Computational Complexity of Weighted Integrity*

Sibabrata Ray and Jitender S. Deogun
Department of Computer Science & Engineering
University of Nebraska
Lincoln, NE 68588-0115

ABSTRACT. *Graph integrity*, a measure of graph vulnerability, has drawn considerable attention of graph theorists in recent years. We have given a set of sufficient conditions for the weighted integrity problem to be NP-Complete a class of graphs. As a corollary of this result we have shown that weighted graph integrity problem is NP-Complete on many common classes of graphs on which the unweighted integrity problem is either polynomial or of unknown complexity. We have shown that weighted graph integrity problem is polynomial for interval graphs.

1. Introduction

Network vulnerability is a concept that has applications in the area of design and analysis of networks. Several graph theoretic models under various assumptions have been proposed for the study of network vulnerability. Graph integrity, introduced by Barefoot et. al. [2, 1], is one of these measures that has received wide attention [5, 6, 7, 10]. Barefoot et. al. studied two measures of network vulnerability, the integrity and the edge integrity of a graph. Recently, Bagga et. al. have introduced a similar measure called pure edge integrity [4]. The concept of graph integrity is motivated by design and analysis of networks under hostile environment. In this model, the basic assumption is that some intelligent enemy is trying to disrupt the network by destroying its elements. The cost on his part is measured by the number of elements he would destroy and his success in incapacitating the network is measured by the order of the largest connected

*Research supported in part by an Office of Naval Research (ONR) grant # N00014-91-J-1693.

component in the remaining network. The enemy of course wants both to be small. Therefore, the minimum attainable sum of these two quantities is considered as a measure of vulnerability of the network. This measure is called graph integrity.

The integrity $I(G)$ of a graph G is defined as

$$I(G) = \min_{S \subseteq V(G)} \{|S| + m(G - S)\},$$

where $m(G - S)$ denotes the order (the number of vertices) of a largest component of $G - S$. The edge-integrity $I'(G)$ of a graph G is defined as

$$I'(G) = \min_{S \subseteq E(G)} \{|S| + m(G - S)\}.$$

Both $I(G)$ and $I'(G)$ turn out to have interesting properties, and they have been extensively studied.

The above definitions have led to a number of interesting results. It is to be noted that all nodes get equal importance in determining the integrity of a graph. In reality, usually different components of a network are of different importance. Therefore, assigning equal importance to every node is not desirable. In this paper, we developed a weighted model of integrity to overcome these criticisms. This leads us to the following definition of weighted integrity.

Definition 1.1. *The weighted integrity of a graph $G = (V, E)$ is defined as*

$$I_w(G) = \min_{X \subseteq V} \{w(X) + m_w(G - X)\}$$

where $w : V \to R^{\geq 0}$ is vertex weight function. $m_w(G - S)$ is the sum of the vertex weights for the maximum vertex weight connected component of $G - S$. It is to be noted that $w(X) = \sum_{v \in X} w(v)$.

In the next section we present a set of sufficient conditions for the weighted integrity problem to be NP-complete. These conditions hold for many commonly known classes. However, we have shown that they do not hold for at least one important class of graphs, viz., cocomparability graphs.

2. Computational Complexity

Let R denote the set of all non-negative rationals, R^+ then denotes the set of all positive rationals.

While providing definitions and notations, it was assumed that some vertices may have zero weights. From an application point of view this means that a node of no importance may exist in a network. In addition, there may exist some nodes that can be destroyed with no effort. From a theoretical view point, this assumption simply means working with a suitable subgraph of the original graph, which is again not desirable. Therefore, it is important to investigate a model where only non-zero weights are allowed. In formal notation, the range of the associated weight function $w : V \rightarrow R$ is changed from R to R^+. Most of the NP-hardness results those can be proved under the assumption of possible zero weights can be proved under the assumption of strictly non-zero weights with some extra effort.

The Weighted Integrity Problem (IP) is defined as follows. Given a graph $G = (V, E)$ and a weight function $w : V \rightarrow R^+$, find $X \subseteq V$ such that $w(X) + m_w,(G - X) = I_w(G)$. The decision version (IPD) of the same problem is: given a graph $G = (V, E)$, a weight function $w : V \rightarrow R^+$ and a rational $k \in R^+$, does there exists $X \subseteq V$ such that $w(X)+m_w(G-X) \leq k$?

