The Tenacity of the Harary Graphs

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ABSTRACT. As a network begins losing links or nodes eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. Many graph theoretical parameters have been used to describe the stability of communication networks, including connectivity, integrity, toughness, tenacity and binding number. Several of these deal with two fundamental questions about the resulting graph. How many vertices can still communicate? How difficult is it to reconnect the graph? For any fixed integers n, p, with $p \geq n+1$, Harary constructed classes of graphs $H_{n,p}$, that are n-connected with the minimum number of edges. Thus Harary graphs are examples of graphs with maximum connectivity. This property makes them useful to network designers and thus it is of interest to study the behavior of other stability parameters for the Harary graphs. In this paper we study the tenacity of the Harary graphs.

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Preliminaries

The stability of a communication network composed of processing nodes and communication links is of prime importance to network designers. As the network begins losing links or nodes eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. Many graph theoretical parameters have been used in the past to describe the stability of communication networks. Most notably, the vertex-connectivity and edge-connectivity have been frequently used. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected. Consequently, a number of other parameters have recently been introduced which attempt to cope with this difficulty. Several of these deal with two fundamental questions. How many vertices can still communicate? How difficult is it to reconnect the graph? In this paper we will deal with both of these issues but first we will give some basic definitions and notation. Any undefined terms can be found in the standard references on graph theory, including Chartrand and Lesniak [3].

Throughout this paper we will let p and q be the number of vertices and edges, respectively, of G. A set of vertices in G is *independent* if no two of them are adjacent. The largest number of vertices in any such set is called the *vertex independence number* of G and is denoted by $\beta(G)$ or β.

Let G be a graph with vertex set V. Let A be a subset of V. We define $G - A$ to be the graph induced by the vertices of $V - A$. Also, for any graph G, $\tau(G)$ is the number of vertices in a largest component of G and $\omega(G)$ is the number of components of G.

A *cut-set* of a graph G is a set of vertices whose removal results in a disconnected graph or the trivial graph K_1. The *connectivity* of G, $\kappa = \kappa(G)$, is the minimum order of a cut-set in G. A graph G is called *n-connected* if $\kappa \geq n$.

The *tenacity* of a graph G was defined in [4] as $T(G) = \min\{\frac{|A|+\tau(G-A)}{\omega(G-A)}\}$, where the minimum is taken over all cut-sets A of G. Note that $T(K_p) = p$. A subset A of $V(G)$ is said to be a *T-set* of G if $T(G) = \frac{|A|+\tau(G-A)}{\omega(G-A)}$. Note that if G is disconnected then the set A may be empty.

Given a graph G, the graph G^r has $V(G^r) = V(G)$ and $uv \in E(G^r)$ if and only if the distance from u to v in G is at most r. Thus, in particular, C_p^r has $V(C_p^r) = \{0, 1, \ldots, p-1\}$ and $E(C_p^r) = \{ij : |i-j| \leq r\}$.

For any fixed integers n, p, with $p \geq n+1$, Harary [5] constructed classes of graphs $H_{n,p}$, that are n-connected with the minimum number of edges on p vertices. Thus Harary graphs are examples of graphs which in some sense have the maximum possible connectivity and hence are of interest as possibly having good stability properties. Also, the Harary graph $H_{n,p}$

with $n = 2r$ is the rth power of the p-cycle, C_p^r, for which both the integrity and toughness have been studied in [1,2]. Here we consider the tenacity, T, of the Harary graphs. $H_{n,p}$ is constructed as follows:

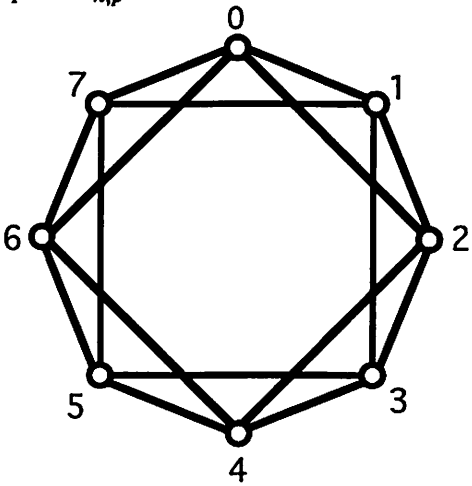

Figure 1.

Case 1: If n is even then let $n = 2r$. Then $H_{n,p}$ has vertices $0, 1, 2, \ldots, p-1$ and two vertices i and j are adjacent if and only if $|i-j| \leq r$ (where addition is taken modulo p). $H_{4,8}$ is shown in Figure 1. Note that this is C_p^r and is n-regular.

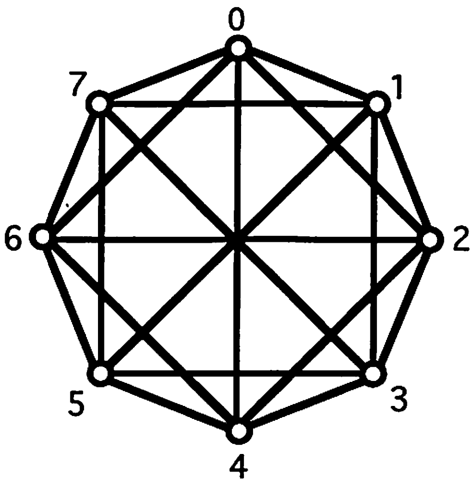

Figure 2.

Case 2: If n is odd ($n > 1$) and p is even. Let $n = 2r + 1$ ($r > 0$). Then

$H_{2r+1,p}$ is constructed by drawing $H_{2r,p}$, and adding edges joining vertex i to vertex $i + \frac{p}{2}$ for $1 \le i \le \frac{p}{2}$. Again note that this is an n-regular graph. $H_{5,8}$ is shown in Figure 2.

Case 3: If n is odd ($n > 1$) and p is odd. Let $n = 2r + 1$ ($r > 0$). then $H_{2r+1,p}$ is constructed by first drawing $H_{2r,p}$, and adding edges joining vertex i to vertex $i + \frac{p+1}{2}$ for $0 \le i \le \frac{p-1}{2}$. Note that under this definition, vertex 0 is adjacent to both vertices $\frac{p+1}{2}$ and $\frac{p-1}{2}$. Again note that all vertices of $H_{n,p}$ have degree n except vertex 0, which has degree $n + 1$. $H_{5,9}$ is shown in Figure 3.

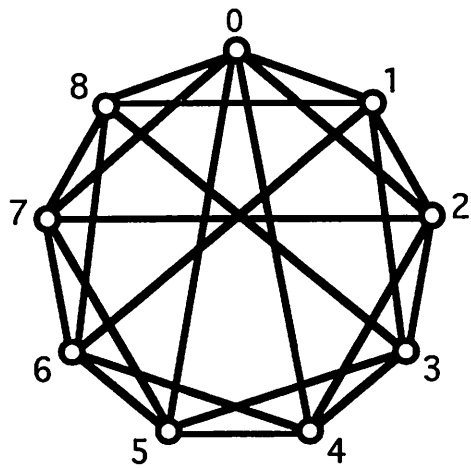

Figure 3.

The tenacity for several classes of graphs is studied in [4] and [6]. The comparison of tenacity to the integrity and toughness of these classes of graphs indicates that tenacity can be a useful measure of graph stability. In this paper we find the exact values or good bounds for the tenacity of the Harary graphs. The following three propositions, along with Theorem 1, were proved in [4].

Proposition 1. *If G is a spanning subgraph of H, then $T(G) \le T(H)$.*

Proposition 2. *For any graph G, $T(G) \ge \frac{\kappa(G)+1}{\beta(G)}$.*

Proposition 3. *If G is not complete, then $T(G) \le \frac{p-\beta(G)+1}{\beta(G)}$.*

Theorem 1. *If G is a bipartite, r-regular, r-connected graph on p vertices then $T(G) = \frac{p+2}{p}$.*

Tenacity of the Harary Graphs

Throughout the rest of this paper we will let the connectivity $n = 2r$ or $n = 2r+1$ and the number of vertices $p = k(r+1) + s$ for $0 \leq s < r+1$. So we can see that $p \equiv s \mod (r+1)$ and $k = \lfloor \frac{p}{r+1} \rfloor$. Also we assume that the graph $H_{n,p}$ is not complete, so $n+1 < p$. Note that this implies that $k \geq 2$. We determine the tenacity first when n is even, then when n is odd and p is even, and lastly we consider the cases when n and p are both odd.

