On Covering Designs with Block Size 5 and Index 6

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Abstract. Let V be a finite set of order v. A (v,κ,λ) covering design of index λ and block size κ is a collection of κ-element subsets, called blocks, such that every 2-subset of V occurs in at least λ blocks. The covering problem is to determine the minimum number of blocks, $\alpha(v,\kappa,\lambda)$, in a covering design. It is well known that $\alpha(v,\kappa,\lambda) \geq \lceil \frac{v}{\kappa} \lceil \frac{v-1}{\kappa-1}\lambda \rceil \rceil = \phi(v,\kappa,\lambda)$ where $\lceil x \rceil$ is the smallest integer satisfying $x \leq \lceil x \rceil$. It is shown here that $\alpha(v,5,6) = \phi(v,5,6)$ for all positive integers $v \geq 5$ with the possible exception of $v = 18$.

1. Introduction

A (v,κ,λ) covering design (or respectively packing design) of order v, block size κ and index λ is a collection β of κ-element subsets, called blocks, of a v-set V such that every 2-subset of V occurs in at least (at most) λ blocks.

Let $\alpha(v,\kappa,\lambda)$ denote the minimum number of blocks in a (v,κ,λ) covering design; and $\sigma(v,\kappa,\lambda)$ denote the maximum number of blocks in a (v,κ,λ) packing design. A (v,κ,λ) covering design with $|\beta| = \alpha(v,\kappa,\lambda)$ is called a minimum covering design. Similarly a (v,κ,λ) packing design with $|\beta| = \sigma(v,\kappa,\lambda)$ will be called a maximum packing design. It is well known that [23]

$$\alpha(v,\kappa,\lambda) \geq \left\lceil \frac{v}{\kappa} \left\lceil \frac{v-1}{\kappa-1}\lambda \right\rceil \right\rceil = \phi(v,\kappa,\lambda)$$

and

$$\sigma(v,\kappa,\lambda) \leq \left\lfloor \frac{v}{\kappa} \left\lfloor \frac{v-1}{\kappa-1}\lambda \right\rfloor \right\rfloor = \psi(v,\kappa,\lambda)$$

where $\lceil x \rceil$ is the smallest integer and $\lfloor x \rfloor$ is the largest integer satisfying $\lfloor x \rfloor \leq x \leq \lceil x \rceil$.

When $\alpha(v,\kappa,\lambda) = \phi(v,\kappa,\lambda)$ the (v,κ,λ) covering design is called minimal covering design. Similarly when $\sigma(v,\kappa,\lambda) = \psi(v,\kappa,\lambda)$ the (v,κ,λ) packing design is called optimal packing design.

Many researchers have been involved in determining the covering numbers known to date (see bibliography) most notably W. H. Mills and R. C. Mullin. In one of their papers they proved the following [22].

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Theorem 1.1. *Let v be an odd integer greater than 5.*

(i) *If $v \equiv 1 \pmod 4$ and $\lambda > 1$, then $\alpha(v,5,\lambda) = \phi(v,5,\lambda) + e$ where $e = 1$ if $\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v \frac{(v-1)}{\kappa-1} \equiv -1 \pmod 5$ and $e = 0$ otherwise with the exceptions that $\alpha(9,5,2) = \phi(9,5,2) + 1$, $\alpha(13,5,2) = \phi(13,5,2) + 1$ and the possible exceptions of the pairs $(v,\lambda) \in \{(53,2),(73,2)\}$, and*

(ii) *If $v \equiv 3 \pmod 4$ and $\lambda \geq 1$ then $\alpha(v,5,\lambda) = \phi(v,5,\lambda) + e$ where e as in (i) with the exceptions $\alpha(15,5,\lambda) = \phi(15,5,\lambda) + 1$ for $\lambda = 1,2$ and the possible exception of the pairs $(v,\lambda) \in \{(63,2),(83,2)\}$.*

Our interest here is in the case $\kappa = 5$ and $\lambda = 6$. Since the case v odd has been treated by Mills and Mullin we only treat v even. Our goal is to prove the following.

Theorem 1.2. *Let $v \geq 5$ be an even integer. Then $\alpha(v,5,6) = \phi(v,5,6)$ with the possible exception of $v = 18$.*

2. Recursive Constructions

In order to describe our recursive constructians we require several other types of combinatorial designs. A balanced incomplete block design, $B[v,\kappa,\lambda]$, is a (v,κ,λ) covering design where every 2-subsets of points is contained in precisely λ blocks. If a $B[v,\kappa,\lambda]$ exists then it is clear that $\alpha(v,\kappa,\lambda) = \lambda v(v-1/\kappa(\kappa-1)$ $= \phi(v,\kappa,\lambda)$ and Hanani, [14], has proved the following existence theorem for $B[v,5,\lambda]$.

