Clique Coverings and Maximal-Clique Partitions: An Example

Sylvia D. Monson and Kevin N. Vander Meulen
Queen's University
Kingston, Ontario
Canada K7L 3N6

Rolf S. Rees
Memorial University
St. John's, Newfoundland
Canada A1C 5S7

A *clique* of a graph G is a complete subgraph of G. A *maximal clique* in a graph G is a clique that is a subgraph of no other clique in G. A family $\mathcal{C}$ of cliques is a *clique covering* (respectively *clique partition*) of G if every edge of G is in at least (resp. exactly) one of the cliques of $\mathcal{C}$. We say that $\mathcal{C}$ *covers* (resp. *partitions*) G. Each clique in any minimum clique covering of G can be replaced by a maximal clique to obtain a clique covering with the same cardinality as $\mathcal{C}$. On the other hand, not every graph G has a clique partition consisting of only maximal cliques (a *maximal-clique partition*). For example, the graph obtained by deleting one edge from the complete graph on four vertices, K_4, has no maximal-clique partition.

The minimum number of cliques needed in a clique covering (resp. clique partition) of G is called the *clique covering number* (resp. *clique partition number*) of G, denoted $\mathrm{cc}(G)$ (resp. $\mathrm{cp}(G)$). When G has one or more maximal-clique partitions, the smallest cardinality of such partitions is called the *maximal-clique partition number* of G and is denoted $\mathrm{mcp}(G)$.

Since every clique partition of G is also a clique covering, we have $\mathrm{cc}(G) \leq \mathrm{cp}(G)$ for all graphs G. Whenever a graph G has a maximal-clique partition, we have

$$\mathrm{cc}(G) \leq \mathrm{cp}(G) \leq \mathrm{mcp}(G).$$

Pullman, Shank, and Wallis [2], discuss maximal-clique partitions and give examples of graphs G for which $\mathrm{cc}(G) = \mathrm{cp}(G) = \mathrm{mcp}(G)$ as well as graphs H for which $\mathrm{cc}(H) < \mathrm{cp}(H) < \mathrm{mcp}(H)$. In this paper, we give an example of a graph G for which $\mathrm{cc}(G) < \mathrm{cp}(G) = \mathrm{mcp}(G)$. Whether or not a graph G exists for which $\mathrm{cc}(G) = \mathrm{cp}(G) < \mathrm{mcp}(G)$ remains an open problem. For a survey of clique covering results, see [1].

JCMCC 16 (1994), pp. 61-63

We will show that the graph G (Figure 2) has $\mathrm{cc}(G) = 8$ and $\mathrm{cp}(G) = \mathrm{mcp}(G) = 10$. The graph G was obtained by replacing the vertex labeled v in Figure 1 (the complement of the Petersen graph) by the two vertices labeled v_1 and v_2 in Figure 2 and partitioning the edges incident to v between the new vertices, v_1 and v_2.