Lemma 1. *If A is a minimal T-set for $H_{n,p}$, $n = 2r$, then A consists of the union of sets of r consecutive vertices such that there exists at least one vertex not in A between any two sets of consecutive vertices in A.*

Proof: We assume $H_{n,p}$ is labeled by $0, 1, 2, \ldots, p-1$. Let A be a minimal T-set for $H_{n,p}$ and j be the least integer such that $S = \{j, j+1, \ldots, j+t-1\}$ is a maximal set of consecutive vertices such that $S \subseteq A$. Re-label the vertices of $H_{n,p}$ as $v_1 = j$, $v_2 = j+1, \ldots, v_t = j+t-1, \ldots, v_p = j-1$. Since $A \neq V(H_{n,p})$, $S \neq V(H_{n,p})$ so v_p does not belong to A. Since A must leave at least two components, $t \neq p-1$, so $v_{t+1} \neq v_p$. Therefore $\{v_{t+1}, v_p\} \cap A = \emptyset$. Choose v_i such that $1 \leq i \leq t$, and delete v_i from A yielding a new set $A' = A - \{v_i\}$ with $|A'| = |A| - 1$. Now suppose $t < r$. The edges v_iv_p and v_iv_{t+1} are in $H_{n,p} - A'$. Consider a vertex v_k adjacent to v_i in $H_{n,p} - A'$, if $k \geq t+1$ then $k < t+r$, so v_k is also adjacent to v_{t+1} in $H_{n,p} - A'$ and if $k < p$ then $k \geq p-r+1$ and v_k is also adjacent to v_p in $H_{n,p} - A'$. Since $t < r$, then v_p and v_{t+1} are adjacent in $H_{n,p} - A$. Therefore we can conclude that deleting vertex v_i from A does not change the number of components, and so $\omega(H_{n,p} - A) = \omega(H_{n,p} - A')$ and the maximum order of a component of $H_{n,p} - A$ is $\tau(H_{n,p} - A') \leq \tau(H_{n,p} - A) + 1$.

Therefore, $\frac{|A'| + \tau(H_{n,p} - A')}{\omega(H_{n,p} - A')} \leq \frac{|A| - 1 + \tau(H_{n,p} - A) + 1}{\omega(H_{n,p} - A)} = T(H_{n,p})$, contrary to our choice of A. Thus we must have $t \geq r$.

Now suppose $t > r$. Delete v_t from the set A yielding a new set $A_1 = A - \{v_t\}$. Since t> r, the edge v_tv_p is not in $H_{n,p} - A_1$. Consider a vertex v_k adjacent to v_t in $H_{n,p} - A_1$. Then $k \geq t+1$ and $k \leq t+r$. So v_k is also adjacent to v_{t+1} in $H_{n,p} - A_1$. Therefore deleting v_t from A yields $\omega(H_{n,p} - A) = \omega(H_{n,p} - A_1)$ and $\tau(H_{n,p} - A_1) \leq \tau(H_{n,p} - A) + 1$. Therefore $\frac{|A_1| + \tau(H_{n,p} - A_1)}{\omega(H_{n,p} - A_1)} \leq \frac{|A| - 1 + \tau(H_{n,p} - A) + 1}{\omega(H_{n,p} - A)}$, again contrary to our choice of A. Thus $t = r$ and so A consists of the union of sets of exactly r consecutive vertices. □

Lemma 1 gives us an indication of the size of the cut-set for the tenacity of Harary graphs when $n = 2r$; the next lemma gives us the size of the largest component.

Lemma 2. *There is a T-set, A, for $H_{n,p}$, $n = 2r$, such that all components of $H_{n,p}$ have order $\tau(H_{n,p} - A)$ or $\tau(H_{n,p} - A) - 1$.*

Proof: Among all minimum order T-sets, consider those sets with maximum order, s, of the minimum order component. Among these sets let A be one with the fewest components of order s. Suppose $s \leq \tau(H_{n,p} - A) - 2$. Note that all of the components must be sets of consecutive vertices. Suppose C_k is a smallest component, so $|V(C_k)| = s$, and without loss of generality let $C_k = \{v_1, v_2, \ldots, v_s\}$. Suppose C_e is a largest component, so $|V(C_e)| = \tau(H_{n,p} - A)$, and $C_e = \{v_j, \ldots, v_{j+\tau(H_{n,p}-A)}\}$. Let $C_1, C_2, \ldots, C_a$ be components with vertices between v_s and v_j, such that $|C_i| = p_i$ for $1 \leq i \leq a$ and $C_i = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{p_i}}\}$. Now construct A' as follows, $A' = A - \{v_{s+1}, v_{1_{p_1}+1}, v_{2_{p_2}+1}, \ldots, v_{a_{p_a}+1}\} \cup \{v_{1_1}, v_{2_1}, \ldots, v_{a_1}, v_j\}$. Therefore $|A'| = |A|$, $\tau(H_{n,p} - A') \leq \tau(H_{n,p} - A)$ and $\omega(H_{n,p} - A') = \omega(H_{n,p} - A)$. So, $\frac{|A|+\tau(H_{n,p}-A')}{\omega(H_{n,p}-A')} \leq \frac{|A|+\tau(H_{n,p}-A)}{\omega(H_{n,p}-A)}$. Therefore $\tau(H_{n,p} - A') = \tau(H_{n,p} - A)$. But $H_{n,p} - A'$ has one less component of order s than $H_{n,p} - A$, and this is a contradiction. Thus all components of $H_{n,p} - A$ have order $\tau(H_{n,p} - A)$ or $\tau(H_{n,p} - A) - 1$. So $\tau(H_{n,p} - A) = \lceil \frac{p-r\omega}{\omega} \rceil$. □

These two lemmas allow us to determine precisely the tenacity of Harary graphs when $n = 2r$.

Theorem 2. *Let $H_{n,p}$ be a Harary graph with $n = 2r$ and $p = k(r+1)+s$, for $0 \leq s < r+1$. Then $T(H_{2r,p}) = r + \frac{1+\lceil \frac{s}{k} \rceil}{k}$.*

Proof: Let A be a minimal T-set of $H_{n,p}$. By Lemma 1 and Lemma 2 $|A| = r\omega$, and $\tau(H_{n,p} - A) = \lceil \frac{p-r\omega}{\omega} \rceil$. Thus, from the definition of tenacity we have $T = \min\{\frac{r\omega + \lceil \frac{p-r\omega}{\omega} \rceil}{\omega} \mid 2 \leq \omega \leq k\}$.

Now consider the function $f(\omega) = \frac{r\omega+\lceil \frac{p-r\omega}{\omega} \rceil}{\omega} = r + \frac{\lceil \frac{p}{\omega} - r \rceil}{\omega}$. Let ω_1 and ω_2 be any two integers in $[2, k]$ with $\omega_1 \leq \omega_2$, then $\lceil \frac{p}{\omega_2} \rceil \leq \lceil \frac{p}{\omega_1} \rceil$. Thus $f(\omega_2) = r + \frac{\lceil \frac{p}{\omega_2} - r \rceil}{\omega_2} \leq r + \frac{\lceil \frac{p}{\omega_1} - r \rceil}{\omega_1} = f(\omega_1)$. Hence the function $f(\omega)$ is a nonincreasing function and the minimum value occurs at the boundary. Thus $\omega = k$ and $\lceil \frac{p-r\omega}{\omega} \rceil = \lceil \frac{k(r+1)+s-rk}{k} \rceil = 1 + \lceil \frac{s}{k} \rceil$. Therefore, $T(H_{2r,p}) = r + \frac{1+\lceil \frac{s}{k} \rceil}{k}$. □

Corollary 1. *Let C_p be the p-cycle, then*

$$T(C_p) = \begin{cases} 1 + \frac{2}{p} & \text{if } p \equiv 0 \mod 2 \\ 1 + \frac{4}{p-1} & \text{if } p \equiv 1 \mod 2 \end{cases}$$

Corollary 2. *Let C_p^r be the power of a cycle with $p = k(r+1)+s$, then*

$$T(C_p^r) = r + \frac{1 + \lceil \frac{s}{k} \rceil}{k}.$$

Lemma 3. *Let $H_{n,p}$ be the Harary graph for p even and n odd, so $n = 2r+1$ for some r. Then $p \equiv 0 \mod (n+1)$ if and only if $s = 0$ and k is even.*

Proof: Let $k = 2q + 1$, for some q. Thus $p = k(r+1) + s = (2q+1)(r+1)+s = q(n+1)+s+r+1$. Since $s+r+1 < n+1, p \not\equiv 0 \mod (n+1)$. Let $k = 2q$. Thus $p = k(r+1)+s = 2q(r+1)+s = q(2r+1+1)+s = q(n+1)+s$. Thus $p \equiv 0 \mod (n+1)$ if and only if $s = 0$. □

Lemma 4. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, and p even, then*

$$\beta(H_{n,p}) = \begin{cases} k & \text{if } p \not\equiv 0 \mod (n+1) \\ k-1 & \text{if } p \equiv 0 \mod (n+1). \end{cases}$$

Proof: Let $G = H_{n,p}$. Since at least r consecutive vertices must be between any two members of an independent set and $s < r+1$, then $\beta(G) \leq k$. Consider the set $B = \{0, r+1, 2(r+1), 3(r+1), \ldots, (k-1)(r+1)\}$. Let $0 \leq s < r+1$ and assume $k = 2q+1$ for some q. Since vertex i is adjacent to vertex $i + \frac{p}{2} = i + q(r+1) + \frac{s+r+1}{2}$, $1 \leq i \leq \frac{p}{2}$, and $\frac{r+1}{2} \leq \frac{s+r+1}{2} < r+1$, then vertex $t(r+1) \in B$, $0 \leq t \leq k-1$, is not adjacent to vertex $x(r+1)$ for any $1 \leq x \leq k-1$. Thus the set B is an independent set and therefore $\beta(G) = k$.