Given a certificate $Y \subseteq V$, it is easy to see that a solution to IPD can be verified in polynomial time. Therefore, IPD $\in$ NP. Further, note that if a polynomial time algorithm exist for IPD, then that algorithm can be used to develop a polynomial time algorithm for IP.

The following are two versions of the knapsack problem. These definitions can be found in any standard text on algorithms [8]. $0 - 1$ Knapsack Problem—Optimization Version (KPO) is defined as follows. Let $p_0, \ldots, p_k$, $w_0, \ldots, w_k$, P be positive integers. Find $I \subseteq \{0, \ldots, k\}$ such that $\sum_{i \in I} p_i \geq P$ and $\sum_{i \in I} w_i$ is minimum. The corresponding decision problem (KPD) is, let $p_0, \ldots, p_k$, $w_0, \ldots, w_k$, P, W be positive integers. Does there exist $I \subseteq \{0, \ldots, k\}$ such that $\sum_{i \in I} p_i \geq P$ and $\sum_{i \in I} w_i \leq W$?

It is known that KPD $\in$ NPC [8]. Further, if there exists a polynomial time algorithm to solve KPO, then it can be used to solve KPD in polynomial time. Therefore, to show that IPD $\in$ NPC, it is enough to show that existence of a polynomial time algorithm for IP implies there exists a polynomial time algorithm for KPO.

Given a graph $G = (V, E)$ and positive vertex weight function w, there exists a subset of V such that the weight of that vertex set (sum of the weights of the vertices in the vertex set) and the weight of the maximum weight connected component after removal of the vertex set is minimum. In other words, if $X \subseteq V$ be such a vertex set, then $w(X) + m_w(G - X)$ will give the weighted integrity of G with respect to the weight function w. Such a subset of V is called an *achieving vertex* cut or simply an *achieving cut*. Further, given any achieving cut X, the vertex set of the maximum weight connected component of $G - X$ is called an *achieving component*. In other words, for an achieving cut X, Y is called an achieving component associated with X, if $w(Y) = m_w(G-X)$ and vertices belonging to Y form a connected component of $G - X$.

Before stating the main NP-completeness result, let us consider an example. Let $T = (V, E)$ be a rooted undirected tree of height two. Let r be the root, $v_0, \ldots, v_k$ be the leaves and $u_0, \ldots, u_k$ be internal vertices of T other than root. The root vertex r is adjacent to $u_0, \ldots, u_k$ and every internal node u_i is adjacent to exactly one leaf node v_i for $i = 0, \ldots, k$. Further, let the weights of the v_i's be large (say of $O(k^4)$), the weight of r be medium (say of $O(k^2)$) and the weights of the u_i's be small (say of $O(1)$). Further, let v_0 be the maximum weight vertex. This graph is shown in Figure 1. For this graph, the only possible achieving component is the singleton $\{v_0\}$. Further, r can not belong to any achieving cut. Therefore, determining the weighted integrity of T reduces to computing a minimum weight subset; Y; of internal vertices such that $w(v_0) = m_w(G-Y)$. Clearly, this is equivalent to a $0-1$ knapsack problem.

The concept of two layered tree can be generalized further. In the intuitive argument presented above, the fact that the root is a single vertex was never used. Thus, the root can be replaced by any connected graph such that every internal vertex is adjacent to some vertex of this connected graph. The modified graph is called a Type 1 subgraph and is shown in Figure 2.

Let $\mathcal{G}$ be an infinite class of graphs. Let every $G \in \mathcal{G}$ have a "big enough" induced subgraph which is either a two layered rooted tree or a Type 1 subgraph. Even if only positive weights are allowed, the weighted integrity problem remains NP-complete on such a class 5. To prove this, we need to assign very small positive weights to the vertices not belonging to the induced subgraph of Type 1. The sum of the weights of these vertices should not exceed some $\epsilon > 0$. The Lemma 2.1 is helpful in determining such an ϵ.