Theorem 2.1. *Necessary and sufficient conditions for the existence of a $B[v,5,\lambda]$ are that $\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v(v-1) \equiv 0 \pmod{20}$ and $(v,\lambda) \neq (15,2)$.*

The following obvious lemma is most useful to us.

Lemma 2.1. *If there exists a $B[v,5,\lambda]$ and $\alpha(v,5,\lambda') = \phi(v,5,\lambda')$ then $\alpha(v,5,\lambda+\lambda') = \phi(v,5,\lambda+\lambda')$.*

Corollary. *Let $v \equiv 0$ or $6 \pmod{10}$ be a positive integer. Then $\alpha(v,5,6) = \phi(v,5,6)$.*

Proof.

For $v \equiv 0$ or $6 \pmod{10}$ a $(v,5,6)$ minimal covering design can be constructed by simply taking the blocks of a $B[v,5,4]$ and a $(v,5,2)$ minimal covering design.

A (v,κ,λ) covering design (or respectively packing design) with a hole of size h is a triple (V,H,β) where V is a v-set, H is a subset of V of cardinality h, and β is a collection of κ-element subsets, called blocks, of V such that

1) no 2-subset of H appears in any block;

2) every other 2-subset of V appears in at least (at most) λ blocks;
3) $|\beta| = \phi(v,\kappa,\lambda) - \phi(h,v,\lambda)$, $(|\beta| = \psi(v,\kappa,\lambda) - \psi(h,\kappa,\lambda))$

Lemma 2.2. *If there exists a* (v,κ,λ) *covering design with a hole of size* $h \geq 5$ *and* $\alpha(h,\kappa,\lambda) = \phi(h,\kappa,\lambda)$ *then* $\alpha(v,\kappa,\lambda) = \phi(v,\kappa,\lambda)$.

Proof.

Form the blocks of an (h,κ,λ) minimal covering design on the points of the hole. Adding the blocks of the covering design with the hole gives a (v,κ,λ) minimal covering design.

We like to mention that in case $\kappa = 5$, and $\lambda(v-1) \equiv 0 \pmod 4$ the existence of a $(v,5,\lambda)$ packing design with a hole of size $h \not\equiv 5$ is equivalent to the existence of a $(v,5,\lambda)$ covering design with a hole of size $h \not\equiv 5$.

In many places through the paper instead of constructing a $(v,5,6)$ minimal covering design we construct a $(v,5,6)$ covering design with a hole of size $h \geq 5$ where $\alpha(h,5,6) = \phi(h,5,6)$ and then apply lemma 2.2.

Let κ, λ and v be positive integers and M be a set of positive integers. A group divisible design GD$[\kappa,\lambda,M,v]$ is a triple (V,β,γ) where V is a set of points with $|V| = v$, and $\gamma = \{G_1,\ldots,G_n\}$ is a partition of V into n sets called groups. The collection β consists of κ-subsets of V, called blocks, with the following properties.

1) $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$;
2) $|G_i| \in M$ for all $G_i \in \gamma$;
3) every 2-subset $\{x,y\}$ of V such x and y belong to distinct groups is contained in exactly λ blocks.

If $M = \{m\}$ then the group divisible design is denoted by GD$[\kappa,\lambda,m,v\}$. A GD$[\kappa,\lambda,m,\kappa m]$ is called a transversal design and denoted by T$[\kappa,\lambda,m]$. It is well known that a T$[\kappa,1,m]$ is equivalent to $\kappa-2$ mutually orthogonal Latin squares of side m.

In the sequel we shall use the following existence theorem for transversal designs. The proof of this result may be found in [1], [11], [12], [14], [22], [24].

Theorem 2.2. *There exists a* T$[6,1,m]$ *for all positive integers* m *with the exception of* $m \in \{2,3,4,6\}$ *and the possible exception of* $m \in \{10,14,18,22,26,30,34,38,42,44\}$.

Theorem 2.3. *If there exists a* GD$[6,6,5,5n]$ *and a* $(20+h,5,\lambda)$ *covering design with a hole of size h then there exists a* $(20(n-1)+4u+h,5,\lambda)$ *covering design with a hole of size* $4u+h$ *where* $0 \leq u \leq 5$.

Proof.

Take a GD$[6,6,5,5n]$ and delete $5-u$ points from the last group. Inflate this design by a factor of 4. On the blocks of size 5 and 6 construct a GD$[5,1,4,20]$

and a GD[5,1,4,24] respectively. Add h points to the groups and on the first $n-1$ groups construct a $(20+h,5,\lambda)$ covering design with a hole of size h, and take the h points with the last group to be the hole.

Theorem 2.4. *If there exists a* GD[6,6,5,5n], *a* $(20+h,5,6)$ *covering design with a hole of size h and a* $(20+h,5,\lambda)$ *minimal covering design then there exists a* $(20n+h,5,6)$ *minimal covering design.*

Proof.