Assume $0 < s < r+1$ and $k = 2q$ for some q. Since vertex i is adjacent to $i + \frac{p}{2} = i + q(r+1) + \frac{s}{2}$, for all $1 \leq i \leq \frac{p}{2}$ and $0 < \frac{s}{2} < r+1$, we see that $t(r+1) \in B$, $0 \leq t \leq k-1$ is not adjacent to $x(r+1)$ for any $1 \leq x \leq k-1$. Thus B is again an independent set and $\beta(G) = k$.

Suppose $s = 0$ and $k = 2q$ for some q. First, consider the set B and assume $\beta(G) = k$. Since $s = 0$, $p = 2q(r+1)$. So there are r consecutive vertices between two members of an independent set. Thus we need to consider edges of the form $\{i, i + \frac{p}{2}\}$. Since $s = 0$, vertex $i(r+1)$ for any $0 \leq i \leq q-1$, is adjacent to $t(r+1)$ for some $q \leq t \leq k-1$, and this is a contradiction to the definition of independent set, and so $\beta(G) < k$. Now, consider the set $C = \{0, r+1, 2(r+1), \ldots, (q-1)(r+1), q(r+1)+1, \ldots, (k-2)(r+1)+1\}$. Since vertex i is adjacent to vertex $i + \frac{p}{2} = i + q(r+1)$ for any $1 \leq i \leq \frac{p}{2}$, then vertex $i(r+1)$, $0 \leq i \leq q-1$, is not adjacent to $t(r+1)+1$ for $q \leq t \leq 2q-2 = k-2$. Thus C is an independent set. Hence $(k-1) \leq \beta(G) < k$. Therefore since $\beta(G)$ and k are integers, we can conclude that $\beta(G) = k-1$. □

The next theorem provides bounds on the tenacity of the Harary graphs. As the corollaries illustrate, it gives precise values for the tenacity in many instances.

Theorem 3. *Let $H_{n,p}$ be a Harary graph for p even, with n odd, $n = 2r+1$, then*

$$r + \frac{1 + \lceil \frac{s}{k} \rceil}{k} \leq T(H_{n,p}) \leq \begin{cases} r + \frac{s+1}{k} & \text{if } p \not\equiv 0 \mod (n+1) \\ \frac{kr+s+2}{k-1} & \text{if } p \equiv 0 \mod (n+1) \end{cases}$$

Proof: Let $G = H_{n,p}$. By Proposition 3, $T(G) \leq \frac{p-\beta(G)+1}{\beta(G)}$. Thus by Lemma 4, if $p \not\equiv 0 \mod (n+1)$, then $T(G) \leq \frac{p-k+1}{k} = r + \frac{s+1}{k}$, and if $p \equiv 0 \mod (n+1)$, then $T(G) \leq \frac{p-(k-1)+1}{k-1} = \frac{kr+s+2}{k-1}$.

Since $V(H_{2r,p}) = V(G)$ and $E(H_{2r,p}) \subseteq E(G)$, then $H_{2r,p}$ is a spanning subgraph of G. By Proposition 1, we have $T(H_{2r,p}) \leq T(G)$. Thus by Theorem 2, we have $r + \frac{1+\lceil \frac{s}{k} \rceil}{k} \leq T(G)$. □

Corollary 3. *If n is odd, p is even and $s = 1$, then $T(H_{n,p}) = r + \frac{2}{k}$*

Corollary 4. *If n is odd, p is even, $s = 0$ and k is odd then $T(H_{n,p}) = r + \frac{1}{k}$.*

Corollary 5. *If n is odd, p is even, $s = 0$ and k is even then $r + \frac{1}{k} \leq T(H_{n,p}) \leq \frac{kr+2}{k-1}$.*

We now have a lemma analogous to Lemma 3 when p is odd.

Lemma 5. *Let $H_{n,p}$ be the Harary graph with p and n both odd, $n = 2r+1$ and $r > 0$. Then $p \equiv 1 \mod (n+1)$ if and only if $s = 1$ and k is even.*

Proof: Let $1 < s < r+1$ and $k = 2q+1$, for some q. Thus $p = k(r+1)+s = q(n+1)+s+r+1$. Since $1 < s+r+1 < n+1, p \not\equiv 1 \mod (n+1)$.

Now suppose $k = 2q$ and $1 < s < r+1$. Thus $p = q(n+1)+s$. Since $1 < s < n+1, p \not\equiv 1 \mod (n+1)$. If $s = 0$, then $p = k(r+1)$. Since p is odd we know that k is odd. Thus $p = q(n+1)+r+1$. Since $1 < r+1 < (n+1)$, then $p \not\equiv 1 \mod (n+1)$. Finally, consider the case when $s = 1$. If k is odd, then $p = q(n+1)+r+2$. Since $1 < r+2 < n+1$, then $p \not\equiv 1 \mod (n+1)$. If k is even, then $p = q(n+1)+1$. Thus $p \equiv 1 \mod (n+1)$. □

From Lemma 4 and the next lemma we see that the independence number of all Harary graphs is either k or $k-1$.

Lemma 6. *Let $H_{n,p}$ be the Harary graph with p and n both odd, $n = 2r+1$ and $r > 0$. Then*

$$\beta(H_{n,p}) = \begin{cases} k & \text{if } p \not\equiv 1 \mod (n+1) \\ k-1 & \text{if } p \equiv 1 \mod (n+10) \end{cases}$$

Proof: The proof is similar to Lemma 4. □

Analogous to Theorem 3 we also have bounds on the tenacity of Harary graphs of odd order.

Theorem 4. *Let $H_{n,p}$ be the Harary graph with p and n odd, and $n = 2r+1$, then*

$$r + \frac{1+\lceil \frac{s}{k} \rceil}{k} \leq T(H_{n,p}) \leq \begin{cases} r + \frac{s+1}{k} & \text{if } p \not\equiv 1 \mod (n+1) \\ \frac{kr+s+2}{k-1} & \text{if } p \equiv 1 \mod (n+1) \end{cases}$$

Corollary 6. *If p and n are odd and $s = 0$, then $T(H_{n,p}) = r + \frac{1}{k}$.*

Corollary 7. *If p and n are odd, $s = 1$ and k is odd, then $T(H_{n,p}) = r + \frac{2}{k}$.*

Corollary 8. *If p and n are odd, $s = 1$ and k is even, then $r + \frac{2}{k} \leq T(H_{n,p}) \leq \frac{kr+3}{k-1}$.*

Now we will investigate the tenacity of Harary graphs with n odd and even order. Note that if $s = 0$ then Corollaries 4 and 5 give the tenacity and bounds on the tenacity of $H_{n,p}$ respectively.

Lemma 7. *Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p even, $r \geq 2$, $0 < s < r+1$, $s < k$, and k odd. Then there is a cut-set A with kr elements such that $\omega(H_{n,p} - A) = k$, and $\tau(H_{n,p} - A) = 2$.*

Proof: We may assume $H_{n,p}$ is labeled by $0, 1, 2, \ldots, p - 1$. Let $s < k$, then $s = k - l$ for some l and so $p = s(r + 2) + l(r + 1)$. Since k is odd, $k = 2q + 1$ for some q.

