. Assume that there is no alternating cycle containing the edge xy, and that both the paths P_1 and P_2 exist between x and y in K_n^c. If the length of P_1 is at least four, then the length of P_2 is four.*

Proof: Assume by contradiction that the length P_1 is at least four and the length of P_2 is at least five, for otherwise we have finished. Suppose without loss of generality that $c(xy) = 1$.

Assume first that there is an edge $q_{i-1}q_i$, $i > 2$, such that $c(q_{i-1}q_i) = 1$. If $c(xq_i) = 1$, then the path $xq_iq_{i+1}...q_my$ is shorter than P_1, a contradiction to the choice of P_1. Assume therefore that $c(xq_i) \neq 1$. If $c(yq_{i-2}) \neq 1$, then if $c(yq_{i-2}) \neq c(q_{i-2}q_{i-3})$, then the path $xq_1q_2...q_{i-2}y$ is shorter than P_1, and if $c(yq_{i-2}) = c(q_{i-2}q_{i-3})$, then the alternating cycle $xq_iq_{i-1}q_{i-2}yx$ contains the edge xy, in both cases a contradiction to our assertions. Consequently, in what follows assume that $c(yq_{i-2}) = 1$.

If some edge f_if_{i+1}, $i < l-1$, is colored 1, then, by interchanging x, y on P_2 and by using the above arguments, we can conclude that $c(yf_i) \neq 1$ and $c(xf_{i+2}) = 1$. Now by setting $P_2 : xq_iq_{i-1}q_{i-2}y$ and $P_1 : xf_{i+2}f_{i+1}f_iy$ we obtain a contradiction to the choice of P_1 and P_2.

Assume on the other hand, that no edge f_if_{i+1}, $i < l-1$, is in color 1. Let $c_\ell \neq 1$ be the color of an edge f_if_{i+1} for some $1 < i < l-1$. Then $c(xf_i) \neq c_\ell$ for otherwise, the path $xf_if_{i+1}\cdots y$ contradits the minimality of P_2. Moreover, $c(yf_i) = c(f_{i-1}f_i$, for otherwise, the cycle $xyf_if_{i-1}f_{i-2}\cdots x$ contains the edge xy, a contradiction to the hypothesis of the lemma. It follows that the cycle $xf_if_{i-1}yx$ contains the edge xy, a contradiction again.

Assume next that there is no edge q_iq_{i+1} on P_1, such that $c(q_iq_{i+1}) = 1$, for all i, $2 \leq i < m$. The proof of this case is essentially based on the following claim.

Claim. *If for all $2 \leq i < m$, there is no edge q_iq_{i+1} on P_1 such that $c(q_iq_{i+1}) = 1$, then $c(xq_i) = c(q_iq_{i+1})$ and $c(q_iy) = 1$.*

Proof of the claim: If for some $2 \leq i < m$, $c(xq_i) \neq c(q_iq_{i+1})$, then, if $c(xq_i) = 1$, then $xq_iq_{i+1}...q_my$ is shorter than P_1 else if $c(xq_i) \neq 1$, then the alternating cycle $xq_iq_{i+1}...q_myx$ contains the edge xy; a contradiction to the hypothesis of the lemma. Furthermore, if $c(yq_i) = c(q_iq_{i-1})$, then the cycle xq_iyx contains the edge xy, again a contradiction to the hypothesis of the lemma. On the other hand, if $c(yq_i) \neq c(q_iq_{i-1})$ and if $c(yq_i) \neq 1$, then the path $xq_1q_2...q_iy$ is shorter than P_1. It follows that $c(q_iy) = 1$ as claimed. This completes the proof of Claim.

Now, if no edge f_if_{i+1}, $1 < i \leq l-1$ on P_2, satisfies $c(f_if_{i+1}) = 1$, then by changing the orientation of P_2 and exchanging x by y, it follows from the above claim that $c(xf_i) = 1$ and $c(f_iy) = c(f_{i-1}f_i)$, for all $i = 2, ..., l-1$. By setting $P_1 : xf_iy$ and $P_2 : xq_jy$, for some i, j, $2 \leq j < m$ and $2 \leq i \leq l-1$, we obtain a contradiction with the choice of P_1 and P_2. On the other hand, if for some i, $2 \leq i \leq l-1$, we have $c(f_if_{i+1}) = 1$, then if $c(xf_i) \neq 1$, then the path $xf_if_{i+1}f_{i+2}...f_ly$ is shorter than P_2, and if $c(f_iy) = 1$, then $xf_1f_2...f_iy$ is again shorter than P_2, in both cases a contradiction to the choice of P_2. Assume therefore that $c(xf_i) = 1$ and $c(f_iy) \neq 1$. Now, if $c(f_iy) \neq c(f_{i-1}f_i)$, then the cycle $xf_1f_2...f_iyx$ contains the edge xy and if $c(f_iy) = c(f_{i-1}f_i)$, then by setting $P_1 : xf_iy$ and $P_2 : xq_jy$, for some j, $2 \leq j \leq m$, in both cases we obtain a contradiction either to the hypothesis of the lemma or to the choice of P_1 and P_2. This completes the proof of the lemma. □

Theorem 2.6. *There is an $O(n^2)$ algorithm for finding a shortest alternating cycle through two given vertices x and y in a 2-edge-colored complete graph K_n^c or else deciding that such a cycle does not exist.*

Proof: Assume for simplicity that the edges of K_n^c are colored red (r) and blue (b). Furthermore, suppose without loss of generality that $c(xy) = r$. Now set now $R = \Gamma_r(x) \cap \Gamma_r(y)$, $B = \Gamma_b(x) \cap \Gamma_b(y)$, $Q = \Gamma_r(x) \cap \Gamma_b(y)$ and $H = \Gamma_b(x) \cap \Gamma_r(y)$.