Take a GD[6,6,5,5n] and inflate this design by a factor of 4. Replace the blocks of this design by the blocks of GD[5,1,4,24]. Add h points to the groups and on the first $(n-1)$ groups construct a $(20+h,5,6)$ covering design with a hole of size h and on the last group construct a $(20+h,5,6)$ minimal covering design. It is readily checked that this construction yields a $(20n+h,5,6)$ minimal covering design.

It is clear that the application of the above theorems require the existence of GD[6,6,5,5n]. Our authority for this is the following lemma of Hanani [14 p 286].

Lemma 2.3. *There exists a* GD[6,6,5,5n] *for* $v=7,8,9,10,12$.

If in the definition of GD[κ,λ,m,v] (similarly T[κ,λ,m]) condition 2 is changed to be read as (2) every 2-subset $\{x,y\}$ of V such that x and y are neither in the same group (column) nor in the same row is contained in exactly λ blocks of β and no block contains two elements of the same row (We may look at the points of V as the points of an array A of size $m\times n$ and then the groups of the modified group divisible design are precisely the columns of A). Then the resultant design is called a modified group divisible design (modified transversal design) and is denoted by MGD[κ,λ,m,v] (MT[κ,λ,m]).

A resolvable modified group divisible design, RMGD[κ,λ,m,v], is a modified group divisible design where its blocks can be partitioned into parallel classes. It is clear that a RMGD[5,1,5,5m] is the same as RT[5,1,m] with one parallel class of blocks singled out, and since RT[5,1,m] is equivalent to T[6,1,m] we have the following.

Theorem 2.5. *There exists a* RMGD[5,1,5,5m] *for all positive integers* m *with the exception of* $m\in\{2,3,4,6\}$ *and the possible exception of* $m\in\{10,14,18,22,26,30,34,38,42,44\}$.

The next theorem is in the form most useful to us.

Theorem 2.6. [4] *If there exists a* RMGD[5,1,5,5m] *and a* GD[5,6,$\{4,s^*\}$, $4m+s$], *where * means there is exactly one group of size* s, *and there exists a* $(20+h,5,6)$ *covering design with a hole of size h then there exists a* $(20m+4u+h+s,5,6)$ *covering design with a hole of size* $4u+h+s$ *where* $0\le u\le m-1$.

It is clear that the application of the above theorem requires the existence of a $GD[5,1,\{4,s^*\},4m+s]$. We observe that we may choose $s = 0$ if $m \equiv 1 \pmod 5$; $s = 4$ if $m \equiv 0$ or $4 \pmod 5$, and $s = \frac{4(m-1)}{3}$ if $m \equiv 1 \pmod 3$ (see [4]). We may also apply the following [14].

Theorem 2.7. *There exists a* $GD[5,1,\{4,8^*\},4m+8]$ *where* $m \equiv 0$ *or* $2 \pmod 5$ $m \geq 7$ *with the possible exception of* $m = 10$.

We close this section with the following notations that will be used later.

A block $\langle \kappa, \kappa+m, \kappa+n, \kappa+y, f(\kappa)\rangle \pmod v$ where $f(\kappa) = a$ if κ is even and $f(\kappa) = b$ if κ is odd is denoted by $\langle 0\ m\ n\ y\rangle \cup \{a,b\} \pmod v$.

3. The Structure of Packing and Covering Designs

Let (V,β) be a (v,κ,λ) packing design, for each 2-subset $e = \{x,y\}$ of V define $m(e)$ to be the number of blocks in β which contain e. Note that by the definition of a packing design we have $m(e) \leq \lambda$ for all e.

The complement of (V,β), denoted by $C(V,\beta)$ is defined to be the graph with vertex set V and edges e occurring with multiplicity $\lambda - m(e)$ for all e. The number of edges (counting multiplicities) in $C(V,\beta)$ is given by $\lambda\binom{v}{2} - |\beta|\binom{\kappa}{2}$.

The degree of the vertex x in $C(V,\beta)$ is $\lambda(v-1) - r_x(\kappa-1)$ where r_x is the number of blocks containing x.

In a similar way we define the excess graph of a (V,β) covering design denoted by $E(V,\beta)$, to be the graph with vertex set V and edges e occurring with multiplicity $m(e) - \lambda$ for all e. The number of edges in $E(V,\beta)$ is given by $|\beta|\binom{\kappa}{2} - \lambda\binom{v}{2}$; and the degree of each vertex is $r_x(\kappa-1) - \lambda(v-1)$ where r_x is as before.

Lemma 3.1. *Let* (V,β) *be a* $(v,5,4)$ *optimal packing design. Then the degree of each vertex of* $C(V,\beta)$ *is divisible by* 4 *and the number of edges in the graph is* $0, 4$ *or* 12 *when* $v \bmod 5 \in \{0,1\}, \{2,4\},$ *or* $\{3\}$.

