Computing star chromatic number from related graph invariants

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Abstract

The concept of the star chromatic number of a graph was introduced by Vince [7], which is a natural generalization of the chromatic number of a graph. In this paper, we will prove that if the complement of a graph G is disconnected, then its star chromatic number is equal to its chromatic number. From this, we derive a number of interesting results. Let G be a graph such that the product of its star chromatic number and its independence ratio is equal to 1. Then for any graph H, the star chromatic number of the lexicographic product of graphs G and H is equal to the product of the star chromatic number of G and the chromatic number of H. In addition, we present many classes of graphs whose star chromatic numbers are equal to their chromatic numbers.

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1 Introduction

The chromatic number of a graph has been well studied in the literature. In 1988, A. Vince [7] introduced the concept of the star chromatic number of a graph, which is a natural generalization of the chromatic number of a graph. His work relies on continuous methods. Later on this concept was studied from a purely combinatorial point of view by Bondy and Hell [2]. We only consider finite simple graphs (without loops or multiple edges) in this paper. Most of our definitions and notation are standard and can be found in [3], others will be defined as needed. Let k and d be positive integers such that $k \geq 2d$. Put $[k] = \{0, 1, \ldots, k-1\}$. A (k, d)-colouring of a graph $G = (V, E)$ is a mapping $c : V \rightarrow [k]$ such that $d \leq |c(u) - c(v)| \leq k - d$, for each edge $uv \in E$. A k-colouring of G is just a $(k, 1)$-colouring of G by this definition. Therefore the chromatic number of G, denoted by $\chi(G)$, is the smallest k for which there is a $(k, 1)$-colouring of G. The star chromatic number of G, denoted by $\chi^*(G)$, is defined as $\chi^*(G) = \inf\{k/d : G \text{ has a } (k, d)\text{-colouring}\}$. If $|V(G)| = n$ and G has a (k, d)-colouring, then there exist integers k' and d' such that G has a (k', d')-colouring with $k'/d' \leq k/d$ and $k' \leq n$ (cf. [2, 7]). Therefore, to calculate $\chi^*(G)$, it is enough to consider those pairs k, d such that $2d \leq k \leq n$. Thus

$$\chi^*(G) = \min\{k/d : G \text{ has a } (k, d)\text{-colouring for } 2d \leq k \leq n\}.$$

In [2, 7], it has been proved that $\chi(G) - 1 < \chi^*(G) \leq \chi(G)$, i.e., $\chi(G) = \lceil \chi^*(G) \rceil$.

We denote by $\alpha(G)$ the *independence number* of G, which is defined as the cardinality of a maximum independent set of G. The *independence ratio* of G is defined to be the fraction $i(G) = \alpha(G)/|V(G)|$. The *lexicographic product* of graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

This paper is organized as follows: In Section 2, we present many classes of graphs whose star chromatic numbers are equal to their chromatic numbers. We prove, in Section 3, that if the complement of G is disconnected, then $\chi^*(G) = \chi(G)$. We also prove that if $\chi^*(G)i(G) = 1$, then $\chi^*(G[H]) = \chi^*(G)\chi(H)$ for any graph H. As a result of these, a number of interesting results are derived.

2 Star chromatic numbers of some graphs

When we write a rational number in the form k/d, we always assume that k and d are coprime integers. For a rational number $k/d \geq 2$, the graph G_k^d has vertex set $V(G) = \{0, 1, 2, \ldots, k-1\}$ and edge set $E(G) = \{ij : d \leq$

$|i-j| \le k-d$, for $i, j \in [k]\}$. A *homomorphism* of a graph G to a graph H is a mapping f of the vertex sets $V(G) \to V(H)$ which preserves the edges, i.e., $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. If such a mapping exists, we say G is homomorphic to H and write $G \mapsto H$.