We notice that if $Q \neq \emptyset$ and $H \neq \emptyset$, then an alternating cycle of length four containing x and y can be easily found. Consequently, in what follows assume that either Q or H, for example H is empty. We distinguish between two cases depending upon Q.

First case. *Q is empty.*

If there is a red edge, say wz in B, then the alternating cycle $xwzyx$ contains both x and y. Assume therefore that each edge in B is blue. Now by the structure of K_c^n in connection with Lemma 2.4, we can conclude that the desired alternating cycle exists in K_n^c if and only if (i) or (ii) below holds.

(i) There is an edge wz in R and two distinct vertices s and t in B such that $c(wz) = b$ and $c(ws) = r = c(zt)$. In this case, the cycle $xyswztx$ is alternating and contains both x and y.

(ii) There are two non-adjacent edges w_1z_1 and w_2z_2 in R such that $c(w_1z_1) = b$, $c(w_2z_2) = b$ and two distinct vertices s and t in B such that $c(sw_1) = r = c(tw_2)$. In this case, the alternating cycle $xsw_1z_1ytw_2z_2x$ goes through x, y.

Second case. *Q is not empty.*

Let us first point out the facts (i)-(iii) below.

(i) As in the First case, we can suppose that each edge inside B is blue.

(ii) If there is a blue edge wz in R and a red edge zu between R and B, then the cycle $xuzwyvx$ is as desired, where v denotes a vertex of Q.

(iii) If there is a red edge wz, $w \in B$, $z \in Q$, then the cycle $xwzyx$ is as desired.

Now we show that the problem of finding an alternating cycle through x and y can be reduced to a shortest path problem from x to y in a directed acyclic graph D defined as follows: The graph D has vertex-set $V(D) = V(K_n^c)$ and arc-set $E(D) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$, where: $E_1 = \{uv \mid u \in B, v \in R, \textit{ and } c(uv) = r\}$,
$E_2 = \{uv \mid u \in R, v \in Q \textit{ and } c(uv) = b\}$,
$E_3 = \{uv \mid u \in Q, v \in R \textit{ and } c(uv) = r\}$,
$E_4 = \{xu \mid u \in B\}$,
$E_5 = \{vy \mid v \in Q \textit{ and there is } v' \in Q \textit{ such that } c(v'v) = r\}$ and
$E_6 = \{vy \mid v \in R \textit{ and there is } v' \in R \textit{ such that } c(v'v) = b\}$.

The remaining part of the proof of the Second case is based on the following claim.

Claim 1. *A shortest alternating cycle C through x and y exists in K_n^c if and only if there is a shortest path P from x to y in D either with even length or else its length is odd and $Q - V(P) \neq \emptyset$.*

Proof of Claim 1: Assume that the alternating cycle C through x and y exists in K_n^c. By the structure of K_n^c in connection with Lemma 2.4, we can easily conclude that C is defined within the following two possible ways (a) and (b).

(a) C passes through the edge xy and therefore it has the form $xb_1r_1q_1r_2q_2r_3q_3r_4...q_kq_{k+1}yx$, where $b_1 \in B$, $r_i \in R$ and $q_i \in Q$.

(b) C does not pass through the edge xy. In this case, C has the form $xb_1r_1q_1r_2q_2r_3q_3r_4...r_kr_{k+1}yq_{k+2}x$, where as above $b_1 \in B$, $r_i \in R$ and $q_i \in Q$.

In both cases (a) and (b), we can easily deduce the existence of a shortest path P from x to y in D fulfilling the conclusion of the claim.

Conversely, let $P : xz_1z_2...z_py$ denote a shortest path from x to y in D. If the length of P is even, then $z_p \in Q$. In this case, C passes through xy and it has vertex-set $V(P) \cup \{z_p'\}$, where z_p' is a vertex of Q identified by v' in E_5. On the other hand, if P has odd length, then $z_p \in R$. Let us define a path P_2 between x and y in K_n^c, where $E(P_2) = E(P) \cup \ z_p z'_p$, where z'_p is identified by v' in E_6. Now, by using Lemma 2.4, we can conclude that an alternating cycle C exists through x and y if and only if there is an alternating path P_1 between x and y in K_n^c (see the definition of P_1 and P_2 before Lemma 2.4). However, by Lemma 2.5, if the path P_1 exists, then it must have length two since P_2 has length at least six. This justifies the fact that the existence of C depends upon the cardinality of $Q - V(P)$. This completes the proof of the claim.

Clearly, the proof of Theorem 2.6 can easily be turned into an algorithm for finding an alternating cycle through two vertices or else decide that such a cycle does not exist in K_n^c. Concerning its complexity, let us notice that in all cases before Claim 1, we check each edge a constant number of times. In the proof of Claim 1, we use Dijkstra's shortest path algorithm [6]. It follows that the whole algorithm terminates within $O(n^2)$ operations, as we claimed.

The proof of the theorem is complete. □

The corollary below is an immediate consequence of Theorem 2.6.

Corollary 2.7. *There is an $O(n^2)$ algorithm for finding a cycle through two given vertices in a bipartite tournament of order n.*

Proof: Let B denote a bipartite tournament with bipartition classes X and Y. Now define K_n^c to be a 2-edge-colored complete graph obtained from B as follows : We replace each XY-arc (resp. YX-arc) by a red edge (resp. by a blue edge) and we fill up X with blue edges and Y with red edges. Clearly K_n^c admits an alternating cycle through x and y if and only if B admits a directed cycle through x and y. □

We notice that Corollary 2.7 improves the $O(n^3)$ algorithm of Y. Manoussakis and Z.Tuza given in [12] for finding a cycle (if any) through two given vertices in a bipartite tournament.

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