The only graph with 4 edges and every vertex of degree divisible by 4 is the graph with four parallel edges connecting two vertices and $v-2$ isolated vertices. Therefore when $v \equiv 2$ or $4 \pmod 5$ a $(v,5,4)$ optimal packing design is the same as, a $(v,5,4)$ packing design with a hole of size 2.

Lemma 3.2. *Let* (V,β) *be a* $(v,5,2)$ *optimal packing design where* $v \equiv 3 \pmod{10}$. *Then the degree of each vertex of* $C(V,\beta)$ *is divisible by* 4 *and the number of edges in the graph is* 6. *Hence* $C(V,\beta)$ *consists of* $v-3$ *isolated vertices and* 3 *other vertices each pair of them is connected by* 2 *edges.*

Lemma 3.3. *Let* (V,β) *be a* $(v,5,4)$ *minimal covering design. Then the degree of each vertex of* $E(V,\beta)$ *is divisible by* 4 *and the number of edges in the graph is* $0, 6$ *or* 8 *when* $v \bmod 5 \in \{0,1\}, \{2,4\},$ *or* $\{3\}$ *respectively.*

The only graph with 6 edges and every vertex of degree divisible by 4 is the graph with $v-3$ isolated vertices and 3 other vertices each one connected to the other 2 by two parallel edges.

The following is very simple but most useful to us.

Theorem 3.1. *If there exists*

1) *A* $(v,5,\lambda)$ *covering design with* $\phi(v,5,\lambda)$ *blocks*
2) *A* $(v,5,\lambda')$ *packing design with* $\psi(v,5,\lambda')$ *blocks*
3) $\phi(v,5,\lambda)+\psi(v,5,\lambda')=\phi(v,5,\lambda+\lambda')$
4) *The complement graph* $C(V,\beta)$ *of the packing design is isomorphic to a subgraph* G *of the excess graph,* $E(V,\beta)$*, of the covering design.*

Then there exists a $(v,5,\lambda+\lambda')$ *covering design with* $\phi(v,5,\lambda+\lambda')$ *blocks, that is, a* $(v,5,\lambda+\lambda')$ *minimal covering design.*

4. Constructions

In this section we distinguish the following cases.

4.1 $v\equiv 4 \pmod{20}$

Lemma 4.1. $\alpha(v,5,6)=\phi(v,5,6)$ *for* $v=24,44,64,84$.

Proof.

For $v=24$ the construction is as follows.

1) Take a $(24,5,4)$ optimal packing design, [7]. In this design each pair appears in exactly four blocks with the exception of one pair, say, $(22,23)$ that appears in zero blocks.
2) Take a $(23,5,1)$ minimal covering design which we construct by taking the blocks of a $B[21,5,1]$ together with new two points, say, $(22,23)$ which we add them to 7 triples that partition the points of $B[21,5,1]$. Without loss of generality we may assume that one of these triples is $\langle 6\ 7\ 8\rangle$.
3) Take a $B[25,5,1]$ and assume we have the block $\langle 1\ 2\ 3\ 24\ 25\rangle$. In this block change 25 to 8 and in all other blocks change 25 to 24.
4) Finally assume in the $(24,5,4)$ optimal packing design we have the block $\langle 1\ 2\ 3\ 7\ 8\rangle$. In this block change 8 to 24.

Since we assume in (2) that we have the triple $\langle 6\ 7\ 8\rangle$, so it is easily checked that the above 4 steps yield the blocks of a $(24,5,6)$ minimal covering design.

For $v=44,64,84$ see the next table. In general the construction in this table and all other tables to come is as follows. Let $X=Z_{v-n}\cup H_n$ or $X=Z_2\times Z_{(v-n)/2}\cup H_n$ where $H_n=\{h_1,\dots,h_n\}$ is the hole. The blocks are constructed by taking the orbits of the tabulated base blocks.

v	Point Set	Base Blocks
44	Z_{44}	⟨0 1 2 4 8⟩ twice ⟨0 3 12 19 32⟩ twice ⟨0 5 15 26 31⟩ twice
		⟨0 6 14 23 33⟩ twice ⟨0 2 5 12 30⟩ ⟨0 4 13 27 33⟩
		⟨0 1 4 20 21⟩ ⟨0 2 7 13 22⟩ ⟨0 4 12 26 34⟩
64	Z_{64}	⟨0 1 3 7 20⟩ 3 times ⟨0 5 14 32 40⟩ twice ⟨0 10 21 33 49⟩ twice
		⟨0 5 16 28 42⟩ ⟨0 8 18 33 42⟩ ⟨0 1 3 11 44⟩ ⟨0 4 16 30 46⟩
		⟨0 6 21 28 45⟩ ⟨0 1 3 26 30⟩ ⟨0 5 17 35 45⟩ ⟨0 6 15 37 50⟩
		⟨0 1 3 7 14⟩ ⟨0 5 15 24 37⟩ ⟨0 5 21 33 44⟩ ⟨0 8 17 38 46⟩
84	Z_{84}	⟨0 1 3 7 22⟩ twice ⟨0 5 31 48 56⟩ twice ⟨0 9 20 44 54⟩ twice
		⟨0 13 27 45 68⟩ twice ⟨0 1 3 15 34⟩ twice ⟨0 4 22 46 59⟩ twice
		⟨0 5 11 21 57⟩ twice ⟨0 7 30 47 56⟩ twice ⟨0 12 20 37 46⟩
		⟨0 1 3 7 15⟩ ⟨0 5 21 48 61⟩ ⟨0 8 18 47 67⟩
		⟨0 11 30 42 62⟩ ⟨0 1 3 8 39⟩ ⟨0 4 13 24 65⟩
		⟨0 6 21 33 55⟩ ⟨0 10 24 40 66⟩