It was proved in [2, 7] that a graph G is (k,d)-colourable if and only if there is a homomorphism from G to G_k^d. In the discussion of star chromatic numbers, these graphs G_k^d take the role of complete graphs as in the discussion of chromatic numbers. It was proved in [2, 7] that $\chi^*(G_k^d) = k/d$. Since any odd cycle C_{2n+1} is isomorphic to G_{2n+1}^n, it is obvious that $\chi^*(C_{2n+1}) = (2n+1)/n$. Perhaps odd cycles of length greater than 3 are the simplest examples of those graphs such that $\chi^* < \chi$, and there are graphs such that $\chi^* = \chi$, such as complete graphs and wheels (though it is not quite easy to see that it is true). A *wheel* W_k is a graph consisting of a cycle $C_k = \{v_0, \ldots, v_{k-1}\}$ with a center vertex v adjacent to all vertices of C_k. A wheel W_k is called *odd* or *even* depending upon the parity of k. It is trivial that every even wheel has star chromatic number 3. For each odd wheel, we will see that its star chromatic number is 4. In both cases, $\chi^*(W_k) = \chi(W_k)$. In this section, we give several sufficient conditions under which $\chi^*(G) = \chi(G)$.

Lemma 2.1 (Vince [7]) *If $\chi(G) = \omega(G)$, then $\chi^*(G) = \chi(G)$.*

Theorem 2.1 *Let G be a graph obtained by deleting a Hamiltonian path from a complete graph. Then $\chi^*(G) = \chi(G) = \lceil n/2 \rceil$.*

Proof. Since $\omega(G) = \chi(G) = \lceil n/2 \rceil$, then we have $\chi^*(G) = \chi(G)$, by Lemma 2.1. ∎

Theorem 2.2 *Let G be a graph obtained by deleting a matching from a complete graph. Then $\chi^*(G) = \chi(G) = \omega(G)$.*

Proof. Suppose that the complete graph has n vertices, and the matching has size m. Then $\chi(G) = \omega(G) = n-m$, and it follows that $\chi^*(G) = \chi(G)$, by Lemma 2.1. ∎

Lemma 2.2 *There is no integer solution k' and d' for*

$$\frac{nd-1}{d} < \frac{k'}{d'} < n \quad \text{and} \quad k' \le nd-1.$$

Proof. Otherwise, we have $d' < d$ from $\frac{nd-1}{d} < \frac{k'}{d'}$ and $k' \le nd-1$. From $k' < nd'$, we have $k' \le nd'-1$. So

$$\frac{k'}{d'} \le \frac{nd'-1}{d'} = n - \frac{1}{d'} < n - \frac{1}{d}$$

leads to a contradiction. ∎

Theorem 2.3 *Let G be the graph obtained by adding edges to G^d_{nd-1} and $\chi(G) = n$. Then $\chi^*(G) = \chi(G)$.*

Proof. We know that $\chi(G^d_{nd-1}) = n$, $\chi^*(G^d_{nd-1}) = n - 1/d$. We also know that adding any edge to G^d_k increases the star chromatic number [8]. Since there is no integer solution k' and d' for

$$\frac{nd-1}{d} < \frac{k'}{d'} < n \text{ and } k' \leq nd - 1.$$

We must have $\chi^*(G) = \chi(G)$. ∎

We write $\nu(G)$ to denote the number of vertices in G, for convenience, in the following context.

Theorem 2.4 *If $\nu(G) < t\,\omega(G)$, then $\chi^*(G)$ can only take one of the following values:*

$$\chi^*(G) = \chi(G) - \frac{j}{i},$$

where $i = 2, 3, \ldots, t-1$; j is an integer between 0 and $i - 1$ inclusively.

Proof. Let K_ω, where $\omega = \omega(G)$, be the maximum clique of G. The restriction of a (k,d)-colouring of G on K_ω is a (k,d)-colouring on K_ω. Since the colour difference of any two vertices of K_ω is at least d, $k \geq d\omega$. If $d \geq t$, then

$$k \geq t\omega > \nu(G).$$

Therefore in evaluating $\chi^*(G)$, we need not to consider the case $d \geq t$. If $d = 1$, this is the ordinary colouring. If $d = i$ ($i = 2, 3, \ldots, t-1$), we need, by [2], that

$$\chi - 1 < \frac{k}{i} \leq \chi, \quad k \leq \nu(G),$$

i.e.,

$$i\chi - i < k \leq i\chi.$$

Therefore k has only i choices: $i\chi - (i-1)$, $i\chi - (i-2)$, ..., $i\chi$, i.e., we only need to consider the $(i\chi - j, i)$-colourability of G for $j = 1, 2, \ldots, i-1$. ∎

