Lattices with series-parallel and interval order and a generalization of Catalan numbers

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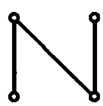
ABSTRACT. We obtain a formula for the number of finite lattices of a given height and cardinality that have a series-parallel and interval order. Our approach is to consider a naturally defined class of m nested intervals on an $m+k$-element chain, and we show that there are $\binom{m-1}{k-1}\gamma(m+1)$ such sets of nested intervals. Here, $\gamma(m+1)$ denotes the Catalan number $\frac{1}{m+1}\binom{2m}{m}$.

1. Introduction

One of the objectives of this paper is to enumerate the non-isomorphic finite lattices that have a series-parallel and interval order. Partially ordered sets (posets) that are series-parallel and those that are interval orders have been studied intensively in recent years. The finite lattices that have both series-parallel and interval orders were fully characterized in [2].

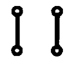
All sets, posets and lattices in this paper are finite.

Recall that a poset A has a *series-parallel* order if A can be constructed from singletons, using only disjoint unions and linear sums as operations, [8]. A has an *interval order* if there exists a map f from A to the intervals of the reals, such that for $a, b \in A$, $a < b$ if and only if the right end of $f(a)$ is to the left of the left end of $f(b)$. Equivalently, A has a series-parallel order if and only if A does not contain



as a subposet [8] in A

JCMCC **16** (1994), pp. 75-85

has an interval order if and only if A does not contain  as a subposet [7]. The papers [1] and [3] deal with asymptotic enumeration problems involving these classes of posets. We focus on lattices. We denote by $\mathcal{SPI}$ the class of finite lattices that have both a series-parallel and interval order.

The problem of the enumeration of the non-isomorphic lattices in $\mathcal{SPI}$ has led us to the introduction of a class of numbers which can be considered as a generalization of the Catalan numbers (e.g. [4], [6], [9]) and which seem to be of interest in themselves. In section 3 we will derive a formula for these "generalized" Catalan numbers. In section 4 we will use the results obtained to achieve our goal of enumerating the members of $\mathcal{SPI}$.

2. Preliminaries

Let C be a chain of n elements, $n \geq 1$. Unless indicated otherwise, we will assume that $C = \{1, 2, \ldots, n\}$ with the natural order and in that case we denote C by $[n]$. An *interval* of $[n]$ is an ordered pair $[i, j]$, $1 \leq i \leq j \leq n$. (Note that our definition of interval differs from the usual one). An interval $[i, j]$ is *proper*, if $i < j$.

Let A be a set of intervals of $[n]$. A is *m-nested* on $[n]$ if

(i) each interval of A is proper;

(ii) $|A| = m \geq 1$;

(iii) for distinct $[i, j]$, $[k, \ell] \in A$ either $j < k$ or $\ell < i$ or $k \leq i < j \leq \ell$ or $i \leq k < \ell \leq j$;

(iv) each element of $[n]$ is either the left end point or the right end point of an interval of A.

Note that if a set A of intervals of $[n]$ only satisfies (i), (ii) and (iii) and not necessarily (iv), then there exists a subchain of $[n]$ on which A is m-nested.

For $n \geq 1$, we denote the set of sets of intervals which are m-nested on $[n]$ by $\Gamma(m, n)$ and we let $\gamma(m, n) = |\Gamma(m, n)|$. Thus $\Gamma(m, n)$ and $\gamma(m, n)$ are defined for $n \geq 1$, $m \geq 1$, but we will adopt the convention that $\Gamma(0, 0) = \{\emptyset\}$ and thus $\gamma(0, 0) = 1$ and also that $\gamma(0, 1) = 1$. Finally, we write $\gamma(m)$ for $\gamma(m-1, m)$, $m \geq 1$. The numbers $\gamma(m)$ are the *Catalan numbers* and it is well-known that

$$\gamma(m) = \frac{1}{m}\binom{2(m-1)}{m-1}, \quad m \geq 1. \tag{2.1}$$

Lemma 2.1. *For $m \geq 1$, $n \geq 1$, $\gamma(m, n) \neq 0$ if and only if $m+1 \leq n \leq 2m$.*