Lemma 4.2. *There exists a* $(24,5,6)$ *covering design, with a hole of size* 4.

Proof.

The blocks of a $(24,5,6)$ covering design with a hole of size 4 are constructed as follows.

1) Take a $B[21,5,1]$.
2) Take a $B[25,5,1]$ and delete the block ⟨21 22 23 24 25⟩ and in all other blocks change 25 to 23 and 24 to 22.
3) Take a $(23,5,1)$ covering design. This design has a block of size 3, say, $\{21,22,23\}$ which we delete, [17].
4) Take 3 copies of $B[25,5,1]$. Assume in each copy we have the block ⟨21 22 23 24 25⟩ which we delete and in all other copies change 25 to 24.

It is readily checked that the above construction yields a $(24,5,6)$ covering design with a hole of size 4.

Notice that the first two steps yield a (23,5,2) covering design with a hole of size 3.

Lemma 4.3. *Let* $v \equiv 4 \pmod{20}$ *be a positive integer* ≥ 24. *Then* $\alpha(v,5,6) = \phi(v,5,6)$.

Proof.

For $v = 24, 44, 64, 84$ the result follows from lemma 4.1. For $v \geq 104$, $v \neq 144, 184, 224$, simple calculations show that v can be written in the form $v = 20m + 4u + h + s$ where m, u, h and s are chosen so that

1) There exists a $RMGD[5,1,5,5m]$.
2) $4u + h + s \equiv 4 \pmod{20}, 24 \leq 4u + h + s \leq 84$.
3) $s \equiv 0 \pmod 4, h = 4$.

Now apply theorem 2.6 and the result follows.

For $v = 144, 184$ apply theorem 2.4 with $n = 7, 9$ and $h = 4$. For $v = 224$ take a GD[6, 6, 5, 60] and delete the last group. Inflate the resultant design by a factor of 4. Add 4 points to the groups and on each group construct a (24, 5, 6) covering with a hole of size 4 except the last group on which we construct a (24, 5, 6) minimal covering design.

For $v = 104$ take a T[5, 6, 20], add 4 points to the groups. On the first group construct a (24, 5, 6) covering with a hole of size 4 and on the other groups construct a (24, 5, 6) minimal covering design.

4.2 $v \equiv 8 \pmod{20}$

Lemma 4.4. $\alpha(v, 5, 6) = \phi(v, 5, 6)$ *for* $v = 8, 28, 48, 68, 88$.

For $v = 8, 28, 48$ see the following table.

v	Point Set	Base Blocks
8	Z_8	$\langle 0\ 2\ 4\ 6\rangle + i, i \in Z_2$ $\langle 0\ 1\ 2\ 4\ 5\rangle$ $\langle 0\ 1\ 2\ 3\ 5\rangle$
28	$Z_{22} U H_6$	$\langle 0\ 2\ 6\ 13\ 21\rangle$ $\langle 0\ 1\ 2\ 7\rangle \cup \{h_1, h_2\}$ $\langle 0\ 3\ 10\ 13\rangle \cup \{h_3, h_4\}$
		$\langle 0\ 4\ 9\ 17\rangle \cup \{h_5, h_6\}$ $\langle 0\ 2\ 8\ 12\ h_1\rangle$ $\langle 0\ 2\ 8\ 14\ h_2\rangle$ $\langle 0\ 1\ 2\ 3\ h_3\rangle$
		$\langle 0\ 3\ 7\ 18 h_4\rangle$ $\langle 0\ 3\ 8\ 13\ h_5\rangle$ $\langle 0\ 4\ 9\ 15\ h_6\rangle$
48	$Z_{40} \cup H_8$	On $Z_{40} \cup H_7$, construct a (47, 5, 2) packing design with a hole of size 7, [8]. Take two copies of this design and the following blocks.
		$\langle 0\ 2\ 6\ 14\ 24\rangle$ $\langle 0\ 1\ 2\ 5\ h_8\rangle$ $\langle 0\ 3\ 11\ 26\rangle \cup \{h_1, h_2\}$
		$\langle 0\ 5\ 18\ 25\rangle \cup \{h_3, h_4\}$ $\langle 0\ 6\ 13\ 29\rangle \cup \{h_5, h_6\}$ $\langle 0\ 9\ 19\ 28\rangle \cup \{h_7, h_8\}$

For $v = 68, 88$ take a GD[6, 3, 5, 5n], $n = 7, 9$ [14], and delete one point from last group. Inflate the resulting design by a factor of 2. Replace all its blocks which are of size 5 and 6, by the blocks of GD[5, 2, 2, 10] and GD[5, 2, 2, 12], [15]. On the first 6 groups construct a (10, 5, 6) minimal covering design and on the last group construct a (8, 5, 6) minimal covering design.