Corollary 2.1 *If $\nu(G) < 3\,\omega(G)$, then $\chi^*(G)$ is either $\chi(G)$ or $\chi(G) - \frac{1}{2}$. Furthermore if G is $(2\chi - 1, 2)$-colourable and $2\chi - 1 \leq \nu(G)$, then $\chi^*(G) = \chi(G) - \frac{1}{2}$; otherwise $\chi^*(G) = \chi(G)$.*

Corollary 2.2 *If $\nu(G) < \min\{3\,\omega(G), 2\chi(G) - 1\}$, then $\chi^*(G) = \chi(G)$.*

Grötzsch [4] proved that any triangle-free planar graph is 3-colourable. Later on, Grünbaum [5] generalized Grötzsch's result and proved that every planar graph with at most 3 triangles is still 3-colourable. Based on this result, we now give a sufficient condition under which $\chi^*(G) = \chi(G) = 3$.

Corollary 2.3 *Let G be any planar graph containing at least one triangle but no more than 3 triangles. Then $\chi(G) = \chi^*(G) = 3$.*

Proof. By Grünbaum's theorem, $\chi(G) \leq 3$. Since G contains at least one triangle, then $\chi(G) = \omega(G) = 3$. Therefore, $\chi(G) = \chi^*(G) = 3$, by Lemma 2.1. ∎

Remark: The converse of the above corollary is not true. For example, $W_{2n+1} - e$ (a subgraph from W_{2n+1} by deleting an edge e) clearly has star chromatic number 3 (W_{2n+1} is edge-critical 4-chromatic), but it contains $2n-1$ or $2n$ triangles.

3 Two sufficient conditions

Vince [7] asked for a characterization of all graphs G having $\chi^*(G) = \chi(G)$. In other words, what determines a graph G whose star chromatic number is an integer? However, Guichard [6] recently showed that the problem to decide whether or not a given graph satisfies $\chi^* = \chi$ is intractable. In spite of this, we will give a sufficient condition for a graph G such that $\chi^*(G)$ is an integer.

Zhu [8] studied some basic properties of star chromatic numbers and relations between the star chromatic numbers of graphs and their products. Zhu introduced the circle chromatic number of G, which is proved to be equivalent to the star chromatic number of G.

Definition 3.1 Let C be a circle in $\mathbb{R}^2$ of length 1, and let $r \geq 1$ be any real number. Denote by $C^{(r)}$ the set of all open intervals of C of length $1/r$. An r-circle colouring of a graph G is a mapping c from $V(G)$ to $C^{(r)}$ such that whenever $(x,y) \in E(G)$, $c(x) \cap c(y) = \emptyset$. If such an r-circle colouring exists, we say that G is r-circle colourable. The *circle-chromatic number* of G, $\chi^c(G) = \inf\{r : G \text{ is } r\text{-circle colourable}\}$.

Zhu also introduced the interval chromatic number of G, and showed that it is equivalent to the chromatic number of G.

Definition 3.2 Let I be a closed interval of length 1, and let $r \geq 1$ be any real number. Denote by $O^{(r)}$ the set of all open intervals of I of length $1/r$. An r-interval colouring of a graph G is a mapping c from $V(G)$ to $O^{(r)}$ such that whenever $(x,y) \in E(G)$, $c(x) \cap c(y) = \emptyset$. If such an r-interval colouring exists, we say that G is r-interval colourable. The *interval-chromatic number* of G, $\chi^i(G) = \inf\{r : G \text{ is } r\text{-interval colourable}\}$.

From the definitions of the circle colouring and of the interval colouring, Zhu [8] gives a nice sufficient condition for $\chi^*(G)$ to be an integer.