Proof: If $\gamma(m,n) \neq 0$, then it is immediate that $n \leq 2m$ and by using induction on m, it is easily shown that $m+1 \leq n$. If $m+1 \leq n \leq 2m$, let

$$A = \{[1,2],[3,4],\ldots,[2k-3,2k-2],[2k-1,n],[2k,n],\ldots,[n-1,n]\}$$

where $k = n-m$. Then $A \in \Gamma(m,n)$. □

It will often be convenient to set $k = n-m$ and thus write $\gamma(m,m+k)$ and $\gamma(m,m+k)$ instead of $\Gamma(m,n)$ and $\gamma(m,n)$ respectively. Thus for $m \geq 1$, $\gamma(m,m+k) \neq 0$ for $1 \leq k \leq m$.

We will prove in section 3 that

$$\gamma(m,m+k) = \binom{m-1}{k-1}\gamma(m+1) \quad \text{for} \quad m \geq 1. \tag{2.2}$$

3. A combinatorial proof of (2.2)

Recall the definition of a *binary tree* [6]. A binary tree T consists of a set of m *nodes*, $m \geq 0$. One node is the *root of* T, denoted root(T). Then $T \sim \text{root}(T)$ consists of two binary trees, the left subtree T_L of root(T) and the right subtree T_R of root(T). We say two binary trees T and T' are *isomorphic* (similar in the sense of [6]) if there is a bijection $f : T \to T'$ such that for $a, b \in T$, $f(a)$ is in the left (right) subtree of $f(b)$ if and only if a is in the left (right) subtree of b. If T and T' are isomorphic we will write $T \simeq T'$. The set of nonisomorphic binary trees with m nodes will be denoted by $\mathcal{T}_m$. It is known that $|\mathcal{T}_m| = \gamma(m+1)$ for all $m \geq 0$, e.g., [6, p. 389]. We define a function

$$\text{Tree} : \bigcup_{\substack{m\geq 1\\ 1\leq k\leq m}} \Gamma(m,m+k) \cup \Gamma(0,0) \to \bigcup_{i\geq 0} \mathcal{T}_i$$

such that for $A \in \Gamma(m,m+k)$, Tree$(A) \in \mathcal{T}_m$.

First, we introduce some additional notation. If $A \in \Gamma(m,m+k)$, then the *major interval* of A, denoted by maj(A), is $[\ell, m+k]$ where $\ell = \min\{j : [j,m+k] \in A\}$. We write $\widetilde{A}$ for $A \sim \text{maj}(A)$. For $A \in \Gamma(m,m+k)$ with $\text{maj}(A) = [\ell, m+k]$, define *cut*($A$) as $\max(\{j : [\ell,j] \in \widetilde{A}\} \cup \{\ell-1\})$. Note that cut($A$) may be 0 and that either $\text{cut}(A) > \ell$ or $\text{cut}(A) = \ell - 1$. Some examples are in figure 1. Let $A_L = \{[i,j] \in A : j \leq \text{cut}(A)\}$ and $A_R = \{[i,j] \in \widetilde{A} : i > \text{cut}(A)\}$. Note that $A = A_L \cup A_R \cup \{\text{maj}(A)\}$ and that A_L or A_R can be empty. We let $m_L = |A_L|$ and $m_R = |A_R|$. So if $A \neq \emptyset$, then $m = m_L + m_R + 1$. We have $A_L \in \Gamma(m_L, m_L + k_L)$ for an integer $k_L \geq 0$. For any integer t and a set of intervals S, let $\delta_t(S)$ denote $\{[i+t, j+t] : [i,j] \in S\}$. Then there exist non-negative integers t and k_R such that $\delta_{-t}(A_R) \in \Gamma(m_R, m_R + k_R)$. For $A \in \Gamma(m,m+k)$ we have that

$\widetilde{A}$ is $(m-1)$-nested on a chain of $m+k$, $m+k-1$, or $m+k-2$ elements, and then $k_L + k_R$ is equal to $k+1$, k, or $k-1$ respectively.