Case 1: If r is even then s is odd and l is even. Therefore $s = k - l \geq 1$ and $l = 2t$ for some t. Hence $k - l = 2q + 1 - 2t \geq 1$ which implies that $q \geq t$. Define the sets W_i for $1 \leq i \leq 2q + 1$ as follows:

$$W_i = \begin{cases} \{ir + i\} & 1 \leq i \leq t \\ \{ir - t + 2i - 1, ir - t + 2i\} & t + 1 \leq i \leq q \\ \{ir - t + q + i\} & q + 1 \leq i \leq q + t \\ \{ir - 2t + 2i - 1, ir - 2t + 2i\} & q + t + 1 \leq i \leq 2q + 1 \end{cases}$$

Let $W = \bigcup_{i=1}^{2q+1} W_i$ and $A = V(G) - W$. The number of vertices in W is equal to $t + 2(q - t) + t + 2(q - t + 1) = 2(2q + 1) - 2t = 2k - l = k + s$, so $|A| = p - k - s = kr$. Now, we can see that for any $1 \leq i \leq 2q + 1$, the elements in W_i differ from those in W_{i+1} by at least $r + 1$. Note that if two vertices differ by $r + 1$ then there are r vertices between them. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \leq i < j \leq 2q + 1$, by an edge in the copy of $H_{2r,p}$ in G. Thus we need only consider edges of the form $\{x, x + \frac{p}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p}{2}$. Hence, since $\frac{p}{2} = qr + 2q - t + 1 + \frac{r}{2} < (q + 1)r - t + q + (q + 1)$, we need to consider only vertices in W_i for $1 \leq i \leq q$.

So consider $W_i = \{ir + i\}$ for $1 \leq i \leq t$. Then $ir + i + \frac{p}{2} = (q + i)r - t + q + (q + i) + 1 + \frac{r}{2} > (q + i)r - t + q + (q + i) = jr - t + q + j$, for $j = q + i$. Also, $ir + i + \frac{p}{2} < (q + i)r - t + q + (q + i) + 1 + r = (q + i + 1)r - t + q + (q + i + 1) = (j + 1)r - t + q + (j + 1)$. Therefore the set $\{ir + i + \frac{p}{2}\}$ is strictly between W_j and W_{j+1} for $j = q + i$, and so it is contained in A.

Finally, consider $W_i = \{ir - t + 2i - 1, ir - t + 2i\}$ for $t + 1 \leq i \leq q$. Then $r - t + 2i - 1 + \frac{p}{2} = (q + i)r - 2t + 2q + 2i + \frac{r}{2} > (q + i)r - 2t +$

$2q + 2i = (q + i)r - 2t + 2(q + i) = jr - 2t + 2j$, for $j = q + i$. Also, $ir-t+2i+\frac{p}{2} = (q+i)r-2t+2(q+i)+1+\frac{r}{2} < (q+i)r-2t+2(q+i)+1+r = (q+i+1)r - 2t + 2(q+i+1) - 1 = (j+1)r - 2t + 2(j+1) - 1$.

Hence the set $\{ir - t + 2i - 1 + \frac{p}{2}, ir - t + 2i + \frac{r}{2}\}$ is strictly between W_j and W_{j+1} for $j = q + i$ and so it is contained in A. Therefore the W_i, $1 \leq i \leq 2q + 1 = k$ are the components of $H_{n,p} - A$, so $\tau(H_{n,p} - A) = 2$, and $\omega(H_{n,p} - A) = k$.

Case 2: If r is odd, then s is even and l is odd. Hence $s = 2h$ for some h. Define the sets W_i for $1 \leq i \leq 2q + 1$ as follows:

$$W_i = \begin{cases} \{ir + 2i - 1, ir + 2\} & 1 \leq i \leq h \\ \{ir + i + h\} & h + 1 \leq i \leq q \\ \{ir + 2i - q + h - 1, ir + 2i - q - h\} & q + 1 \leq i \leq q + h \\ \{ir + i + 2h\} & q + h + 1 \leq i \leq 2q + 1 \end{cases}$$

Let $W = \bigcup_{i=1}^{2q+1} W_i$ and $A = V(G) - W$. The number of vertices in W is equal to $2h + (q - h) + 2h + q - h + 1 = 2q + 1 + 2h = k + s$, so $|A| = p - k - s = kr$. Now, we can see that for any $1 \leq i \leq 2q + 1$, the elements in W_i differ from those in W_{i+1} by at least $r+1$. Hence, no vertex in W_i is adjacent to a vertex in W_j, $1 \leq i < j \leq 2q+1$, by an edge in the copy of $H_{2r,p}$ in G. Thus we need only consider edges of the form $\{x, x+\frac{p}{2}\}$. In fact, we need to consider only such edges when x is at most $\frac{p}{2}$. Hence, since $\frac{p}{2} = qr + q + h + \frac{r}{2} + \frac{1}{2} < (q+1)r + 2(q+1) - q + h - 1$, we need only consider vertices in W_i for $1 \leq i \leq q$.

So consider $W_i = \{ir + 2i - 1, ir + 2i\}$ for $1 \leq i \leq h$. Since $r \geq 2$, then $ir+2i-1+\frac{p}{2} = (q+i)r+2(q+i)-q+h+\frac{r}{2}-\frac{1}{2} > (q+i)r+2(q+i)-q+h = jr+2j-q+h$, for $j = q+i$. Also, $ir+2i+\frac{p}{2} = (q+i)r+2(q+i)-q+h+\frac{r}{2}+\frac{1}{2} < (q+i+1)r + 2(q+i+1) - q + h - 1 = (j+1)r + 2(j+1) - q + h - 1$. Therefore the set $\{ir + 2i - 1 + \frac{p}{2}, ir + 2i + \frac{p}{2}\}$ is strictly between W_j and W_{j+1} for $j = q + i$, and so it is contained in A.

Finally, consider $W_i = \{ir+i+h\}$ for $h+1 \leq i \leq q$. Then $ir+i+h+\frac{p}{2} = (q+i)r+(q+i)+2h+\frac{r}{2}+\frac{1}{2} > (q+i)r+(q+i)+2h = jr+j+2h$, for $j = q+i$. Also, $ir+i+h+\frac{p}{2} < (q+i+1)r+(q+i+1)+2h = (j+1)r+(j+1)+2h$. Hence the set $\{ir+i+h+\frac{p}{2}\}$ is strictly between W_j and W_{j+1} for $j = q+i$ and so it is contained in A. Therefore the W_i, $1 \leq i \leq 2q + 1 = k$ are the components of $H_{n,p} - A$, so $\tau(H_{n,p} - A) = 2$, and $\omega(H_{n,p} - A) = k$. □

Using this result, we have the first theorem for even order Harary graphs with odd n.

Theorem 5. *Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p even, k odd, $0 < s < r + 1$ and $s < k$. Then $T(H_{n,p}) = r + \frac{2}{k}$.*

Proof: Note first that if $r = 1$ then $s = 1$ and so $p = 2k+1$, a contradiction. Thus we may assume that $r \geq 2$. By Theorem 3, we have $r + \frac{1+\lceil\frac{s}{2}\rceil}{k} \leq$

$T(H_{n,p})$. Hence, since $s < k$, we have $r + \frac{2}{k} \leq T(H_{n,p})$. By Lemma 7, there is a cut-set A of $H_{n,p}$ with kr elements such that $\tau(H_{n,p} - A) = 2$ and $\omega(H_{n,p} - A) = k$. Therefore, the tenacity attains the lower bound using A, so $(H_{n,p}) = r + \frac{2}{k}$. □

Now we consider the cases when $s \geq k$. The following two lemmas are needed in the proofs of lemmas 10, 11, and 12.

Lemma 8. *Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p even, $0 < s < r + 1$, k odd and $s \geq k$, where $s = ak + b$, for some a and b.*

Case 1: For $0 < b < k$ then

(i) if r is even, then

$$r \geq \begin{cases} 6 & a \text{ odd} \\ 8 & a \text{ even} \end{cases}$$

and

(ii) if r is odd, then

$$r \geq \begin{cases} 5 & a \text{ odd} \\ 9 & a \text{ even} \end{cases}$$

Case 2: For $b = 0$, so $s = ak$, then

(i) if r is even then $r \geq 4$ and

(ii) if r is odd then $r \geq 7$.

Proof: We will prove only the case when r is even and $0 < b < k$ and leave the remaining cases to the reader. Hence s is odd.

If a is odd then b is even and the minimum value for b is 2. Since a and k are odd and $b < k$, the minimum values for k and a are 3 and 1 respectively. Therefore the minimum value for s is $1(3) + 2 = 5$. Since r is even and $s < r + 1$, we have $r \geq 6$.

Similarly, if a is even then b is odd and the minimum value for b is 1. Since a is even, k is odd and $b < k$, the minimum values for k and a are 3 and 2 respectively. Therefore the minimum value for s is $2(3) + 1 = 7$. Since r is even and $s < r + 1$, we have $r \geq 8$. □

Lemma 9. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p even, $0 < s < r+1$, and k odd. If $s = ak+b$ for some a and b, $0 < b < k$, then $a+1 < \frac{r}{2}$. If $b = 0$, so $s = ak$, then $a + 1 \leq \frac{r}{2}$.*

Proof: Again we will prove only the case when r is even and $0 < b < k$ and leave the remaining cases to the reader. Hence s is odd, so $s < r$.