Proposition 3.1 (Zhu [8]) *For any graph G, $\chi(G) = \chi^*(G)$ if and only if for any real number r, if G is r-circle colourable then there is an r-circle colouring c of G and an $x \in C$ such that $x \notin c(g)$ for any $g \in V(G)$.*

An immediate consequence of this is the following. A vertex is called *universal* if it is adjacent to all other vertices in a graph.

Theorem 3.1 (Zhu [8]) *If G has a universal vertex then $\chi^*(G) = \chi(G)$.*

As a corollary, one can see that $\chi^*(W_{2n+1}) = \chi(W_{2n+1}) = 4$ for $n \geq 1$. The sufficient condition in Theorem 3.1 can be strengthened a bit by relying on Zhu's results (cf. [8]). The following theorem was independently proved by Abbott and Zhou [1] recently.

Theorem 3.2 *Let G be a graph such that its complement is disconnected. Then $\chi^*(G) = \chi(G)$.*

Proof. We may assume that $V(G) = V_1 \cup V_2$, where there are edges between every vertex of V_1 and every vertex of V_2. Suppose c is an r-circle colouring of G for some rational number r. Take $v_1 \in V_1$ and $v_2 \in V_2$. Let $c(v_1) = (a_1, b_1)$, $c(v_2) = (a_2, b_2)$, then $(a_1, b_1) \cap (a_2, b_2) = \emptyset$, where (a_i, b_i) $(i = 1, 2)$ are intervals on the unit length circle C in $\mathbb{R}^2$. We may assume a_1, b_1, a_2 and b_2 appear on C in clockwise order. We may further define an order $\prec$ as the clockwise order on the vertices of the circle C from a_1 to b_2 (i.e., regard a_1 as the smallest, and b_2 as the largest). Let

$$t = \max\{b : (a, b) = c(v) \text{ for } v \in V_1, b \prec a_2\}.$$

Then t does not belong to any $c(v)$ $(v \in V_1)$, for otherwise it contradicts the maximality. The vertex t does not belong to any $c(v)$ $(v \in V_2)$ either since there is an edge between each vertex of V_1 and each vertex of V_2. Therefore, $\chi^*(G) = \chi(G)$ by Proposition 3.1. ∎

Theorem 3.3 *Let G be a graph obtained by deleting a 2-factor F from a complete graph, then*

$$\chi^*(G) = \begin{cases} \chi(G) - \frac{1}{2} & \textit{if } F \textit{ is a Hamiltonian cycle of odd length,} \\ \chi(G) & \textit{otherwise.} \end{cases}$$

Proof. In fact F is the complement of G. If F has more than one cycle, then F is disconnected, so $\chi(G) = \chi^*(G)$, by Theorem 3.2. If F has only one cycle, then G is isomorphic to $G^2_{\nu(G)}$, so $\chi^\star(G) = \frac{\nu(G)}{2}$, which equals to $\chi(G)$ if $\nu(G)$ is even, and equals to $\chi(G) - 1/2$ if $\nu(G)$ is odd. ∎

It was proved in [8] that $\chi^\star(G[H]) \leq \chi^\star(G)\chi(H)$ for any two graphs G and H, and that $\chi^\star(G[H]) = \chi^\star(G[K_n])$ if G contains at least one edge and $\chi(H) = n$.

In the following, we present a sufficient condition under which $\chi^\star(G[H]) = \chi^\star(G)\chi(H)$.

Theorem 3.4 *Let G be a graph containing at least one edge and satisfying $\chi^\star(G)i(G) = 1$. Then*

$$\chi^\star(G[H]) = \chi^\star(G)\chi(H).$$

Proof. Let $\chi(H) = m$. It is known (cf. [8]) that $\chi^\star(G[K_m]) \leq \chi^\star(G)\chi(K_m)$. The following two facts are well-known, and also easy to prove:

$$\alpha(G) = \alpha(G[K_m]), \quad \chi^\star(G)i(G) \geq 1.$$

Thus, we have

$$\chi^\star(G[K_m]) \geq \frac{1}{i(G[K_m])} = \frac{|V(G[K_m])|}{\alpha(G[K_m])} = \frac{|V(G)|}{\alpha(G)}m = \frac{\chi(K_m)}{i(G)}.$$