We now define the function Tree inductively. We let $\text{Tree}(\emptyset) = \emptyset$. For $1 \leq k \leq m$ and $A \in \Gamma(m, m+k)$, the nodes of Tree(A) are the m intervals of A, $\text{root}(\text{Tree}(A)) = \text{maj}(A)$, $(\text{Tree}(A))_L \simeq \text{Tree}(A_L)$, and $(\text{Tree}(A))_R \simeq \text{Tree}(\delta_{-t}(A_R))$. Figure 2 contains some examples.

Lemma 3.1. *For each $m \geq 0$, the function Tree is one-to-one on $\Gamma(m, 2m)$ and onto $\mathcal{T}_m$. It follows that $\gamma(m, 2m) = \frac{1}{m+1}\binom{2m}{m} = \gamma(m+1)$ for $m \geq 0$.*

Proof: The lemma obviously holds for $m = 0$ and $m = 1$. Suppose it holds for all $m' < m$. Let $A \in \Gamma(m, 2m)$ with $A_L \in \Gamma(m_L, 2m_L)$. Then $\text{maj}(A) = [2m_L+1, 2m]$ and $\text{cut}(A) = 2m_L$. We have $\delta_{-t}(A_R) \in \Gamma(m_R, 2m_R)$ for $t = 2m_L+1$. Now $m_L + m_R + 1 = m$ so m_L and m_R are each less than m. Suppose $A, A' \in \Gamma(m, 2m)$ and that $\text{Tree}(A) \simeq \text{Tree}(A')$. By induction we see that $A_L = A'_L$ and $A_R = A'_R$, and thus $A = A'$. Next, consider $T \in \mathcal{T}_m$ with $m \geq 1$. Suppose T_L has m_L nodes and T_R has m_R nodes. Then $m_L, m_R < m$, and by induction there exist $B \in \Gamma(m_L, 2m_L)$ and $C = \Gamma(m_R, 2m_R)$ such that $\text{Tree}(B) \simeq T_L$ and $\text{Tree}(C) \simeq T_R$. If $A = B \cup \delta_{2m_L+1}(C) \cup \{[2m_L+1, 2m]\}$, then $A \in \Gamma(m, 2m)$ and $\text{Tree}(A) \simeq T$. □

We use the following notation. For a set X and $0 \leq q \leq |X|$, let $\binom{X}{q}$ denote the set of all q element subsets of X.

Theorem 3.2. $\gamma(m, m+k) = \binom{m-1}{k-1}\gamma(m+1)$ *for all $m, k \geq 1$.*

Proof: We will define for each $m \geq 1$ and $0 \leq q \leq m-1$ and for $m = 0$ and $q = 0$ a function

$$\beta : \left\{(A, S) : A \in \Gamma(m, 2m),\ S \in \binom{\widetilde{A}}{q}\right\} \to \Gamma(m, 2m-q)$$

such that β is one-to-one and onto $\Gamma(m, 2m-q)$ and also such that $\text{Tree}(\beta(A, S)) \simeq \text{Tree}(A)$. From the existence of such a β and Lemma 3.1 we see that $\binom{m-1}{q}\gamma(m+1) = \gamma(m, 2m-q)$. Letting $k = m-q$ proves the theorem.

The function β will be defined inductively. The idea is to identify one endpoint of each interval in S with an endpoint of another interval. Suppose $[p, q]$ and $[p', q']$ are intervals of A such that $[p, q] \in S$ and in Tree(A) the node v corresponding to $[p, q]$ is a child of the node v' corresponding to $[p', q']$. If v is a left child of v' then we replace $[p', q']$ by $[p, q']$ and if v is a right child of v' we replace $[p', q']$ by $[p', q]$. We also shift to the left as necessary to maintain condition (iv) in the definition of m-nested intervals. Examples are given in Figure 2.

For $m = q = 0$, let $\beta(\emptyset, \emptyset) = \emptyset$. For $m \geq 1$ and $q = 0$, define $\beta(A, \emptyset) = A$ for every $A \in \Gamma(m, 2m)$.