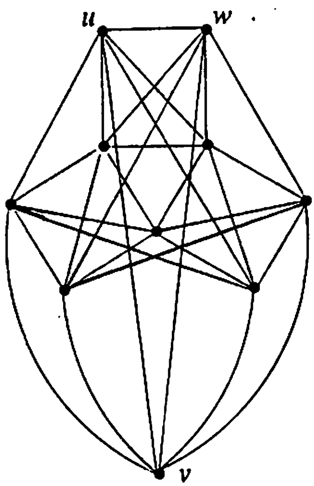

Figure 1. Complement of the Petersen Graph

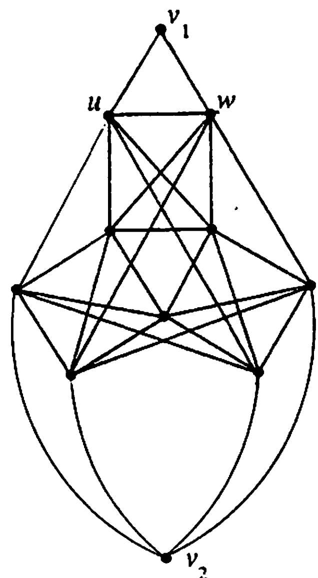

Figure 2. The graph G

Theorem 1. *There exists a graph G for which $cc(G) < cp(G) = mcp(G)$.*

Proof: Let G be the graph of Figure 2. Let $\mathcal{C}$ be a minimum clique covering of G using maximal cliques. Since the complement of the Petersen graph contains exactly five K_4's (see [2], p.351), two of which were dismantled when we replaced vertex v, we observe that G contains only three cliques on four vertices. All of the other maximal cliques are K_3;s. If $\mathcal{C}$ contains 0, 1, or 2 of the K_4's, then an edge count implies $\mathcal{C}$ has at least 10, 8, or 6 K_3's respectively, and hence $|\mathcal{C}| \geq 8$. If $\mathcal{C}$ contains all 3 of the K_4's, then $\mathcal{C}$ must also contain the triangle v_1uw, and the remaining ten edges require at least four more K_3's. In fact, the graph can be covered using three K_4's and five K_3's. Hence $\mathrm{cc}(G) = 8$.

The edge $v_1 i$ is in only one maximal clique, and so the triangle v_1uw belongs to any maximal-clique partition of G. Then the K_4 containing uw cannot belong to any maximal-clique partition. However, for each of the remaining five edges of this K_4, there is exactly one other maximal clique, a triangle, containing it. These five K_3's contain some of the edges of each of the other two K_4's. Consequently, none of the K_4's belongs to a maximal-clique partition of G. The remaining twelve edges of G can be partitioned into four maximal K_3's. Hence $\text{mcp}(G) = 10$.

Suppose $\text{cp}(G) < 10$. Let d_i be the number of cliques of size i in a minimum clique partition $\mathcal{C}$ of G. Hence:

$$
\begin{aligned}
d_4 &\leq 3 && (G \text{ contains only three } K_4\text{'s}) \\
d_2 + d_3 + d_4 &\leq 9 && (\text{by assumption}) && (1) \\
d_2 + 3d_3 + 6d_4 &= 30 && (G \text{ has 30 edges}) && (2)
\end{aligned}
$$

Note that if a clique C is in $\mathcal{C}$ then the graph obtained from G by removing the edges of C will be partitioned by $\mathcal{C}\backslash\{C\}$. The graph remaining after removing the three K_4's from G has clique partition number 8, implying $\text{cp}(G) = 11$ if $d_4 = 3$. Hence

$$d_4 \leq 2. \qquad (3)$$

Combining (1) and (2) we note that $d_4 \geq 1$. Observe that the graph obtained from G by removing any one or two of the K_4's has at least 3 independent vertices of odd degree. Then $\mathcal{C}$ contains at least 3 K_2's. Hence:

$$
\begin{aligned}
d_2 &\geq 3 && && (4) \\
3d_2 + 3d_3 + 3d_4 &\leq 27 && (\text{from } (1)) \\
3d_4 - 2d_2 &\geq 3 && (\text{subtracting } (2)) \\
3d_4 &\geq 0 && (\text{using } (4))
\end{aligned}
$$

Thus $d_4 \geq 3$ which contradicts (3). Therefore $\text{cp}(G) \geq 10$. But $\text{cp}(G) \leq \text{mcp}(G)$, so $\text{cp}(G) = 10$. □

Replacing the K_3 labeled v_1uw in Figure 2 by a K_n, $n \geq 3$, gives a graph H on $n + 8$ vertices which also has $\text{cc}(H) = 8$ and $\text{cp}(H) = \text{mcp}(H) = 10$. Hence:

Theorem 2. *For every $n \geq 11$, there is a graph G on n vertices having a maximal-clique partition for which $cc(G) < cp(G) = mcp(G)$.*

References

[1] Sylvia D. Monson, Norman J. Pullman, and Rolf S. Rees, A Survey of Clique and Biclique Coverings, and Factorizations of (0,1)-Matrices, *Bull. of the I.C.A.*, to apppear.

[2] N.J. Pullman, H. Shank, and W.D. Wallis, Clique Coverings of Graphs V: Maximal-Clique Partitions, *Bull. Australia. Math. Soc.*, **25** (1982) 337–356.