If a is odd then b is even and the minimum values for b and k are 2 and 3 respectively. Thus $3a+2 \leq ak+b=s$. Since $2a+2<3a+2$, we have $2a+2<s<r$. Therefore $a+1<\frac{r}{2}$.

Similarly, if a is even then b is odd and the minimum value for b is 1. Since $b<k$ and k is odd, the minimum value for k is 3. Thus $3a+1 \leq ak+b=s$. Since a is even, we have $2a+2<3a+1$. Thus $2a+2<s<r$ and so $a+1<\frac{r}{2}$. □

Now, we consider the cases when p is a multiple of k.

Lemma 10. *Let $H_{n,p}$ be the Harary graph with $n=2r+1$, p even, k odd, and $s=ak$ for some a. Then there is a cut-set A with kr elements such that $\omega(H_{n,p}-A)=k$, and $\tau(H_{n,p}-A)=a+1$.*

Proof: Let $s=ak$ and $k=2q+1$ for some q. The proof of this lemma is similar to Lemma 7 with $W_i=\{i(r+1)+(i-1)a,\ldots,i(r+1)+ia\}$ for $1 \leq i \leq 2q+1$. □

If $H_{n,p}$ is an even order Harary graph with $n=2r+1$ and p is a multiple of k then we have the following theorem.

Theorem 6. *Let $H_{n,p}$ be the Harary graph with $n=2r+1$, p even, $0<s<r+1$, k odd, and $s=ak$ for some a. Then $T(H_{n,p})=r+\frac{1+a}{k}$.*

Proof: By Theorem 3, we have $r+\frac{1+a}{k} \leq T(H_{n,p})$. Set A of Lemma 10 achieves this lower bound, since $|A|=kr$, $\tau(H_{n,p}-A)=a+1$, and $\omega(H_{n,p}-A)=k$. Hence $\frac{|A|+\tau(H_{n,p}-A)}{\omega(H_{n,p}-A)}=r+\frac{1+a}{k}$ and the result follows.□

If p is not a multiple of k, in particular, then we have the following lemma.

Lemma 11. *Let $H_{n,p}$ be the Harary graph with $n=2r+1$, p even, $0<s<r+1$, k odd and $s>k$, where $s=ak+b$ for some a and b, $0<b<k$. Then there is a cut-set A with kr elements such that $\omega(H_{n,p}-A)=k$, and $\tau(H_{n,p}-A)=a+2$.*

Proof: Let $s>k$, where $s=ak+b$ and $0<b<k$. Thus $p=kr+(k-b)(a+1)+b(a+2)$.

Write $p=k(r+1)+s$. If r is even, then s is odd. In this case a is odd if and only if b is even. If r is odd, then s is even. In this case a is even if and only if b is even. The remainder of this proof is similar to the proof of Lemma 7 with W_i defined as given in the following cases.

Case 1: Let r be even and a odd, or r odd and a even. Hence b is even.

Therefore $b = 2h$ for some h. Define

$$W_i = \begin{cases} \{i(r+2)+(i-1)a-1,\ldots,i(r+2)+ia\} \\ \quad 1 \le i \le h \\ \{i(r+1)+(i-1)a+h,\ldots,i(r+1)+ia+h\} \\ \quad h+1 \le i \le q \\ \{i(r+2)+(i-1)a-q+h-1,\ldots,i(r+2)+ia-q+h\} \\ \quad q+1 \le i \le q+h \\ \{i(r+1)+(i-1)a+2h,\ldots,i(r+1)+ia+2h\} \\ \quad q+h+1 \le i \le 2q+1 \end{cases}$$

Case 2: Let r and a both be even or r and a both be odd. Then b is odd and hence $k-b$ is even. Therefore $k-b=2t$ for some t. Define

$$W_i = \begin{cases} \{i(r+1)+(i-1)a,\ldots,i(r+1)+ia\} \\ \quad 1 \le i \le t \\ \{i(r+2)+(i-1)a-t-1,\ldots,i(r+2)+ia-t\} \\ \quad t+1 \le i \le q \\ \{i(r+1)+(i-1)a+q-t,\ldots,i(r+1)+ia+q-t\} \\ \quad q+1 \le i \le q+t \\ \{i(r+2)+(i-1)a-2t-1,\ldots,i(r+2)+ia-2t\} \\ \quad q+t+1 \le i \le 2q+1 \end{cases}$$

□

Now, we can determine precisely the tenacity for the remaining Harary graphs, when p is even and k is odd.

Theorem 7. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p even, k odd, $0 < s < r+1$, and $s > k$, where $s = ak+b$ for some a and b and $0 < b < k$. Then $T(H_{n,p}) = r + \frac{a+2}{k}$.*

Proof: By Lemma 11, there is a cut-set A of $H_{n,p}$ with kr elements. The number of components of $H_{n,p} - A$ is k and the largest component of $H_{n,p} - A$ has cardinality $a+2$. Hence, $\frac{|A|+\tau(H_{n,p}-A)}{\omega(H_{n,p}-A)} = r + \frac{a+2}{k}$. By Theorem 3, we have $r + \frac{1+\lceil \frac{s}{k} \rceil}{k} \le T(H_{n,p})$. Since $s = ak+b$ for $0 < b < k$, then $\lceil \frac{s}{k} \rceil = a+1$. Hence $r + \frac{a+2}{k} \le T(H_{n,p})$, and the theorem follows. □

Lemma 12. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both even, $r \ge 2$, $0 < s < r+1$, and $s < k$. Then there is a cut-set A with $kr+1$ elements such that $\omega(H_{n,p} - A) = k$, and $\tau(H_{n,p} - A) = 2$.*

Proof: We may assume $H_{n,p}$ is labeled by $0, 1, 2, \ldots, p-1$. Let $s < k$, so $s = k - l$ for some l. Since p and k are even, then s is even and so l is even.

Hence $l = 2t$ and $k = 2q$, for some t and q. Thus $s = k - l = 2q - 2t \geq 2$, which implies that $q \geq t + 1$. The remainder of this proof is similar to the proof of Lemma 7 with W_i defined as given in the following cases.

Case 1: Let $s = 2$, so $q = t + 1$. Define $W_i = \{i(r + 1)\}$ for $0 \leq i \leq 2q$.

Case 2: Let $q > t + 1$. Define

$$W_i = \begin{cases} \{i(r+2)-1, i(r+2)\} & 1 \leq i \leq q-t-1 \\ \{i(r+1)+q-t-1\} & q-t \leq i \leq q+1 \\ \{i(r+2)-t-3, i(r+2)-t-2\} & q+2 \leq i \leq 2q-t \\ \{i(r+1)+2q-2t-2\} & 2q-t+1 \leq i \leq 2q \end{cases}.$$

□

This lemma provides us with tight bounds on the tenacity of even order Harary graphs.

Theorem 8. *Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p and k both even, $0 < s < r + 1$, and $s < k$. Then $r + \frac{2}{k} \leq T(H_{n,p}) \leq r + \frac{3}{k}$.*

Proof: Again note that if $r = 1$ then $s = 1$ and so $p = 2k + 1$, a contradiction. Thus we may assume that $r \geq 2$. By Theorem 3, we have $r + \frac{1 + \lceil \frac{s}{k} \rceil}{k} \leq T(H_{n,p})$. Since $s < k$, we have $\lceil \frac{s}{k} \rceil = 1$. Thus $r + \frac{2}{k} \leq T(H_{n,p})$. By Lemma 12, there is a cut-set A of $H_{n,p}$ with $kr+1$ elements. The largest component of $H_{n,p} - A$, has cardinality 2 and there are k components in $H_{n,p} - A$. Hence $\frac{|A| + \tau(H_{n,p} - A)}{\omega(H_{n,p} - A)} = r + \frac{3}{k}$. Therefore, $r + \frac{2}{k} \leq T(H_{n,p}) \leq r + \frac{3}{k}$.
□

For example, $H_{25,1308}$ has $12.02 \leq T(H_{25,1308}) \leq 12.03$.

As before the following two lemmas are required in the proofs of lemmas 14, 16, and 17.

Lemma 13. *Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p and k both even, $0 < s < r + 1$, $s > k$ where $s = ak + b$, for some a and b.*

Case 1: If $0 < b < k$, then

$$r \geq \begin{cases} 6 & a \text{ odd} \\ 10 & a \text{ even} \end{cases}.$$

Case 2: If $b = 0$, so $s = ak$, then

$$r \geq \begin{cases} 2 & a \text{ odd} \\ 4 & a \text{ even} \end{cases}.$$

In addition, if $k > 2$ and $b = 0$ then

$$r \geq \begin{cases} 4 & a \text{ odd} \\ 8 & a \text{ even} \end{cases}.$$

Proof: The proof of this is similar to the proof of Lemma 8. □

Note that if r is odd then all of the bounds in Lemma 13 are increased by one.