Lemma 2.1. *Let $r_i = p_i/q_i$; $i = 1, \ldots, k$ be positive rations. Then there exists an $\epsilon > 0$ such that for any $I, J \subseteq \{1, \ldots, k\}$, $|\sum_{i\in I} r_i - \sum_{i\in J} r_i| \neq 0$ implies, $|\sum_{i\in I} r_i - \sum_{i\in J} r_i| > \epsilon$.*

Proof: Let $Q = q_1 \ldots q_k$ and $Q_i = Q/q_i$ for $i = 1, \ldots, k$. Now, $|\sum_{i\in I} r_i - \sum_{i\in J} r_i| \neq 0 \Rightarrow |\sum_{i\in I} r_i i \sum_{i\in J} r_i| = \frac{\sum_{i\in I} p_i Q_i - \sum_{i\in J} p_i Q_i}{Q} \geq \frac{1}{Q} > \frac{1}{Q+1} = \epsilon$.

The concept "big enough" means the size of the Type 1 subgraph should be polynomially related to the size of the original graph. The reason behind this is the size of the KPD considered is less than the size of the Type 1 subgraph. Further, the graph may be allowed to be at most polynomially bigger than the KPD. Therefore, the graph may be only polynomially bigger than the Type 1 subgraph.

Further, to ensure that for any KPD, there will be a graph at most polynomially bigger than it with a suitable Type 1 subgraph, it is required that $\{|G| : G \in \mathcal{G}\}$ forms a dense set. Let $n_0 < n_1 < \ldots$ be an infinite sorted list of positive integers. The set $\{n_i \mid i = 0, 1, \ldots\}$ is called dense if there exists a positive real α such that $n_{i+1} \leq n_i^\alpha$ for all i.

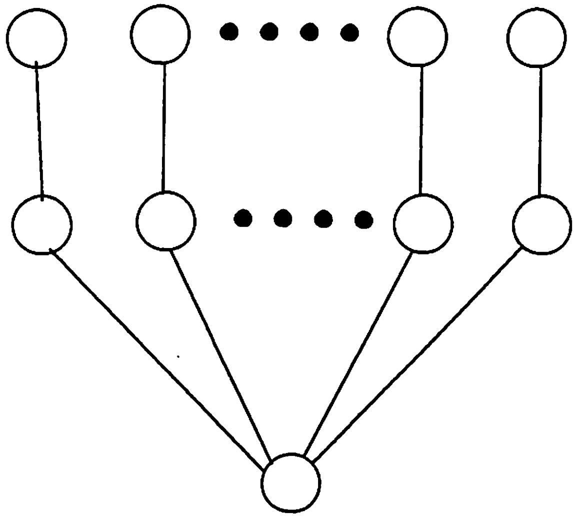

Figure 1

Theorem 2.1. *Let $\mathcal{G}$ be an infinite class of graphs such that $\{|G| : G \in \mathcal{G}\}$ is a dense set. If for all $G \in \mathcal{G}$, there exists an induced subgraph H' of G satisfying the following conditions, then the weighted (vertex) integrity problem is NP-hard on $\mathcal{G}$.*

1. *There exists an independent set $\{v_0, \ldots, v_k\}$ of H' such that $\deg_{H'}(v_i) = 1$ for $i = 0, \ldots, k$ and $k \geq |G|^{\alpha}$.*
2. *For each v_i, there exists a distinct $u_i \in H'$ adjacent to v_i for $i = 0, \ldots, k$.*
3. *All v_i in the set $\{v_i \mid i \notin I\}$ belong to the same connected component of $H' - \{v_j \mid j \in I\}$, for all $I \subset \{0, \ldots, k\}$.*

Proof: Let $p_1, \ldots, p_k, w_1, \ldots, w_k, P$ be an instance of a KPO. Choose a graph G from $\mathcal{G}$ that has a Type 1 induced subgraph H with $k+1$ leaves and $|G|$ is bounded by some fixed polynomial of k for the class $\mathcal{G}$.

Let $v_0, \ldots, v_k$ be the leaves of H and u_i be the only vertex in H adjacent to v_i for $i = 0, \ldots, k$. The vertices $v_0, \ldots, v_k$ are high weight vertices. All vertices in $V(H) - \{v_0, \ldots, v_k, u_0, \ldots, u_k\}$ are medium weight vertices and $u_1, \ldots, u_k$ are small weight vertices and u_0 and all other vertices in $V(G) - V(H)$ are negligibly small weight vertices.