Let $m \geq 2$ and suppose β has been defined for all $1 \leq m' < m$ and all q', $0 \leq q' \leq m' - 1$ so that $\text{Tree}(\beta(A', S')) \simeq \text{Tree}(A')$ for $A' \in \Gamma(m', 2m')$ and $0 \leq |S'| = q' \leq m' - 1$. Suppose $A \in \Gamma(m, 2m)$, $0 \leq q \leq m-1$, and $S \in \binom{\widetilde{A}}{q}$. Let $|A_L| = m_L$, $|\widetilde{A_L} \cap S| = q_L$, $|A_R| = m_R$, and $|\widetilde{A_R} \cap S| = q_R$. By the induction hypothesis we have $\beta(A_L, \widetilde{A_L} \cap S) = A'_L$ and $\beta\big(\delta_{-(2m_L+1)}(A_R), \delta_{-(2m_L+1)}(\widetilde{A_R} \cap S)\big) = A'_R$ with $A'_L \in \Gamma(m_L, 2m_L - q_L)$, $A'_R \in \Gamma(m_R, 2m_R - q_R)$, $\text{Tree}(A_L) \simeq \text{Tree}(A'_L)$, and $\text{Tree}(A_R) \simeq \text{Tree}(A'_R)$. There are four cases:

(i) $\text{maj}(A_L) \in S$, $\text{maj}(A_R) \in S$, and hence $q_L + q_R = q - 2$;

(ii) $\text{maj}(A_L) \in S$, $\text{maj}(A_R) \notin S$, and hence $q_L + q_R = q - 1$;

(iii) $\text{maj}(A_L) \notin S$, $\text{maj}(A_R) \in S$, and hence $q_L + q_R = q - 1$;

(iv) $\text{maj}(A_L) \notin S$, $\text{maj}(A_R) \notin S$, and hence $q_L + q_R = q$.

In cases (i) and (ii) we define $\beta(A, S) = A'_L \cup \delta_{2m_L - q_L}(A'_R) \cup \big\{[\ell, 2m - q]\big\}$, where ℓ is the left endpoint of $\text{maj}(A'_L)$, and in cases (iii) and (iv) we define

$$\beta(A, S) = A'_L \cup \delta_{2m_L - q_L + 1}(A'_R) \cup \big\{[2m_L - q_L + 1, 2m - q]\big\}.$$

It is not difficult to show that in all cases $\beta(A, S)$ is an m-nested set of intervals on $[2m - q]$ and thus $\beta(A, S) \in \Gamma(m, 2m - q)$. Note that for cases (i) and (ii), $\text{maj}\big(\beta(A, S)\big) = [\ell, 2m - q]$ and for cases (iii) and (iv), $\text{maj}\big(\beta(A, S)\big) = [2m_L - q_L + 1, 2m - q]$. It follows that $\beta(A, S)_L = A'_L$ in all cases, $\beta(A, S)_R = \delta_{2m_L - q_L}(A'_R)$ in cases (i) and (ii), and $\beta(A, S)_R = \delta_{2m_L - q_L + 1}(A'_R)$ in cases (iii) and (iv). By induction, $\text{Tree}\big(\beta(A, S)_L\big) \simeq \text{Tree}(A'_L) \simeq \text{Tree}(A_L)$ and $\text{Tree}\big(\beta(A, S)_R\big) \simeq \text{Tree}(A'_R) \simeq \text{Tree}(A_R)$. It follows that $\text{Tree}\big(\beta(A, S)\big) \simeq \text{Tree}(A)$.

To show β is one-to-one, let $A, B \in \Gamma(m, 2m)$ and suppose $\beta(A, S) = \beta(B, T)$. We have $\text{Tree}(A) \simeq \text{Tree}\big(\beta(A, S)\big) \simeq \text{Tree}\big(\beta(B, T)\big) \simeq \text{Tree}(B)$. But $A, B \in \Gamma(m, 2m)$, thus $A = B$ by Lemma 3.1. Also, from $\beta(A, S) \simeq \beta(B, T)$ we get $2m - |S| = 2m - |T|$, so $|S| = |T|$. We have $\widetilde{A_L} \cap S = \widetilde{A_L} \cap T$ and $\widetilde{A_R} \cap S = \widetilde{A_R} \cap T$ since $A = B$. From $\text{maj}\big(\beta(A, S)\big)$ we can determine which of cases (i), (ii), (iii) or (iv) holds, and thus $\text{maj}(A_L) \in S$ if and only if $\text{maj}(A_L) \in T$ and $\text{maj}(A_R) \in S$ if and only if $\text{maj}(A_R) \in T$. It follows that $S = T$.