Lemma 14. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k are even, $0 < s < r+1$, then the following hold.*

Case 1: If $s > k$, so $s = ak+b$, for some a and b, $0 < b < k$, then $a+1 < \frac{r}{2}$.

Case 2: If $s = ak$, for some a, then $a \leq \frac{r}{2}$.

Case 3: If $s = ak$ and $k > 2$, then $a \leq \frac{r}{4}$.

Proof: The proof of this lemma is similar to the proof of Lemma 9. □

If p and k are both even and p is a multiple of k, we have the following lemma.

Lemma 15. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both even, $k > 2$, and $0 < s < r+1$. Write $s = ak$ for some a, and $k = 2q$ for some q. Then there is a cut-set A with $kr+1$ elements such that $\omega(H_{n,p} - A) = k$, and $\tau(H_{n,p} - A) = a + 2 + z$, where $z = \lfloor \frac{a-1}{q-1} \rfloor$.*

Proof: The proof of this lemma is similar to the proof of Lemma 7 with W_i defined as given in the following cases.

Case 1: Suppose $a < q$ and so $z = 0$. Define

$$W_i = \begin{cases} \{ir + (i-1)a + 2i - 1, \ldots, ir + ia + 2i\} \\ \quad 1 \leq i \leq a-1 \\ \{ir + ia + i - 1, \ldots, ir + (i+1)a + i - 1\} \\ \quad a \leq i \leq q-1 \\ \{ir + qa + i - 1\} \\ \quad q \leq i \leq q+1 \\ \{ir + (i-2)a + 2(i-1) - q - 1, \ldots, ir + (i-1)a + 2(i-1) - q\} \\ \quad q+2 \leq i \leq q+a \\ \{ir + (i-1)a + (i-1) - 1, \ldots, ir + ia + (i-1) - 1\} \\ \quad q+a+1 \leq i \leq 2q \end{cases}.$$

Case 2: Now, suppose $a = q$ and so $z = 1$. Define

$$W_i = \begin{cases} \{ir + (i-1)a + 2i - 1, \ldots, ir + ia + 2i\} \\ \quad 1 \le i \le q-1 \\ \{ir + qa + i - 1\} \\ \quad q \le i \le q+1 \\ \{ir + (i-2)a + 2i - q - 3, \ldots, ir + (i-1)a + 2i - q - 2\} \\ \quad q+2 \le i \le 2q \end{cases}.$$

Case 3: Finally, suppose $a - 1 = z(q-1) + u$ for some integers z and u, $0 < u < q-1$, which implies that $a > q$. Define

$$W_i = \begin{cases} \{i(r+2)+(i-1)(a+z)-1, \ldots, i(r+2)+i(a+z)\} \\ \quad 1 \le i \le u \\ \{i(r+1)+(i-1)(a+z)+u, \ldots, i(r+1)+i(a+z)+u\} \\ \quad u+1 \le i \le q-1 \\ \{i(r+1)+qa-1\} \\ \quad q \le i \le q+1 \\ \{i(r+2)+(i-3)(a+z)-q+u-2, \ldots, i(r+2)+(i-2)(a+z)-q+u-1\} \\ \quad q+2 \le i \le q+u+1 \\ \{i(r+1)+(i-3)(a+z)+2u, \ldots, i(r+1)+(i-2)(a+z)+2u\} \\ \quad q+u+2 \le i \le 2q \end{cases}.$$

□

Theorem 9. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both even, $k > 2$, $0 < s < r+1$, where $s = ak$ for some a, and $k = 2q$ for some q, then $r + \frac{a+2}{k} \le T(H_{n,p}) \le r + \frac{a+3+z}{k}$, where $z = \lfloor \frac{a-1}{q-1} \rfloor$.*

Proof: By Theorem 3, we have $r + \frac{1+\lceil \frac{s}{k} \rceil}{k} \le T(H_{n,p})$. Since $s = ak$, then $\lceil \frac{s}{k} \rceil = a$. Hence $r + \frac{a+2}{k} \le T(H_{n,p})$. By Lemma 15, there is a cut-set A of $H_{n,p}$ with $kr+1$ elements. The number of components of $H_{n,p} - A$ is k and the largest component of $H_{n,p} - A$ has cardinality $a + 2 + z$. Hence, $\frac{|A| + \tau(H_{n,p}-A)}{\omega(H_{n,p}-A)} = r + \frac{a+3+z}{k}$, and the theorem follows. □

For example, $H_{65,296}$ has $32.750 \le T(H_{65,296}) \le 33$.

Lemma 16. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both even, $k > 2$, $0 < s < r+1$ and $s > k$. Write $s = ak + b$ for some a and b. Then there is a cut-set A with $kr+1$ elements such that $\omega(H_{n,p} - A) = k$, and $\tau(H_{n,p} - A) = a + 2 + z$, where $z = \lfloor \frac{2a+b-2}{k-2} \rfloor$.*

Proof: Write $s = ak + b$, for $0 < b < k$. Since p and k are even, then s is even. Hence b is even. Thus $b = 2t$ and $k = 2q$ for some t and q. The proof of this lemma is similar to the proof of Lemma 7 with W_i defined as given in the following cases.

Case 1: Suppose $a+t<q$ and so $z=0$. Define

$$W_i=\begin{cases}\{ir+(i-1)a+2i-1,\dots,ir+ia+2i\} \\ \quad 1\le i\le a+t-1 \\ \{ir+ia+i+t-1,\dots,ir+(i+1)a+i+t-1\} \\ \quad a+t\le i\le q-1 \\ \{ir+qa+t+i-1\} \\ \quad q\le i\le q+1 \\ \{ir+(i-2)a+2(i-1)+t-q-1,\dots,ir+(i-1)a+2(i-1)+t-q\} \\ \quad q+2\le i\le q+a+t \\ \{ir+(i-1)a+(i-1)+2t-1,\dots,ir+ia+(i-1)+2t-1\} \\ \quad q+a+t+1\le i\le 2q\end{cases}.$$

Case 2: Suppose $a+t-1=z(q-1)$ for some integer z. Define

$$W_i=\begin{cases}\{ir+(i-1)(a+z)+i,\dots,ir+i(a+z)+i\} & 1\le i\le q-1 \\ \{ir+(q-1)a+(q-1)z+i\} & q\le i\le q+1 \\ \{ir+(i-3)(a+z)+i,\dots,ir+(i-2)(a+z)+i\} & q+2\le i\le 2q\end{cases}.$$

Case 3: Suppose that $a+t-1=z(q-1)+c$ for some integer z and c, $0<c<q-1$. Define

$$W_i=\begin{cases}\{ir+(i-1)(a+z)+2i-1,\dots,ir+i(a+z)+2i\} \\ \quad 1\le i\le c \\ \{ir+(i-1)(a+z)+i+c,\dots,ir+i(a+z)+i+c\} \\ \quad c+1\le i\le q-1 \\ \{ir+(q-1)(a+z)+i+c\} \\ \quad q\le i\le q+1 \\ \{ir+(i-3)(a+z)+2i-q+c-2,\dots,ir+(i-2)(a+z)+2i-q+c-1\} \\ \quad q+2\le i\le q+c+1 \\ \{ir+(i-3)(a+z)+i+2c,\dots,ir+(i-2)(a+z)+i+2c\} \\ \quad q+c+2\le i\le 2q\end{cases}.$$

□

Theorem 10. *Let $H_{n,p}$ be the Harary graph with $n=2r+1$, p and k both even, $k>2$, $0<s<r+1$, and $s>k$, where $s=ak+b$ for some a and b, and $0<b<k$. Then $r+\frac{a+2}{k}\le T(H_{n,p})\le r+\frac{a+3+z}{k}$, where $z=\lfloor\frac{2a+b-2}{k-2}\rfloor$.*

Proof: By Lemma 16, there is a cut-set A of $H_{n,p}$ with $kr+1$ elements with $\omega(H_{n,p}-A)=k$ and $\tau(H_{n,p}-A)=a+2+z$. Hence $\frac{|A|+\tau(H_{n,p}-A)}{\omega(H_{n,p}-A)}=r+\frac{a+3+z}{k}$. Also, since $s=ak+b$ for $0<b<k$, we have $\lceil\frac{s}{k}\rceil=a+1$. Hence, by Theorem 3, $r+\frac{a+2}{k}\leq T(H_{n,p})$. □

For example, $H_{93,422}$ has $46.875\leq T(H_{93,422})\leq 47.125$.