It is readily checked that this construction yields a (68, 5, 6) and (88, 5, 6) minimal covering design.

Lemma 4.5. *Let* $v \equiv 8 \pmod{20}$ *be a positive integer. Then* $\alpha(v, 5, 6) = \phi(v, 5, 6)$.

Proof.

For $v = 8, 28, 48, 68, 88$ the result is given in lemma 4.4. For $v \geq 108$, $v \neq 128$, write $v = 20m + 4u + h + s$ where m, u, h, s are chosen the same as in lemma 4.3 with one difference that $4u + h + s \equiv 8 \pmod{20}$. Now apply theorem 2.6 and the result follows.

For $v = 128$ apply theorem 2.3 with $n = 7$, $h = 4$ and $u = 1$.

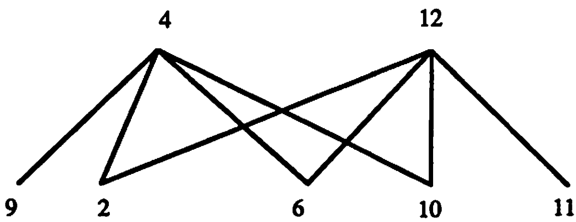
4.3 $v \equiv 12 \pmod{20}$

Lemma 4.6. $\alpha(v,5,6) = \phi(v,5,6)$ *for* $v = 12, 32, 52, 72, 92$.

Proof.

For $v = 12$ the construction is as follows.

1) Take a $(12, 5, 2)$ minimal covering design. The excess graph of this design has the following bipartite subgraph, say, at $\{2, 4, 6, 9, 10, 11, 12\}$. Furthermore assume in this design we have the block $\langle 1\ 5\ 10\ 12\ 11 \rangle$ where $\{1, 5, 10\}$ are arbitrary numbers. In this block change 11 to 4.



2) Take a $(12, 5, 4)$ minimal covering design. The excess graph of this design consists of 9 isolated vertices and 3 other vertices, say, $\{2, 6, 10\}$ each pair of them is connected by 2 edges. Furthermore assume in this design we have the block $\langle 1\ 5\ 10\ 9\ 4 \rangle$. In this block change 4 to 11. Assume also in this design we have the block $\langle 2\ 4\ 6\ 10\ 12 \rangle$ which we delete.

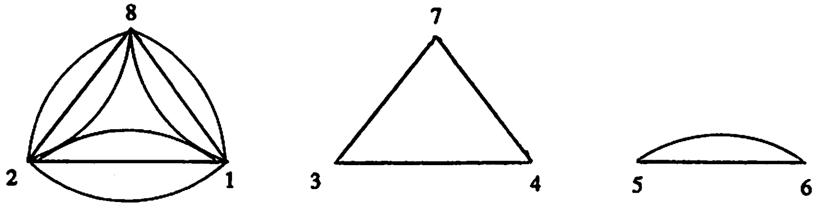
Now it is easily checked that the above construction yields a $(12, 5, 6)$ minimal covering design.

For $v = 32, 52$ the construction is as follows.

1) Take a $B[v-1, 5, 2]$.
2) Take a $(v+1, 5, 2)$ optimal packing design, [5]. The complement graph of this design consists of $v-3$ isolated vertices and three other vertices, say, $\{1, 2, v+1\}$ each two of them are joined by two edges. Furthermore, assume in this design we have the blocks $\langle 3\ 4\ 5\ v\ v{+}1 \rangle$ $\langle 6\ 7\ 8\ v\ v{+}1 \rangle$ In the first block change $v+1$ to 10 and in the second block change $v+1$ to 11 where $\{3, 4, 5, 6, 7, 8, 10, 11\}$ are arbitrary numbers. In all other blocks change $v+1$ to v.
3) Take a $(v, 5, 2)$, $v = 32, 52$, minimal covering design [19].

Assume in this design we have the blocks $\langle 3\ 4\ 5\ 14\ 10 \rangle$ $\langle 6\ 7\ 8\ 14\ 11 \rangle$ where 14 is an arbitrary number. In the first block change 10 to v and in the second change 11 to v.

But the $(v, 5, 2)$ minimal covering design, $v = 32, 52$, is constructed [19] by taking a hole (block) of size 8. So the excess graph of $(v, 5, 2)$ minimal covering design, $v = 32, 52$, has a subgraph that is isomorphic to the following graph.