Finally, to show β is onto, we let $B \in \Gamma(m, 2m - q)$, $0 \leq q \leq m - 1$. Let $A \in \Gamma(m, 2m)$ be such that $\text{Tree}(A) \simeq \text{Tree}(B)$. By induction, there exist $S_L \subseteq \widetilde{A_L}$ and $S_R \subseteq \widetilde{A_R}$ such that $\beta(A_L, S_L) = B_L$ and $\beta\big(\delta_{-t}(A_R), \delta_{-t}(S_R)\big) = B_R$, for $t = 2m_L + 1$. Let $S = S_L \cup S_R \cup S_0$, where S_0 is defined as follows. $S_0 = \emptyset$ except if the left endpoint of $\text{maj}(B_L)$ is the left endpoint of $\text{maj}(B)$, then include $\text{maj}(B_L)$ in S_0, or if the right endpoint of $\text{maj}(B_R)$ is $2m - q$, then include $\text{maj}(B_R)$ in S_0. It follows from the definition that $\beta(A, S) = B$. □

4. Enumeration of lattices with series-parallel and interval orders

We first recall the characterization of the lattices in $\mathcal{SPI}$ given in [2]. For a poset P, $a, b \in P$, $a \prec b$ means that b covers a, that is, $a \leq e < b$ implies $a = e$ for $e \in P$. A poset P has height n if a longest chain in P has $n+1$ elements.

Let $\mathcal{SPI}_n$ denote the class of lattices in $\mathcal{SPI}$ of height n. The following is in [2].

Theorem 4.1. *Let L be a finite lattice. L is in $\mathcal{SPI}_n$ if and only if the following conditions are satisfied.*

(i) L has height n and C is a chain in L with $|C| = n+1$;

(ii) for every $a \in L \sim C$, there exist $\underline{a}, \overline{a} \in C$ such that $\underline{a} \prec a \prec \overline{a}$ and there exists $c \in C$ such that $\underline{a} < c < \overline{a}$;

(iii) for $a, b \in L \sim C$, $\underline{a} < \underline{b} < \overline{a}$ implies $\overline{b} \leq \overline{a}$.

Note that for a, $b \in L \sim C$ if $a < b$ then $\underline{a} < \underline{b}$. Otherwise $\underline{b} \leq \underline{a} < a < b \to \underline{b} \not\prec b$. Similarly, $a < b$ implies $\overline{a} < \overline{b}$. Conversely, if for a, $b \in L \sim C$, $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$ then $a < b$. Indeed, if $\overline{a} \leq \underline{b}$ then $a < \overline{a} \leq \underline{b} < b$ so $a < b$ and if $\underline{b} < \overline{a}$ then $\underline{a} < \underline{b} < \overline{a}$ thus $\overline{b} \leq \overline{a}$.

It follows from Theorem 4.1 that lattices in $\mathcal{SPI}$ of height n, $n \geq 0$ can be constructed as follows.

Start with a chain C of $n+1$ elements. Adjoint to C a set B of elements and assign to each element $a \in B$ two elements $\underline{a}$ and $\overline{a}$ of C, satisfying the conditions $\underline{a} \not\prec \overline{a}$ and for a, $b \in B$, $\underline{a} < \underline{b} < \overline{a} \to \overline{b} \leq \overline{a}$.

Extend the ordering of C to $C \cup B$ by letting $\underline{a} < a < \overline{a}$ for all $a \in B$. If $L = C \cup B$, then $B = L \sim C$, and if $|B| = |L \sim C| = m$, then $|L| = m+n+1$.