Finally, we will investigate the tenacity of Harary graphs with n odd and odd order. First note that if $s=0$ then $p\equiv 0 \mod (r+1)$ and so $T(H_{n,p})=r+\frac{1}{k}$ by Corollary 6. Also note that if $s=1$ then $T(H_{n,p})$ is given by Corollaries 7 and 8.

Lemma 17. *Let $H_{n,p}$ be the Harary graph with $n=2r+1$, p and k both odd, $r\geq 2$, $1<s<r+1$, and $s<k$. Then there is a cut-set A with kr elements such that $\omega(H_{n,p}-A)=k$, and $\tau(H_{n,p}-A)=2$.*

Proof: We may assume $H_{n,p}$ is labeled by $0,1,2,\ldots,p-1$. Let $s<k$, then $s=k-l$ for some l. Thus $p=s(r+2)+l(r+1)$. Since k is odd then $k=2q+1$ for some q. The proof of this lemma is similar to the proof of Lemma 7 with W_i defined as given in the following cases.

Case 1: If r is odd then s is odd and l is even. Then $s=k-l\geq 1$ and $l=2t$ for some t. Hence $k-l=2q+1-2t\geq 1$ which implies that $q\geq t$. Define

$$W_i=\begin{cases}\{ir+i\} & 1\leq i\leq t\\ \{ir-t+2i-1, ir-t+2i\} & t+1\leq i\leq q\\ \{ir-t+q+i\} & q+1\leq i\leq q+t\\ \{ir-2t+2i-1, ir-2t+2i\} & q+t+1\leq i\leq 2q+1\end{cases}.$$

Case 2: If r is even then s is even and l is odd. Hence $s=2h$ for some h. Define

$$W_i=\begin{cases}\{ir+2i-1, ir+2i\} & 1\leq i\leq h\\ \{ir+i+h\} & h+1\leq i\leq q\\ \{ir+2i-q+h-1, ir+2i-q+h\} & q+1\leq i\leq q+h\\ \{ir+i+2h\} & q+h+1\leq i\leq 2q+1\end{cases}.$$

□

Using this result, we have the first theorem for odd order Harary graphs with odd n.

Theorem 11. *Let $H_{n,p}$ be the Harary graph with $n=2r+1$, p and k both odd, $1<s<r+1$ and $s<k$. Then $T(H_{n,p})=r+\frac{2}{k}$.*

Proof: First we see that $r\geq 2$, since if $r=1$, $1<s<2$. By Theorem 3, we have $r+\frac{1+\lceil\frac{s}{k}\rceil}{k}\leq T(H_{n,p})$. Hence, if $s<k$, then $(r+\frac{2}{k})\leq T(H_{n,p})$. By

Lemma 17, there is a cut-set A of $H_{n,p} - A$ with kr elements. The largest component of $H_{n,p} - A$ has cardinality 2 and there are k components in $H_{n,p} - A$. Therefore, $T(H_{n,p}) = r + \frac{2}{k}$. □

Now we consider the cases when $s \geq k$ and p and k are odd. Again as before the following two lemmas are required in the proofs of Lemmas 20, 21, and 22.

Lemma 18. *Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p and k both odd, $1 < s < r + 1$, and $s \geq k$, where $s = ak + b$, for some a and b.*

Case 1: For $0 < b < k$,

(i) if r is even, then

$$r \geq \begin{cases} 8 & a \text{ even} \\ 4 & a \text{ odd} \end{cases}.$$

(ii) if r is odd, then

$$r \geq \begin{cases} 7 & a \text{ even} \\ 5 & a \text{ odd} \end{cases}.$$

Case 2: For $b = 0$, $s = ak$,

(i) if r is even, then $r \geq 6$, and

(ii) if r is odd, then $r \geq 3$.

Proof: The proof of this is similar to the proof of Lemma 8. □

Lemma 19. *Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p and k both odd, $1 < s < r + 1$, $s \geq k$, where $s = ak + b$ for some a and b. If $0 < b < k$ then $a + 1 \leq \frac{r}{2}$ with equality possible only if r is even and a is odd. If $b = 0$, so that $s = ak$, then $a + 1 \leq \lceil \frac{r}{2} \rceil$.*

Proof: The proof of this lemma is similar to the proof of Lemma 9. □

Lemma 20. *Let $H_{n,p}$ be the Harary graph with $n = 2r + 1$, p and k both odd, $1 < s < r + 1$, and $s = ak$ for some a. Then there is a cut-set A with kr elements such that $\tau(H_{n,p} - A) = a + 1$, and $\omega(H_{n,p} - A) = k$.*

Proof: Let $s = ak$ and $k = 2q + 1$ for some q. The proof of this lemma is similar to the proof of Lemma 7 with $W_i = \{ir + (i-1)a + i, \ldots, ir + ia + i\}$ for $1 \leq i \leq 2q + 1$. □

Theorem 12. *Let $H_{n,p}$ be the Harary graphs with $n = 2r + 1$, p and k both odd, $1 < s < r + 1$ and $s = ak$ for some a. Then $T(H_{n,p}) = r + \frac{1+a}{k}$.*

Proof: By Theorem 3, we have $r + \frac{1+a}{k} \leq T(H_{n,p})$. Set A of Lemma 20 achieves this lower bound, since $|A| = kr$, $\tau(H_{n,p} - A) = a+1$, and $\omega(H_{n,p} - A) = k$. Therefore $T(H_{n,p}) = r + \frac{1+a}{k}$ when s is a multiple of k and k is odd. □

If p is not a multiple of k, in particular, then we have the following lemma.

Lemma 21. *Let $H_{n,p}$ be the Harary graph with p and k both odd, $n = 2r+1$, $1 < s < r+1$, and $s > k$, where $s = ak+b$ for some a and b, $0 < b < k$. Then there is a cut-set A with kr elements such that $\omega(H_{n,p} - A) = k$, and $\tau(H_{n,p} - A) = a + 2$.*

Proof: Let $s > k$, and $s = ak + b$ for $0 < b < k$. Thus $p = kr + (k - b)(a + 1) + b(a + 2)$. Write $p = k(r + 1) + s$. If r is even, then s is even. In this case a is even if and only if b is even. If r is odd, then s is odd. In this case a is odd if and only if b is even. Thus we have the following two cases, both of whose proofs are similar to the proof of Lemma 7 with W_i as defined.

Case 1: Let r and a both be even, or r and a both odd. Hence b is even. Therefore $b = 2h$ for some h. Since k is odd, $k = 2q + 1$ for some q. Hence $k - b = 2q + 1 - 2h \geq 1$ which implies that $q \geq h$. Define

$$W_i = \begin{cases} \{i(r+2) + (i-1)a - 1, \ldots, i(r+2) + ia\} \\ \quad 1 \leq i \leq h \\ \{i(r+1) + (i-1)a + h, \ldots, i(r+1) + ia + h\} \\ \quad h+1 \leq i \leq q \\ \{i(r+2) + (i-1)a - q + h - 1, \ldots, i(r+2) + ia - q + h \\ \quad q+1 \leq i \leq q+h \\ \{i(r+1) + (i-1)a + 2h, \ldots, i(r+1) + ia + 2h\} \\ \quad q+h+1 \leq i \leq 2q+1 \end{cases}.$$

Case 2: If r is even and a is odd, or r is odd and a is even, then b is odd and hence $k - b$ is even. Therefore $k - b = 2t$ for some t. Define

$$W_i = \begin{cases} \{i(r+1) + (i-1)a, \ldots, i(r+1) + ia\} \\ \quad 1 \leq i \leq t \\ \{i(r+2) + (i-1)a - t - 1, \ldots, i(r+2) + ia - t\} \\ \quad t+1 \leq i \leq q \\ \{i(r+1) + (i-1)a + q - t, \ldots, i(r+1) + ia + q - t\} \\ \quad q+1 \leq i \leq q+t \\ \{i(r+2) + (i-1)a - 2t - 1, \ldots, i(r+2) + ia - 2t\} \\ \quad q+t+1 \leq i \leq 2q+1 \end{cases}.$$

□

Theorem 13. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p and k both odd, $1 < s < r+1$, and $s > k$, where $s = ak + b$ for some a and b, $0 < b < k$. Then $T(H_{n,p}) = r + \frac{2+a}{k}$.*

Proof: By Theorem 3, we have $r + \frac{2+a}{k} \leq T(H_{n,p})$, since $s = ak + b$ for $0 < b < k$. Set A of Lemma 21 achieves this lower bound, since $|A| = kr$, $\tau(H_{n,p} - A) = a+2$, and $\omega(H_{n,p} - A) = k$. Therefore $T(H_{n,p}) = r + \frac{2+a}{k}$ when s is a multiple of k and k is odd. □