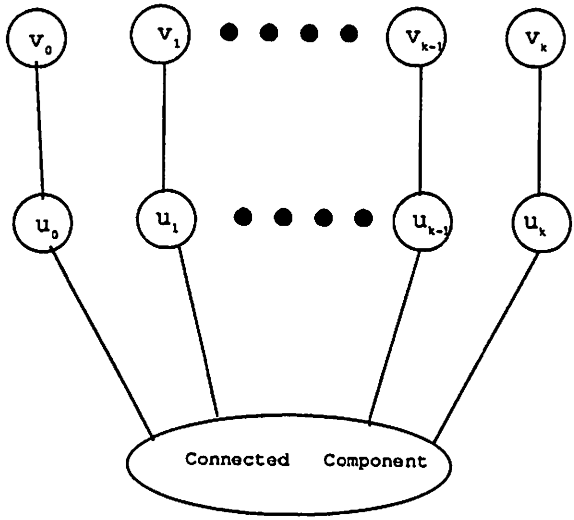

Figure 2

Given the KOP, a weight function $w : V(G) \rightarrow R^+$ is constructed as follows. For vertex v_i, weight of v_i, $w(v_i) = p_i$, $i = 1, \ldots, k$ and $w(v_0) = \sum p_i - P + 1$. W.l.o.g, it may be assumed that $\sum p_i - P \geq \max_i p_i$ and therefore, in turn $\sum p_i - P + 1 \geq \max_i p_i$.

The assignment of weights of vertices is made in such a way that the sum of weights of all vertices except the high weight vertices is less than 1. Further, the sum of the weights of all small weight and negligibly small weight vertices should be less than the weight of any medium vertex. The weights of small weight vertices should be proportional to the w_i's in the KOP. The weights of the negligibly small weight vertices should satisfy one more condition. If $I, J \subseteq \{1, \ldots, k\}$, then from Lemma 2.1 there exists $\epsilon > 0$ such that if $|\sum_I w(u_i) - \sum_J w(u_i)| \neq 0$ then $|\sum_I w(u_i) - \sum_J w(u_i)| > \epsilon$. The sum of the weights of negligibly small weight vertices should be less than this ϵ. Existence of ϵ is gurranteed by Lemma 2.1. The proof of the theorem will work for any assignment of weights that satisfy these properties. Here we present one such assignment.

- Assign weight $W = \frac{1}{2(|V(H)| - 2(k+1)|}$ to every medium weight vertex. Note that the sum of the weights of all medium weight vertices is 2.
- Assign weights $W_i = W \frac{w_i}{2(\sum w_i + 1)}$ to u_i for $i = 1, \ldots, k$. Note that the sum of the weights of all small weight vertices is less than $\frac{W}{2}$.

- Further, assign weight $W' = \frac{W}{2(\sum w_i+1)|V(G)|}$ to all negligibly small weight vertices.

It is easy to check that this weight assignment satisfies all the conditions.

If $X \subseteq V(G)$ then the weighted integrity of G is less than or equal to $w(X)+m_w(G-X)$. Consider a vertex cut X consisting of all small weight and negligibly small weight vertices. Note that $w(X) < W$ and the maximum weight connected component of $G-X$ is $\{v_0\}$. Therefore, the weighted integrity of G is strictly less than $\sum p_i - P + 1 + W$. Therefore, it may be noted that medium weight vertices neither belong to the corresponding achieving component nor any achieving cut. Therefore, an achieving component may contain only v_0 and possibly some negligibly small weight vertices. Hence, if S is an achieving cut, then the maximum weight connected component of $G - S$ must contain v_0 and possibly some negligibly small weight vertices.

If S is a vertex cut such that the maximum weight connected component of $G-S$ contains only v_0 and some negligibly small weight vertices then S is a feasible cut. Following the above discussion it is clear that any achieving cut is a feasible cut. Further, for $I \subseteq \{1, \ldots, k\}$ if $\sum_I p_i \geq P$ then I is called a feasible solution of the KPO.