In order to enumerate the non-isomorphic lattices in $\mathcal{SPI}$, we start with solving this problem for those lattices belonging to this class and which are *elementary* in the following sense. If $L \in \mathcal{SPI}$ and C is a subchain satisfying the conditions of Theorem 4.1, then L is *elementary*, if for a, $b \in L \sim C$, $\underline{a} = \underline{b}$, $\overline{a} = \overline{b} \to a = b$. (It can be shown that the property of being elementary is independent of the choice of C). Figure 3 contains some examples of lattices in $\mathcal{SPI}$; the elements of C are not shaded. Note that in the examples in Figure 3 the second and the third are elementary. For $n \geq 0$ and $m \geq 0$, let $\alpha(m, n)$ denote the cardinality of the set of non-isomorphic elementary lattices L in $\mathcal{SPI}$ of height n and $|L| = m+n+1$. Obviously, $\alpha(0, n) = 1$ for $n \geq 0$. We will also see that for $n \geq 1$, $\alpha(m, n) \neq$

$0 \to m \le n-1$. We will assume $m \ge 1$, $n \ge 2$. If $L \in \mathcal{SPI}$ has height n, then we have seen that there exists a chain C in L, $|C| = n+1$, such that for $a \in L \sim C$, there exists $\underline{a}$, $\overline{a} \in C$ satisfying the conditions of Theorem 4.1. We will assume that $C = \{0 < 1 < \cdots < n\}$

Let $\mathcal{L}$ be the set of non-isomorphic lattices in $\mathcal{SPI}$ that are of height n, elementary and of cardinality $n+m+1$. Thus, $|\mathcal{L}| = \alpha(m,n)$. Let $\mathcal{B}$ be the set of all sets of intervals of $[n]$ which are m-nested on a subchain of $[n]$. If $B \in \mathcal{B}$ and B is m-nested in a subchain of $[n]$ of $m+k$ elements, then by Lemma 2.1, $1 \le k \le m$. Also, $m+k \le n$, thus $1 \le k \le \min(m, n-m)$. Since there are $\binom{n}{m+k}$ subchains of $[n]$ of $m+k$ elements, it follows that

$$|\mathcal{B}| = \sum_{k=1} \binom{n}{m+k} \gamma(m, m+k).$$

Lemma 4.2. $\alpha(m,n) = |\mathcal{B}|$.

Proof: For each $L \in \mathcal{L}$ choose and fix an $(n+1)$-element chain C in L. Note that if C' is any other $(n+1)$-element chain in L, then there is a lattice automorphism of L taking C to C'. We define a map $f : \mathcal{L} \to \mathcal{B}$ as follows. For $L \in \mathcal{L}$ let $f(L) = \{[\underline{a}+1, \overline{a}] : a \in L \sim C\}$. Obviously, $[\underline{a}+1, \overline{a}]$ is a proper interval of $[n]$, since $0 \le \underline{a} < \underline{a}+1 < \overline{a} \le n$. The function f is one-one since if $f(L) = f(L')$, then $L \simeq L'$. Thus $|f(L)| = m$.

Next, we show that $f(L)$ satisfies condition (iii) of section 2. Suppose $a, b \in L \sim C$ and let $(\underline{a}, \overline{a}) = (i, j)$ and $(\underline{b}, \overline{b}) = (k, \ell)$. Thus $[i+1, j]$ and $[k+1, \ell]$ are elements of $f(L)$. We must show, that either $j < k+1$ or $\ell < i+1$ or $k+1 \le i+1 < j \le \ell$ or $i+1 \le k+1 < \ell \le j$. We assume $\underline{b} \le \underline{a}$ (the case $\underline{b} \ge \underline{a}$ is analogous). First suppose $\underline{b} = \underline{a}$, then $\overline{b} \ge \overline{a}$ or $\overline{b} \le \overline{a}$. Now $\underline{b} = \underline{a}$ and $\overline{b} \ge \overline{a} \to k+1 = i+1 < j \le \ell$. Again $\underline{b} = \underline{a}$ and $\overline{b} \le \overline{a} \to i+1 = k+1 < \ell \le j$. Next, assume $\underline{b} < \underline{a}$, then $\overline{b} \le \underline{a}$ or $\overline{b} \ge \overline{a}$. But $\overline{b} \le \underline{a} \to \ell < i+1$ and $\underline{b} < \underline{a}$ and $\overline{b} \ge \overline{a} \to k+1 < i+1 < j \le \ell$.