From the assumption that $\chi^\star(G)i(G) = 1$ and the two inequalities above, it follows that $\chi^\star(G[K_m]) = \chi^\star(G)\chi(K_m)$. Thus, we obtain the equality $\chi^\star(G[H]) = \chi^\star(G)\chi(H)$ provided that $\chi^\star(G)i(G) = 1$. This completes the proof. ∎

Corollary 3.1 *For any graph H, two positive integers k and d, $k \geq 2d$, we have*

$$\chi^\star(G^d_k[H]) = \chi^\star(G^d_k)\chi(H) = k\chi(H)/d.$$

Proof. For the circulant graph G^d_k, a subset $\{0, 1, \ldots, d-1\}$ of $V(G^d_k)$ is an independent set, in other words, $\alpha(G^d_k) \geq d$. Suppose that the equality does not hold, that is, $\alpha(G^d_k) > d$. Let S be an independent set whose cardinality is $\alpha(G^d_k) > d$. Then there exist two vertices u and v in S such that $d \leq |u - v| \leq k - d$, which implies that u and v are adjacent, a contradiction. Thus, we have $\alpha(G^d_k) \leq d$. Therefore, we have proved that $i(G^d_k) = d/k$, which implies that $\chi^\star(G^d_k)i(G^d_k) = 1$. Thus the corollary follows from Theorem 3.4. ∎

Theorem 3.5 *Let H_i be any graph, and G_i a graph satisfying $\chi^\star(G_i)i(G_i) = 1$, for $i = 1, 2, \ldots, s$. Let F be the graph obtained from $K_s = \{v_1, v_2, \ldots, v_s\}$ with the replacement of v_i by $G_i[H_i]$. Then*

$$\chi^\star(F) = \lceil \chi^\star(G_1)\chi(H_1) \rceil + \lceil \chi^\star(G_2)\chi(H_2) \rceil + \cdots + \lceil \chi^\star(G_s)\chi(H_s) \rceil.$$

Proof.

$$\begin{aligned}
\chi^*(F) &= \chi^*(G_1[H_1] + G_2[H_2] + \cdots + G_s[H_s]) \\
&= \chi(G_1[H_1] + G_2[H_2] + \cdots + G_s[H_s]) \qquad \text{(By Theorem 3.2)} \\
&= \chi(G_1[H_1]) + \chi(G_2[H_2]) + \cdots + \chi(G_s[H_s])) \\
&= \sum_{i=1}^{s} \lceil \chi^*(G_i[H_i]) \rceil \qquad \text{(By [2, 7])} \\
&= \sum_{i=1}^{s} \lceil \chi^*(G_i)\chi(H_i) \rceil . \qquad \text{(By Theorem 3.4)}
\end{aligned}$$

∎

Corollary 3.2 *In Theorem 3.5, for $i = 1, 2, \ldots, s$, if G_i is replaced by $G_{k_i}^{d_i}$, H_i is replaced by K_{t_i}, then*

$$\chi^*(F) = \sum_{i=1}^{s} \left\lceil \frac{k_i}{d_i} t_i \right\rceil .$$

In Theorem 3.5, for $i = 1, 2, \ldots, s$, G_i can be degenerated to a single vertex or a single edge.

Corollary 3.3 *For any integers $n, m \geq 1$, if $\chi(H) = m$, then*

$$\chi^*(W_{2n+1}[H]) = \chi(W_{2n+1}[H]) = 3m + \lceil m/n \rceil .$$

Proof. In Theorem 3.5, set $s = 2$, let G_1 be a single vertex, and G_2 be an odd cycle C_{2n+1}, set $H = H_1 = H_2$. Then the resulting graph F is clearly $W_{2n+1}[H]$. By applying Theorem 3.5, we have

$$\begin{aligned}
\chi^*(W_{2n+1}[H]) &= \lceil \chi(H) \rceil + \lceil \chi^*(C_{2n+1}[H]) \rceil \\
&= m + \lceil (2n+1)m/n \rceil \\
&= 3m + \lceil m/n \rceil .
\end{aligned}$$

Thus, the proof is completed. ∎

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