So without loss of generality we may assume that the triple $\langle 1\ 2\ 32 \rangle$ appears 5 times and $\langle 10, 11, 14 \rangle$ appears 3 times in the blocks of $(v, 5, 2)$ minimal covering design $v = 32, 52$.

It is easy to check that the above construction produces a $(v, 5, 6)$ minimal covering design for $v = 32, 52$.

We now construct $(v, 5, 6)$ minimal covering design, $v = 72, 92$, the same way as $v = 32, 52$. Since there exists a $(v + 1, 5, 2)$ optimal packing design, [5] for $v = 72, 92$ and since there exists a $B[v - 1, 5, 2]$ for $v = 72, 92$ so we only need to show that there exists a $(v, 5, 2)$ covering design with a hole of size 8 for $v = 72, 92$.

For $v = 72$ let $X = Z_{64} \cup H_8$. The blocks are the following under the action of Z_{64}. $\langle 0\ 2\ 6\ 14\ 30 \rangle$ $\langle 0\ 1\ 6\ 16\ 42 \rangle$ $\langle 0\ 3\ 17\ 37\ 46 \rangle$ $\langle 0\ 7\ 20\ 31\ 39 \rangle$ $\langle 0\ 1\ 4\ 11 \rangle \cup \{h_1, h_2\}$ $\langle 0\ 2\ 17\ 29 \rangle \cup \{h_3, h_4\}$ $\langle 0\ 5\ 18\ 43 \rangle \cup \{h_5, h_6\}$ $\langle 0\ 9\ 23\ 42 \rangle \cup \{h_7, h_6\}$.

For $v = 92$ let $X = Z_{84} \cup H_8$. Then the required blocks are the following under the action of Z_{84}. $\langle 0\ 4\ 12\ 26\ 42 \rangle$ $\langle 0\ 1\ 3\ 7\ 22 \rangle$ $\langle 0\ 5\ 31\ 48\ 56 \rangle$ $\langle 0\ 9\ 20\ 44\ 54 \rangle$ $\langle 0\ 13\ 27\ 45\ 68 \rangle$ $\langle 0\ 1\ 3\ 15\ 49 \rangle$ $\langle 0\ 5\ 24\ 33 \rangle \cup \{h_1, h_2\}$ $\langle 0\ 6\ 13\ 31 \rangle \cup \{h_3, h_4\}$ $\langle 0\ 10\ 27\ 47 \rangle \cup \{h_5, h_6\}$ $\langle 0\ 11\ 32\ 55 \rangle \cup \{h_7, h_8\}$.

Lemma 4.7. *Let* $v \equiv 12 \pmod{20}$ *be a positive integer. Then* $\alpha(v, 5, 6) = \phi(v, 5, 6)$.

Proof.

For $v = 12, 32, 52, 72, 92$ the result follows from lemma 4.6. For $v \geq 112$, $v \neq 132$, write $v = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 4.3 with the difference that $4u + h + s \equiv 12 \pmod{20}$, $12 \leq 4u + h + s \leq 92$.

Now apply theorem 2.6 and the result follows.

For $v = 132$ apply theorem 2.3 with $n = 7$, $h = 4$ and $u = 3$.

4.4 $v \equiv 2 \pmod{20}$

Lemma 4.8. *Let* $v \equiv 2 \pmod{20}$ *be a positive integer. Then* $\alpha(v, 5, 6) = \phi(v, 5, 6)$*; b) There exists a* $(22, 5, 6)$ *covering design with a hole of size* 2.

Proof.

The blocks of a $(v,5,6)$ minimal covering design for all $v \equiv 2 \pmod{20}$, $v \not\equiv 22$ can be constructed as follows.

1) Take a $(v,5,4)$ minimal covering design which exists for all $v \equiv 2 \pmod{20}$ with the possible exception of $v = 22$, [9]. In this design there is a triple, say $\{v-2, v-1, v\}$ that appears in 6 blocks.
2) Take a $B[v-1,5,1]$.
3) Take a $(v+1,5,1)$ minimal covering design, [17]. This design has a block of size 3, say $\langle v-1, v, v+1 \rangle$ which we delete and in all other blocks change $v+1$ to v.

Now it is readily checked that the above three steps give the blocks of a $(v,5,6)$ minimal covering design for all $v \equiv 2 \pmod{20}$, $v \not\equiv 22$.

For $v = 22$ the construction is as follows.