It follows that $f(L) \in \mathcal{B}$. The map f is also onto $\mathcal{B}$. Indeed, suppose $B \in \mathcal{B}$. Construct an element of $\mathcal{L}$, using the construction of lattices which have a series-parallel and interval order as outlined above. Start with a chain C of $n+1$ elements, $C = \{0 < 1 < \cdots < n\}$. Adjoin to C a set $\{a_{ij} : [i,j] \in B\}$ of elements such that $\underline{a_{ij}} = i-1$, $\overline{a_{ij}} = j$. Since $i < j$, we have $\underline{a_{ij}} \not< \overline{a_{ij}}$. Furthermore, $\underline{a_{ij}} < \underline{a_{k\ell}} < \overline{a_{ij}} \to \ell \le j$ and thus $\overline{a}_{k\ell} \le \overline{a}_{ij}$. Let $L = C \cup \{a_{ij} : [i,j] \in B\}$ and extend the linear ordering of C to a partial ordering of L by letting $\underline{a_{ij}} < a_{ij} < \overline{a_{ij}}$ for all $[i,j] \in B$. Then $L \in \mathcal{L}$ and it is easy to see that $f(L) = B$. It follows that f is a bijection and hence $\alpha(m,n) = |\mathcal{B}|$. □

Remark: It follows from Lemma 2.1 that $m+1 \le n$.

We infer from Lemma 4.2 and Theorem 3.2

Theorem 4.3. $\alpha(m,n) = \sum_{k=1} \binom{n}{m+k}\gamma(m, m+k) =$

$$\sum_{k=1} \binom{n}{m+k}\binom{m-1}{k-1}\gamma(m+1) \quad \textit{for} \quad 1 \le m \le n-1$$

and $\alpha(0,n) = 1$ *for* $n \ge 0$.

Using (2.1) and applying some binomial coefficient manipulations we derive from Theorem 4.3 the following formula for $\alpha(m,n)$.

$$\alpha(m,n) = \frac{1}{m+1}\binom{n-1+m}{n-1}\binom{n-1}{m} \quad \text{for } n \ge 1,\ m \ge 0 \text{ and } \alpha(0,0) = 1. \tag{4.1}$$

Recall that we required in Theorem 4.3 that $1 \le m \le n-1$. But (4.1) also holds for $m = 0, n \ge 1$ and for $m = 1$, $n = 1$ since $\alpha(0,n) = 1$ for $n \ge 1$ and $\alpha(1,1) = 0$. Thus (4.1) holds for $n \ge 1$, $m \ge 0$.

It is also easy to verify, using (4.1) that the following formula holds for $\alpha(m,n)$.

$$\alpha(m,n) = \frac{1}{n}\binom{n-1+m}{m}\binom{n}{m+1} \quad \text{for } n \ge 1,\ m \ge 0 \text{ and } \alpha(0,0) = 1.$$

Let k be fixed and vary m to obtain the sequence $\alpha(m, m+k)$, $m = 1, 2, \ldots$. For $k = 1$ we get the Catalan numbers and for $k = 2$ we have $\alpha(m, m+2) = \binom{2m+1}{m+1}$. Other small values of k yield sequences that appear in [9]. From [9] we see that the expression in (4.1) appears in a completely different context in [5, p. 449].

We will now treat the general case where the lattice does not need to be elementary. Thus we wish to enumerate the number of non-isomorphic lattices in $\mathcal{SPI}$ having height n, $n \ge 0$, and cardinality $m+n+1$. Let $\mathcal{B}(m,n)$ be a family of non-isomorphic representatives and let $\beta(m,n) = |\mathcal{B}(m,n)|$. Suppose each lattice L in $\mathcal{B}(m,n)$ has the same subchain C, with $|C| = n+1$. We determine $\beta(m,n)$ as follows. Note $\beta(0,n) = \alpha(0,n) = 1$ for $n \ge 0$ and $\beta(1,n) = \alpha(1,n) = 0$ for $n \le 1$. Therefore assume $m \ge 1$, $n \ge 2$. We partition $\mathcal{B}(m,n)$ into classes so that two lattices are in the same class if they have the same set of pairs $(\underline{a}, \overline{a})$ in C^2. Each class in this partition of $\mathcal{B}(m,n)$ contains exactly one elementary lattice from $\mathcal{SPI}_n$. So for each integer s, with $1 \le s \le \min(n-1, m)$, there are $\alpha(s,n)$ classes that have precisely s pairs $(\underline{a}, \overline{a})$. The number of ways to assign the m elements in $L \sim C$ to these s pairs is $\binom{m-1}{s-1}$. Then