Let S be a feasible cut. Claim 1, $I = \{i \mid u_i \in S\}$ is a feasible solution of the KPO. Let $\{v_0\} \cup X$ be the maximum weight connected component of $G-S$. Note that, there is a connected component of $G-S$, which consists of $\{v_i \mid i \notin I\}$, $\{u_i \mid i \notin I\}$ and some medium weight vertices. From the definition of a feasible cut it follows that

$$w(v_0) + w(X) \geq \sum_{I^c} p_i + \sum_{I^c} W_i + W.$$

Note that, $\sum_{I^c} W_i + W - w(X)$ is a positive proper fraction. Therefore,

$$w(v_) \geq \sum_{I^c} p_i + \text{ a proper fraction.}$$

But $w(v_0)$ is an integer. Therefore, the equality can not hold. Hence

$$\sum p_i - P + 1 > \sum_{I^c} p_i + \text{ a proper fraction.}$$

Therefore,

$$\sum p_i - P \geq \sum_{I^c} p_i,$$

or,

$$\sum p_i - \sum_{I^c} p_i \geq P,$$

or,

$$\sum_I p_i \geq P.$$

This completes the proof of claim 1.

Similarly, it may be shown that if I is a feasible solution then $S = \{u_i \mid i \in I\} \cup \{v \mid v \text{ is a negligibly small weight vertex}\}$ is a feasible cut.

Claim, if S is an achieving cut, then $I = \{i \mid u_i \in S\}$ is a solution to KPO. Consider another feasible cut $S' = \{u_i \mid i \in J\} \cup \{v \mid v \text{ is a negligibly small weight vertex}\}$ where J is a solution of the KPO. It suffices to show that if I is not a solution of the KPO, then $w(S) > w(S')$.

Note $w(S) \geq \sum_I w(u_i)$. If I is not a solution to KPO, then by the properties of the weights chosen, $\sum_I w(u_i) > \sum_J (u_i) + \varepsilon'$, where ε' is the sum of the weights of all negligibly small weight vertices. Therefore,

$$w(S) > \sum_I (u_i) + \varepsilon' = w(S').$$

Thus the proof.

Therefore it is easy to show that,

Corollary 2.1. *The weighted integrity problem is NP-complete on*

1. *trees,*
2. *meshes,*
3. *hypercubes, and*
4. *regular graphs.*

Further from [3] we know that,

Theorem 2.2. *The weighted integrity problem is polynomial for Interval graphs.*

It is interesting to note that the set of sufficient conditions presented in this paper do not hold for the class of cocomparability graphs, a super class for interval graphs. Therefore, we conjecture that,

Conjecture: The weighted integrity problem is polynomial for cocomparability graphs. It will be interesting to see whether the weighted integrity problem is polynomial for permutation graphs, an important subclass of the class of cocomparability graphs.

References

[1] C.A. Barefoot, R.C. Entringer and H. Swart, Integrity of trees and powers of cycles. *Congressus Numerantium* **58**(1987), 103–114.

[2] C.A. Barefoot, R.C. Entringer and H. Swart, Vulnerability in graphs—a comparative survey, *J. Combin. Math. Combin. Comput.* **1**(1987), 13–22.

[3] S. Ray, H. Jiang and J.S. Deogun, Weighted Integrity is Polynomial for Interval Graphs, submitted to the *IEEE Trans. on Computers.*

[4] K.S. Bagga and J.S. Deogun, A variation on the edge integrity, *Congressus Numer.* **91**(1992), 207–211.

[5] K.S. Bagga, L.W. Beineke, W.D. Goddard, M.J. Lipman, and R.E. Pippert, A survey of integrity, *Discrete Appl. Math.* **37/38**(1992), 13–28.

[6] L.H. Clark, R.C. Entringer and M.R. Fellows, Computational complexity of integrity, *J. Combin. Math. Combin. Comput.* **2**(1987), 179–191.

[7] M.R. Fellows and S. Stueckle, The immersion order, forbidden subgraphs and the complexity of network integrity, *J. Combin. Math. Combin. Computing* **6**(1989), 23–32.

[8] M.R. Garey and D.S. Johnson, *Computers and Intractibility: A Guide to the Theory of NP-Completeness*, W. H. Freeman, San Fransisco, 1979.

[9] W. Goddard, On the Vulnerability of Graphs, Ph.D Thesis, Univ. of Natal, Durban, South Africa, 1989.

[10] W. Goddard and H.C. Swart, On the integrity of combinations of graphs, *J. Combin. Math. Combin. Computing.* **4**(1988), 3–18.