1) Take a $(22,5,4)$ optimal packing design. In this design each pair appears in precisely 4 blocks except one pair, say, $(9,10)$ that appears in zero blocks.
2) Take a $B[21,5,1]$.
3) Take a $(23,5,1)$ minimal covering design as constructed in lemma 4.1. So without loss of generality we may assume that the pair $(9,10)$ appears in 7 blocks and that the pair $(22,23)$ appears in precisely one block, say, $\langle 1\ 2\ 3\ 22\ 23 \rangle$. In this block change 23 to a number, say, 4 and in all other blocks change 23 to 22. Further assume the pair $(4,21)$ appears twice in $(23,5,1)$ minimal covering design.
4) Assume in the $(22,5,4)$ optimal packing design we have the block $\langle 1\ 2\ 3\ 21\ 4 \rangle$ In this block change the point 4 to 22.

The blocks of a $(22,5,6)$ covering design with a hole of size 2 is constructed as follows.

1) Take a $(22,5,4)$ optimal packing design, [7]. In this design each pair appears in precisely 4 blocks except one pair, say, $\{21,22\}$ that appears in zero blocks.
2) Take a $B[21,5,1]$.
3) Take a $(23,5,1)$ minimal covering design, [17]. This design has a block of size 3, say, $\langle 21\ 22\ 23 \rangle$ which we delete and in all other blocks change 23 to 22.

It is readily checked that the above 3 steps give the blocks of a $(22,5,6)$ covering design with a hole of size 2.

4.5 $v \equiv 14 \pmod{20}$

Lemma 4.9. $\alpha(v,5,6) = \phi(v,5,6)$ *for* $v = 14, 34, 54, 74, 94$.

Proof.

The blocks of a $(v, 5, 6)$ minimal covering design are the blocks of a $(v, 5, 3)$ minimal covering design each taken twice. But $\alpha(v,5,3) = \phi(v,5,3)$ for $v = 14, 34, 54, 74, 94$, [6], hence $\alpha(v,5,6) = \phi(v,5,6)$ for $v = 14, 34, 54, 74, 94$.

Lemma 4.10. *Let* $v \equiv 14 \pmod{20}$ *be a positive integer. Then* $\alpha(v,5,6) = \phi(v,5,6)$.

Proof.

For $v = 14, 34, 54, 74, 94$ the result is given in lemma 4.9. For $v \geq 114$, $v \neq 134$, write $v = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 4.3 with the difference that $h = 2$ and $4u + h + s \equiv 14 \pmod{20}$. Apply theorem 2.6 and the result follows.

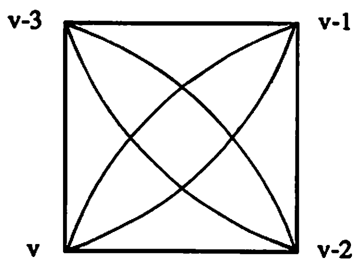
For $v = 134$ apply theorem 2.3 with $n = 7$, $h = 2$ and $u = 3$.

4.6 $v \equiv \pmod{20}$

Lemma 4.11. *Let* $v \equiv 18 \pmod{20}$ *be a positive integer. Then* $\alpha(v,5,6) = \phi(v,5,6)$ *with the possible exception of* $v = 18$.

Proof.

In [**10 lemma 5.9**] we have shown that there exists a $(v, 5, 4)$ minimal covering design for $v \equiv 18 \pmod{20}$ $v \neq 18, 178$ such that the excess graph of this design consists of $v - 4$ isolated vertices and the following graph on the remaining 4 vertices, say, $\{v-3, v-2, v-1, v\}$.



We now construct a $(v, 5, 6)$ minimal covering design for $v \equiv 18 \pmod{20}$, $v \neq 18, 178$ as follows.

1) Take a $(v, 5, 4)$ minimal covering design $v \neq 18, 178$ such that its excess graph satisfies the above.
2) Take a $B[v+3, 5, 1]$ and assume we have the following blocks $\langle v-1\ v\ v+1\ v+2\ v+3 \rangle$ $\langle 1\ 2\ 3\ v-2\ v+1 \rangle$ where $\{1, 2, 3\}$ are arbitrary numbers. Delete

the first block and in the second change $v+1$ to a number, say, 5. In all other blocks change $v+3$ to v, $v+2$ to $v-1$ and $v+1$ to $v-2$.

3) Take a $(v-3,5,1)$ minimal covering design, [16]. Assume in this design we have the block $\langle 1\ 2\ 3\ 4\ 5\rangle$ and assume that $(4,5)$ appears at least twice in the blocks of this design. In this block change 5 to $v-2$. Now it is readily checked that the above 3 steps yield the blocks of a $(v,5,6)$ minimal covering design for all $v \equiv 18 \pmod{20}$ $v \neq 18, 178$. For $v = 178$ apply theorem 2.3 with $n = 9$, $h = 2$ and $u = 4$.

5. Conclusion

We have shown that $\alpha(v,5,6) = \phi(v,5,6)$ for all positive integers $v \geq 5$ with the possible exception of $v = 18$: v odd follows from 1.1; $v \equiv 0$ or $1 \pmod 5$ follows from the corollary; for all other values see lemmas 4.3, 4.5, 4.7, 4.8, 4.10, and 4.11.

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