$$\beta(0,n) = 1 \quad \text{for all} \quad n,$$
$$\beta(1,0) = \beta(1,1) = 0$$

and

$$\beta(m,n) = \sum_{s=1} \binom{m-1}{s-1} \alpha(s,n) \quad \text{for} \quad m \geq 1,\ n \geq 2,$$
$$= \frac{1}{n} \sum_{s=1} \binom{m-1}{s-1} \binom{n-1+s}{s} \binom{n}{s+1}$$

give the cardinality of $\mathcal{B}(m,n)$.

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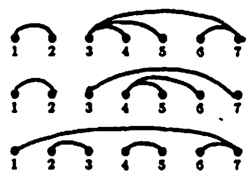


	m	k	maj(A)	cut(A)	m_L	m_R	k_L	k_R
A_1:	5	2	[3, 7]	5	3	1	2	1
A_2:	4	3	[3, 7]	2	1	2	1	1
A_3:	4	3	[1, 7]	0	0	3	0	3

Figure 1: Examples of maj(A) and cut(A)

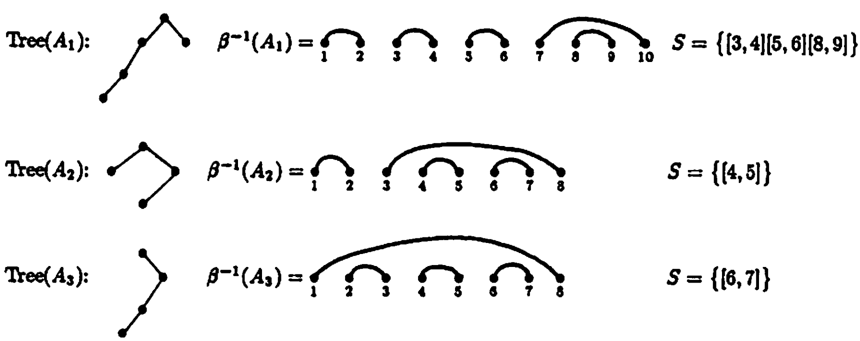
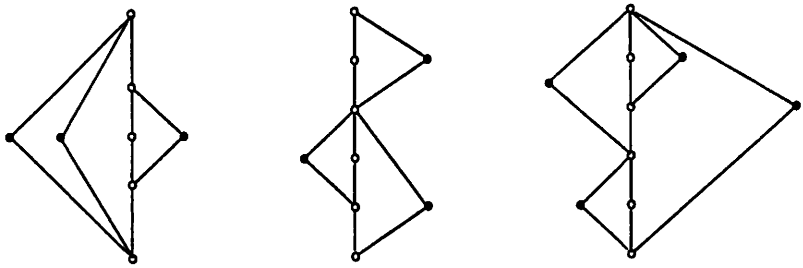
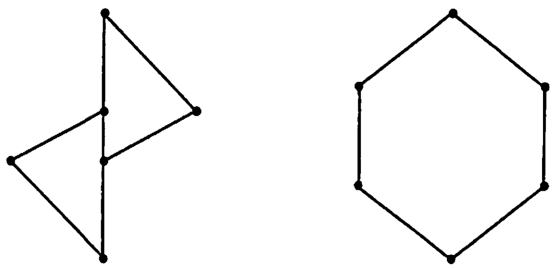


Figure 2: Examples of Tree and of β in Theorem 3.2



Examples of lattices which have a series-parallel and interval order.



Examples of lattices which are not series-parallel or not interval.
Figure 3