Lemma 22. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p odd, $r \geq 2$, $1 < s < r+1$, $s < k$, and k even. Then there is a cut-set with $kr+1$ elements such that $\omega(H_{n,p} - A) = k$ and $\tau(H_{n,p} - A) = 2$.*

Proof: We may assume $H_{n,p}$ is labeled by $0, 1, 2, \ldots, p-1$. Let $s < k$, then $s = k - l$ for some l. Since p is odd and k is even, then s is odd. Hence $l = 2t+1$ and $k = 2q$, for some t and q. Thus $s = k - l = 2q - 2t - 1 > 1$, which implies that $q > t+1$. The proof of this lemma is similar to the proof of Lemma 7 with

$$W_i = \begin{cases} \{i(r+2)-1, i(r+2)\} & 1 \leq i \leq q-t-1 \\ \{i(r+1)+q-t-1\} & q-t \leq i \leq q+1 \\ \{i(r+2)-t-3, i(r+2)-t-2\} & q+2 \leq i \leq 2q-t \\ \{i(r+1)+2q-2t-2\} & 2q-t+1 \leq i \leq 2q \end{cases}.$$

□

This lemma provides us with tight bounds on the tenacity of some odd order Harary graphs.

Theorem 14. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p odd, k even, $1 < s < r+1$ and $s < k$. Then $r + \frac{2}{k} \leq T(H_{n,p}) \leq r + \frac{3}{k}$.*

Proof: First note that if $r = 1$ then $1 < s < 2$, a contradiction. Hence we have $r \geq 2$. By Theorem 3, we have $r + \frac{2}{k} \leq T(H_{n,p})$. By Lemma 22, there is a cut-set A with $kr+1$ elements. The largest component of $H_{n,p} - A$ has cardinality 2 and $\omega(H_{n,p} - A) = k$. Hence, $\frac{|A| + \tau(H_{n,p}-A)}{\omega(H_{n,p}-A)} = r + \frac{3}{k}$. Therefore $r + \frac{2}{k} \leq T(H_{n,p}) \leq r + \frac{3}{k}$. □

For example, $H_{15,325}$ has $7.050 \leq T(H_{15,325}) \leq 7.075$. Now we consider the cases when $s > k$, k is even, and p is odd. The following two lemmas are needed in the proof of Lemma 25.

Lemma 23. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p odd, k even, $k > 2$, $1 < s < r+1$, and $s > k$ where $s = ak + b$, for some a and b, $0 < b < k$. Then*

$$r \geq \begin{cases} 5 & a \text{ odd} \\ 9 & a \text{ even} \end{cases}.$$

Proof: The proof of this lemma is similar to the proof of Lemma 8. □

Note that if r is even then the bounds in the previous lemma can be increased by 1.

Lemma 24. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p odd, k even, $k > 2$, $1 < s < r+1$ and $s > k$ where $s = ak+b$, for some a and b, $0 < b < k$, then $a+1 < \frac{r}{2}$.*

Proof: The proof of this lemma is similar to the proof of Lemma 9. □

Since k is even and p is odd, then p is not a multiple of k, so $s \neq ak$. Hence we have our final lemma.

Lemma 25. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p odd, k even, $k > 2$, $1 < s < r+1$ and $s > k$. Write $s = ak+b$ for some a and b, $0 < b < k$. Then there is an cut-set A with $kr+1$ elements such that $\omega(H_{n,p} - A) = k$, and $\tau(H_{n,p} - A) = a+1+z$, where $z = \lceil \frac{2a+b-1}{k-2} \rceil$.*

Proof: Let $s > k$, then $s = ak+b$, for $0 < b < k$. Since p is odd and k is even, then s is odd. Hence b is odd. Thus $b = 2t-1$ and $k = 2q$ for some t and q. The remainder of this proof is similar to the proof of Lemma 7 with W_i defined as given in the following cases.

Case 1: Suppose $a+t < q$, so $z = \lceil \frac{2a+b-1}{2(q-1)} \rceil = 1$. Define

$$W_i = \begin{cases} \{i(r+2)+(i-1)a-1, \ldots, i(r+2)+ia\} \\ \quad 1 \le i \le a+t-1 \\ \{i(r+1)+ia+t-1, \ldots, i(r+1)+(i+1)a+t-1\} \\ \quad a+t \le i \le q-1 \\ \{i(r+1)+qa+t-1\} \\ \quad q \le i \le q+1 \\ \{i(r+2)+(i-2)a+t-q-3, \ldots, i(r+2)+(i-1)a+t-q-2\} \\ \quad q+2 \le i \le q+a+t \\ \{i(r+1)+(i-1)a+2t-2, \ldots, i(r+1)+ia+2t-2\} \\ \quad q+a+t+1 \le i \le 2q \end{cases}.$$

Case 2: Suppose $a+t-1 = z(q-1)$ for some integer z and so $q-1$ divides $a + \frac{b-1}{2}$. Define

$$W_i = \begin{cases} \{i(r+1)+(i-1)(a+z), \ldots, i(r+1)+i(a+z)\} \\ \quad 1 \le i \le q-1 \\ \{i(r+1)+(q-1)(a+z)\} \\ \quad q \le i \le q+1 \\ \{i(r+1)+(i-3)(a+z), \ldots, i(r+1)+(i-2)(a+z)\} \\ \quad q+2 \le i \le 2q \end{cases}.$$

Case 3: Suppose that $a+t-1 = z(q-1)+c$ for $0 < c < q-1$ and so $q-1$ does not divide $a+\frac{b-1}{2}$. Define

$$W_i = \begin{cases} \{i(r+2)+(i-1)(a+z)-1, \ldots, i(r+2)+i(a+z)\} \\ \quad 1 \le i \le c \\ \{i(r+1)+(i-1)(a+z)+c, \ldots, i(r+1)+i(a+z)+c\} \\ \quad c+1 \le i \le q-1 \\ \{i(r+1)+(q-1)(a+z)+c\} \\ \quad q \le i \le q+1 \\ \{i(r+2)+(i-3)(a+z)-q+c-2, \ldots, i(r+2)+(i-2)(a+z)-q+c-1\} \\ \quad q+2 \le i \le q+c+1 \\ \{i(r+1)+(i-3)(a+z)+2c, \ldots, i(r+1)+(i-2)(a+z)+2c\} \\ \quad q+c+2 \le i \le 2q \end{cases}.$$

□

Finally, we have the following theorem.

Theorem 15. *Let $H_{n,p}$ be the Harary graph with $n = 2r+1$, p odd, k even, $k > 2$, $1 < s < r+1$ and $s > k$. Write $s = ak+b$ for some a and b, $0 < b < k$. Then $r + \frac{a+2}{k} \le T(H_{n,m}) \le r + \frac{a+2+z}{k}$, where $z = \lceil \frac{2a+b-1}{k-2} \rceil$.*

Proof: By Theorem 3, we have $r + \frac{a+2}{k} \le T(H_{n,p})$, since $s = ak+b$ for $0 < b < k$. Also, by Lemma 25, there is a cut-set A of $H_{n,p}$ with $kr+1$ elements. The number of components of $H_{n,p} - A$ is k and the largest component of $H_{n,p} - A$, has cardinality $a+1+z$, where $z = \lceil \frac{2a+b-1}{2(q-1)}$. Hence we have $\frac{|A|+\tau(H_{n,p}-A)}{\omega(H_{n,p}-A)} = r + \frac{a+2+z}{k}$. □

We can summarize what we have proved about the Harary graphs as follows, where $n = 2r$ or $n = 2r+1$ and $p = k(r+1)+s$, for $0 \le s < r+1$:

(1) If n is even or if n is odd and k is odd then $T(H_{n,p}) = r + \frac{1+\lceil \frac{s}{k} \rceil}{k}$.

(2) If n is odd and k is even then

$$r + \frac{1+\lceil \frac{s}{k} \rceil}{k} \le T(H_{n,p}) \le \begin{cases} \frac{kr+2+s}{k-1} & s = 0,1 \\ r + \frac{3}{k} & 1 < s < k \\ r + \frac{a+3+x}{k} & 2 < k \le s = ak+b, \ p \text{ even} \\ r + \frac{a+2+y}{k} & 2 < k \le s = ak+b, \ p \text{ odd} \end{cases},$$

where $x = \lfloor \frac{2a+b-2}{k-2} \rfloor$ and $y = \lceil \frac{2a+b-1}{k-2} \rceil$.

Note that the best bounds we have on the tenacity of the Harary graphs with n odd and $k = 2$ are given by Theorems 3 and 4.

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